THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

31 May, 2017
Part II: 9 am - 12 pm

Instructions: Part II covers slightly more advanced material and you have the choice of 4 of 8 questions. In Part II, 60% is required for a passing grade.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files.

PART II: Do 4 of the following 8 questions. If you attempt more than 4, only the first four to be marked will count.

1. **Partial Differential Equations**

   Consider the integral conservation law for a density of a chemical $u(x,t) \geq 0$ with associated flux ϕ and reaction term $f(u)$ in a one dimensional thin tube is given by:

$$\frac{\partial}{\partial t}\int_a^b u\,dx = -\left[\phi\right]_a^b + \int_a^b f(u)\,dx \tag{1}$$

   (a) Justify the integral form of the conservation law given above.

   (b) Show the differential form of the above conservation law (1) for $\phi = \phi(u)$ is given by

$$u_t + \phi'(u)u_x = f(u) \tag{2}$$

   State any assumptions you made. What type of solutions of (1) with smooth initial condition $u(x,0) = u_0(x)$ are not given by solutions of (2) with the same initial condition?

   (c) Consider the case $\phi = -u_x$ where differential form of the conservation law (1) yields IVP:

$$u_t = u_{xx} + f(u), \quad u(x,0) = u_0(x) \text{ for } x \in \mathbb{R} \tag{3}$$

   Apply the Fourier Transform $\mathcal{F}(v(x,t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} v(x,t)e^{-ikx}\,dx$ to this IVP where you assume that $f(u(x,t)) = F(x,t)$. Solve for the transformed variable $\hat{u} = \mathcal{F}[u]$, but don't simplify the inverse transform.

   (d) Simplify the inverse transform (see the attached Table of Fourier Transforms). Use your result to reformulate (3) as a nonlinear integral equation of form $u = \psi(u)$. Briefly [10 words max, point form OK] what is an application of this integral equation?

2. **Numerical Methods**

Consider the $n \times n$ upper bidiagonal system $A_n x = b$ where

$$A_n = \begin{bmatrix} 1 & -2 & & & \\ & 1 & -2 & & \\ & & 1 & \ddots & \\ & & & \ddots & -2 \\ & & & & 1 \end{bmatrix}.$$

(a) Perhaps by inverting the cases $n = 2, 3, 4$, guess the pattern and deduce A_n^{-1} (which is Toeplitz, like A_n) for general dimension n.

(b) Compute a condition number for $A_n x = b$ using your result from part (a). You may use any matrix norm: the 1–norm and the ∞–norm are the easiest, but the Frobenius norm is almost as easy. The 2–norm is more difficult but by choosing wisely you can bound $\|A_n^{-1}\|_2$ by using $\|A_n^{-1}b\|_2 \leq \|A_n^{-1}\|_2 \, \|b\|_2$. Whatever norm you use, name it correctly.

(c) Compare the cost of solving $A_n x = b$ by back substitution to the cost of solving $A_n x = b$ by computing $x = A_n^{-1}b$ by multiplying the explicit inverse. [Just count flops: don't worry about memory use].

(d) The solution of a triangular system is componentwise backward stable. Explain what that means, and explain why x might not be accurate even so if $n \geq 52$.

3. **Numerical Solution of Differential Equations**

Consider the boundary value problem:

$$u'' = -(x+1)u; +2u + (1 - x^2)e^{-x} \quad (4)$$

$$u(0) = 1, \qquad u(1) = 0. \quad (5)$$

(a) Divide the interval $[0, 1]$ into N equally spaced intervals of width h. Construct a second-order finite difference method for solving this equation by replacing the first and second derivatives by central difference approximations.

(b) Rewrite your result as a linear system of equations $A\mathbf{w} = \mathbf{b}$, giving the first and last three rows/columns of A, $\mathbf{w}$, and $\mathbf{b}$.

(c) Write out the matrix equation explicitly for $h = 1/4$.

(d) Describe a numerical algorithm that could be used to solve the linear system found for arbitrary $h < 1$. (You do not have to write out the algorithm explicitly, just note any symmetries of the system of equations that might be useful, whether one needs to consider pivoting, and what algorithms you might suggest someone use to solve this system.)

4. **Dynamical Systems**

   Consider the stability of the origin of the Lorenz System:

$$\begin{aligned}\dot{x} &= a\,(y - x)\\ \dot{y} &= c\,x - y - x\,z\\ \dot{z} &= x\,y - b\,z,\end{aligned}$$

   where $a > 0,\ b > 0$.

   (a) Apply linearization to show that the origin is asymptotically stable for $c < 1$.

   (b) Use a Lyapunov function to show that the origin is globally asymptotically stable for $-3 < c < 1$.

5. **Colloquia:** Comment intelligently (i.e. something not in the abstract and enough to convince us that you attended), in a sentence or two, on 7 of the following colloquium topics.

   (a) Riemann surfaces and Branch cuts

   (b) Toward real-time cardiac simulation

   (c) Markov chains, mark-recapture, and misidentification: why I am Bayesian and how you could be too

   (d) Structure-Revealing Condensed Forms for Matrix Polynomials

   (e) Critical transitions in coupled behaviour-disease systems: applying dynamical systems theory to data science problems

   (f) Viruses, computers, and the shapes of trees

   (g) The Lambert W function in ecological and evolutional models

   (h) Challenges in multivalued matrix functions

   (i) Exponential polynomials and generalized Lambert W function

   (j) Consistent chiral kinetic theory in Weyl materials: chiral magnetic plasmons and helicons

   (k) A couple of Histroric Comments About the W Function and How Mathematical Modelling Is Producing Weird and Incorrect Results in Science

   (l) Regularisation Based Time-adaptive Solution For a Class of Highly Degenerate Diffusion-Reaction Systems With Gradient Blow-up

   (m) A Hamiltonian for the zeros of the Riemann zeta function

   (n) Stationary Dirac Concentrations in an Integro-PDE arising from Evolution of Dispersal

6. **Evolutionary Genetics**

Consider a dominant advantageous allele A with selection coefficient s. You may assume s is small.

(a) Give an expression for the change in allele frequency p in a single generation.

(b) Find the equilibrium gene frequency. Consider the stability of possible equilibria.

(c) Assume further that allele A mutates to a at rate u. Neglecting back mutation, find the equilibrium gene frequency. Assume both s and u are sufficiently small that terms of order u^2 *etc.* are negligible.

(d) What is the name for the principle you have derived?

7. **Mathematical Biology**

The purple butterfly, *Fictitious westerni*, inhabits a patchy habitat with constant total number of habitat patches. In each habitat patch the local butterfly population either (1) has disappeared, (2) has become threatened, or (3) is thriving. Each threatened local population disappears at positive rate α. Each thriving local population becomes threatened also at positive rate α. Because of overcrowding, groups of butterflies emigrate from each thriving local population at positive rate γ. Each group then settles on a new habitat patch chosen uniformly at random from all patches in the environment (including the one it left). If the group settles on a patch where the local population had disappeared, then that local population is instantaneously classified as threatened. If the group settles on a patch where the local population had been classified as threatened, then the local population instantaneously becomes thriving. If the group settles on a patch where the local population had been classified as thriving, no change to that local population's status occurs.

(a) Write down a system of three ODEs that describes how the number of the various kinds of local populations, n_i for $i = 1, 2, 3$, changes over time t.

(b) Reduce the system of three ODEs to a dimensionless system of two ODEs. Make sure to rescale time so that $a = \alpha/\gamma$ is the only parameter.

(c) Show that total extinction corresponds to a locally asymptotically stable equilibrium solution of the system in (b).

(d) Show that, under certain conditions, there are two distinct equilbrium solutions of the system in (b) that occur in the interior of the phase space. Show that one of the equilibria is locally asymptotically stable and the other is not.

(e) Sketch a bifurcation diagram that summarizes your findings in (c) and (d).

8. **Computer Algebra**

(a) Given two polynomials

$$p(x) = \sum_{k=0}^{n} p_k x^k \ ,$$
$$q(x) = \sum_{k=0}^{m} q_k x^k \ ,$$

describe how the *Sylvester matrix* is formed, and how it is used to decide whether $p(x)$ and $q(x)$ have a common root.

(b) For the following polynomials

$$p(x) = 6x^2 + 13x - 5 \ ,$$
$$q(x) = 6x^2 + 17x + 5 \ ,$$

use the Sylvester matrix to decide whether there is a common root. [Obviously, solving the polynomials using the quadratic formula gets no credit.]

(c) Given bivariate polynomials $P(x, y)$ and $Q(x, y)$, describe how the Sylvester matrix can be used to solve for x and y.

(d) Consider the polynomials

$$P(x, y) = 12x^2y + 13x - 10y \ ,$$
$$Q(x, y) = 24y^2x^2 + 34xy + 5 \ .$$

Using the Sylvester matrix, eliminate x and obtain an equation only in y.

(e) [**Bonus**] Comment on the solutions obtained by this method.