Ph.D. Comprehensive Examination (Algebra)
Department of Mathematics
The University of Western Ontario
May 2023 3 hours

Instructions: More credit may be given for a complete solution than for several partial solutions, so you should aim to solve some problems completely and correctly rather than attempting every problem. You should attempt at least one question from each topic. Justify all of your answers. Start each solution on a new page.

Linear Algebra.

(1) Let A be a non-zero real symmetric $n \times n$ matrix with trace zero. Let $f(x) = (x-1)(x+5)^3$ and suppose that $f(A) = 0$.

(a) Determine the minimal polynomial of A.

(b) Determine the characteristic polynomial of A as a function of n. Identify any constraints you observe about the possible values of n.

(2) (a) Let $n \geq 1$. Show that there exists a matrix $A \in M_n(\mathbb{Q})$ such that for any nonzero $v \in \mathbb{Q}^n$, the set $\{v, Av, A^2v, \dots, A^{n-1}v\}$ is a basis of $\mathbb{Q}^n$. Hint: you can assume that there is an irreducible polynomial of degree n over $\mathbb{Q}$ for every $n \geq 1$.

(b) For which n is this possible if $\mathbb{Q}$ is replaced by $\mathbb{C}$ everywhere? Explain.

Rings and modules.

(3) (a) Show that $\mathbb{Z}[x]/(3, x^4 + x^3 - x - 1) \cong \mathbb{F}_3[x]/(x^4 + x^3 - x - 1)$ as rings.

(b) Describe all ideals of the ring $\mathbb{Z}[x]/(3, x^4 + x^3 - x - 1)$.

(4) Let R be a principal ideal domain. Let M be a finitely generated torsion R-module.

(a) Let $m \in M$ have annihilator ideal $\mathrm{Ann}_R(m) = (b)$ and let $a \in R$. Prove that

$$\mathrm{Ann}_R(am) = \left(\frac{b}{\gcd(a,b)}\right)$$

In particular, show that $a(Rm) = 0$ if b divides a, and show that $a(Rm) = Rm$ if $\gcd(a,b) = 1$.

(b) Let p be a prime in R. Let $k \in \mathbb{N}$. Let

$$\{p^{c_i} : i = 1, \dots, s, c_1 \geq c_2 \geq \dots \geq c_s\}$$

be the set of p-elementary divisors for M. Show that $\dim_{R/(p)}(p^{k-1}M/p^kM) = r_k$, where

$$r_k = |\{1 \leq i \leq s : c_i \geq k\}|$$

Hint: First write $M = \oplus_{i=1}^t Rm_i$ where $\mathrm{Ann}_R(m_i) = (q_i^{c_i})$ is an elementary divisor of M with respect to some prime q_i of R.

Group theory.

(5) Consider the group $\mathrm{SL}_2(\mathbb{F}_3)$ of 2×2 matrices with determinant 1 over $\mathbb{F}_3$.

(a) Determine the order of $\mathrm{SL}_2(\mathbb{F}_3)$.

(b) The subgroup $H = \{I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\}$ is a Sylow 3-subgroup. Determine how many Sylow 3-subgroups $\mathrm{SL}_2(\mathbb{F}_3)$ has, and show that the normalizer of H is the subgroup U of upper triangular matrices with determinant 1.

(6) (a) Let $G = G_1 \times G_2$ be a finite group with $\gcd(|G_1|, |G_2|) = 1$. Prove that every subgroup H of G is of the form $H = H_1 \times H_2$, where H_i is a subgroup of G_i for $i = 1, 2$. (The claim is about *equality* of subgroups, not isomorphism.)

(b) Show that this is false without the assumption that $\gcd(|G_1|, |G_2|) = 1$.

Field theory.

(7) Let $\alpha = \sqrt{a + \sqrt{b}}$ where $a, b \in \mathbb{Q}$ and b is not a square in $\mathbb{Q}$.

(a) Find the minimal polynomial $m_{\alpha^2,\mathbb{Q}}(x)$ of $\alpha^2 = a + \sqrt{b}$ over $\mathbb{Q}$. Set $f(x) = m_{\alpha^2,\mathbb{Q}}(x^2)$ and find its roots $\pm\alpha, \pm\beta$ in $\mathbb{C}$. Compute $(\alpha \pm \beta)^2$, $\alpha \pm \beta$ and $\alpha\beta$ in terms of $a, b \in \mathbb{Q}$.

(b) Let K be the splitting field of $f(x) \in \mathbb{Q}[x]$ contained in $\mathbb{C}$. Find the possible factorizations of $f(x) \in \mathbb{Q}[x]$ as a product of irreducibles in $\mathbb{Q}[x]$.
Hint: Consider the factorizations of $f(x)$ and $m_{\alpha,\mathbb{Q}}(x)$ in $K[x]$.

(8) Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible quartic with Galois group $G \cong A_4$. Let K be the splitting field of $f(x) \in \mathbb{Q}[x]$. Let E be the splitting field of the resolvent cubic $h(x)$ of $f(x)$ and let $H = \mathrm{Gal}(K/E)$.
Note: If $\alpha_1, \ldots, \alpha_4$ are the roots of a quartic polynomial, then

$$\theta_1 = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4), \theta_2 = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4), \theta_3 = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3)$$

are the roots of its resolvent cubic.

(a) Determine $\mathrm{Gal}(E/\mathbb{Q})$ and $\mathrm{Gal}(K/E)$ up to isomorphism. Show that there exists a subgroup H of $\mathrm{Gal}(K/\mathbb{Q})$ such that $\mathrm{res}_E : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Gal}(E/\mathbb{Q}), \sigma \to \sigma_E$ maps H isomorphically onto $\mathrm{Gal}(E/\mathbb{Q})$.

(b) Show that $\mathrm{Gal}(K/\mathbb{Q}) = H\,\mathrm{Gal}(K/E)$ and that every element $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ can be written as $\sigma = h\tau$ for unique $h \in H$, $\tau \in \mathrm{Gal}(K/E)$.

Do not forget to justify your answers!