THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

25 May 2015
Part II: 1:00 pm - 3:30 pm

Instructions: Part II covers slightly more advanced material and you have the choice of 4 of 7 questions. In Part II, 60% is required for a passing grade.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

PART II: Do 4 of the following 7 questions. If you attempt more than 4, only the first four to be marked will count.

1. **Partial Differential Equations**

Suppose $u_0(x)$, $f(u)$ are continuous bounded functions on $\mathbb{R}$ and

$$u_t = u_{xx} + f(u), \quad u(x,0) = u_0(x), \quad \text{for} \quad x \in \mathbb{R} \tag{1}$$

where f in (1) satisfies a Lipschitz condition on $\mathbb{R}$ with constant L:

$$|f(u) - f(v)| \le L|u - v| \quad \text{for all} \quad u, v \in \mathbb{R}. \tag{2}$$

You can assume that equation (1) can be reformulated as the equivalent integral equation of form $u(x,t) = \mathscr{F}(u(x,t))$ where

$$\mathscr{F}(u(x,t)) := \int_{-\infty}^{+\infty} G(x-\tilde{x},t)u_0(\tilde{x})\, d\tilde{x} + \int_0^t \int_{-\infty}^{+\infty} G(x-\tilde{x},t-\tau) f(u(\tilde{x},\tau))\, d\tilde{x} d\tau \tag{3}$$

and $G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-x^2/(4t))$ satisfies $\int_{-\infty}^{+\infty} G(x-\tilde{x},t)\, d\tilde{x} = 1$.

(a) Show that in an appropriate norm:

$$|\mathscr{F}(u) - \mathscr{F}(v)| \le Kt\|u - v\| \tag{4}$$

(b) The Banach fixed point theorem states that a mapping $\mathscr{H}$ which is a contraction

$$\|\mathscr{H}(w) - \mathscr{H}(z)\| \le C\|w - z\|$$

for some $0 < C < 1$ on a Banach space has a unique fixed point. Use this result and (4) to prove a local existence and uniqueness theorem for solutions of the IVP.

2. **Numerical Methods**

   (a) Give the central difference formula for the second derivative. Explicity *derive* the truncation error associated with this formula.

   (b) Using the results of (a), derive a finite diference scheme for solving $\partial_t^2 u - c^2 \partial_x^2 u = 0$, $t > 0$, $0 \leq x \leq 1$. Don't worry about starting conditions or boundary conditions; just give the method for propagating forward after the initial step. What is the order of accuracy of this scheme (don't just state the answer, show why it is the case)?

   (c) What is the CFL condition and how does it apply to the method you derived in (b)? Hint: What are the characteristic curves for this equation?

   (d) Derive the amplification factor for the method in (b), assuming c is a constant.

3. **Statistical Mechanics**

   The motion of a particle in three dimensions is a series of independent steps of length a. Each step makes an angle θ with the z axis with a probability $p(\theta) = (2/\pi)\cos^2(\theta/2)$, $\theta \in [0, \pi]$; while the azimuthal angle ϕ is uniformily distributed between 0 and 2π. (Note that the solid angle factor of $\sin\theta$ is already included in the definition of $p(\theta)$, which is correctly normalized to unity.) The particle starts at the origin and makes a *large* number of steps N.

   (a) Calculate the expectation values of x, y, z, their variances, and covariances after N steps.

   (b) Use the central limit theorem to estimate the probability density $p(x, y, z)$ for the particle to end up at the point (x, y, z) after N steps. Explain why it is appropriate to use the central limit theorem here.

4. **Advanced Linear Algebra**

   (a) i. Use spectral factorization to diagonalize the quadratic form $p(x) = x^2 - 3xy + 5y^2$.

   ii. Describe in words another method that you could have used to diagonalize $p(x)$. Would the answer be the same?

   (b) Use the Gram-Schmidt process to determine an orthonormal basis for the corange of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix}$, using the weighted inner product $< \vec{v}, \vec{w} >= 3v_1w_1 + 2v_2w_2 + v_3w_3$.

   (c) Prove that every 2×2 symmetric matrix which has positive diagonals and a positive determinant is positive definite.

5. **Colloquia:** Comment intelligently (i.e. something not in the abstract and enough to convince us that you attended), in a few sentences, on 10 of the following colloquium topics.

   (a) Malaria dynamics in seasonal environment with long incubation period in hosts.
   (b) Academic wages, singularities, phase transitions and pyramid schemes.
   (c) Rich bifurcation structure of disease transmission models in patchy environment.
   (d) Supintegrability in Classical and Quantum Mechanics.
   (e) An immersed boundary method for mass transfer across permeable moving interfaces.
   (f) Modeling of Contact Tracing in Epidemic Populations by Disease Age.
   (g) Time of death estimation from temperature readings only: a Laplace transform approach.
   (h) Equation-Based Modeling: Structural Analysis and Hybrid Systems.
   (i) Compartmental modeling of lysogenic and lytic cycles during phage-bacteria interaction.
   (j) Specifying Nodes as Sets of Choices.
   (k) Waldegrave's Problems in Probability.
   (l) Facial Reduction for Cone Optimization with Applications to Systems of Polynomial Equations, Sensor Network Localization, and Molecular Conformation.
   (m) Effective simulation of stochastic models of biochemical systems.
   (n) The homotopy analysis method: rigorous proof of convergence and its applications.
   (o) Optimal Backward Error and Dahlquist Test Problem.

6. **Mathematical Biology**

   Consider a bacterial strain that consumes some resource. At time t and location x the density of the bacteria is given by $b(t, x)$, and the density the resource is given by $a(t, x)$. Assume that the resource does not move through space, is non-renewable, and is consumed by bacteria at per-capita rate $k > 0$. In addition, ignore death and birth of bacteria, and model its movement as the superposition of diffusive flux and a chemotatic response to resource density. To be clear, model bacterial movement as $J = -D\, b_x + \chi_0(a) b a_x$, where $D > 0$ is the diffusion-rate constant, and $\chi_0(a) = \chi/a$ for $\chi > 0$ constant (and $\chi \neq D$) implies a chemotatic response that increases with decreasing resource density. Write down a pair of partial differential equations for $b(t, x)$ and $a(t, x)$, and show that the equations admit a travelling wave solution that also satisfies $\lim_{|z|\to\infty} b = 0$, $\lim_{z\to-\infty} a = 0$, and $\lim_{z\to\infty} a = 1$.

7. **Quantum Mechanics**

(a) Using $\vec{p} = -i\hbar\vec{\nabla}$, as in the coordinate-space (Schrödinger) representation of quantum mechanics, show that $[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$.
Hint: consider $[x, p_x]\psi(x, y, z)$.

(b) If $\vec{L} = \vec{r} \times \vec{p}$, use the above results to show that $[L_x, L_y] = i\hbar L_z$.

(c) Generalize your results of parts (a) and (b) to show that $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$.

(d) Define $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$, and show that $[L_z, L_+] = \hbar L_+$ and $[L_z, L_-] = -\hbar L_-$.

(e) If $|m\rangle$ is an eigenstate of L_z with eigenvalue $m\hbar$, $L_z|m\rangle = m\hbar|m\rangle$, use the commutation relations of part (d) to show that $L_+|m\rangle$ is proportional to the state $|m+1\rangle$ and that $L_-|m\rangle$ is proportional to the state $|m-1\rangle$.

(f) Derive the relation $L_+L_- = \vec{L}^2 - L_z^2 + \hbar L_z$.