THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

31 May 2017
Part I: 9 am - 12 pm

Instructions: The exam consists of Part I only. Part I consists of mandatory problems and covers basic material. In Part I, 80% is required for a passing grade.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

PART I: Solve ALL PROBLEMS presented below.

1. **Linear Algebra**

   (a) Let $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{pmatrix}$. Diagonalize A, that is, express A as the product PDP^{-1} where D is a diagonal matrix.

   (b) If $P = [a \quad 0 \quad -3]$ and $Q = [a-1 \quad 1 \quad 10]$, find a such that $\overrightarrow{PQ}$ and $\overrightarrow{RQ}$ are perpendicular.

   (c) B is an orthogonal matrix, that is, $B^{-1} = B^T$. What are the possible values for the determinant of B?

2. **Calculus**

   (a) Suppose $a, b \in \mathbb{R}$ with $a < b$. To what value does

   $$\lim_{n\to\infty}\left[\sum_{i=1}^{n} e^{a+i\frac{b-a}{n}}\,\frac{(b-a)}{n}\right]$$

   converge? Explain briefly.

   (b) Use integration by parts to evaluate $\int x \sin x \, dx$.

   (c) Find the radius of convergence of the following power series:

   $$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{n^2(x+2)^{n-1}}{3^{n+1}}$$

   You do not need to find the exact interval convergence.

   (d) Consider the function $f(x, y) = x^3y + 2x^2y^2 - xy^3$.

   i. What is the rate of change of f at $(2, 3)$ in the direction $(1, -1)$?

   ii. We can plot the level curves of f as contour lines in the x, y-plane. What vector field, $g : \mathbb{R}^2 \to \mathbb{R}^2$, is perpendicular to the level curves of f? Explain briefly.

Continued on reverse

3. **Ordinary Differential Equations**

   (a) Solve, $y' = \dfrac{\cos x}{\sec^2 y}$ with $y(0) = \dfrac{\pi}{4}$. *Hint:* what is the derivative of tan?

   (b) Use the Bernoulli substitution, $u = y^{-2}$ to solve $x^2 y' + 2xy - y^3 = 0$ for $x > 0$.

   (c) Solve $y'' + y = 3\cos 2x$ with $y(0) = y'(0) = 1$.

   (d) Recall that the Laplace transform $\mathcal{L}$ is linear, as is its inverse. Recall also that $\mathcal{L}(u'(t)) = s\mathcal{L}(u(t)) - u(0)$, and $\mathcal{L}(e^{ct}) = \dfrac{1}{s-c}$ for $s > c$. Use the Laplace transform to solve $u'(t) = -2u(t) + 3e^{-3t}$ with $u(0) = 2$. *Hint:* partial fractions can help you invert.

4. **Numerical Methods** Explicitly show how you obtain your numerical answers in the following.

   (a) i. What issues arise in numerically evaluating $\ln(x+1) - \ln x$ for large x?

       ii. Rearrange the function to avoid this problem at large x.

   (b) One potential way of computing an integral with an integrand that lacks an antiderivative is to use analytic substitution. For example,

   $$I = \int_0^1 e^{-x^2}\,dx \approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!}\right) dx. \qquad (1)$$

   Bound the error in this expression using the remainder formula for the Taylor polynomial being used.

   (c) i. Perform 4 iterations on Newton's method for the function

   $$f(x) = \pi/2 + x - \cos x, \qquad (2)$$

   using $x_0 = -1$ as a starting point.

       ii. Comment on the apparent rate of convergence for the iterates you found and what the implications of this might be.

   (d) A function has been evaluated at 3 points, $(x_i, f(x_i))$ with values $(0, 1)$, $(0.25, 0.5)$, and $(0.75, 0.25)$. Estimate a value for $f'(0.5)$ and the integral of $f(x)$ from 0 to 0.75 as accurately as possible. (Hint: Constructing an interpolating function first will make this easier.)