THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

Monday 10, June 10, 2019
9:00 am - 12:00

The exam consists of two parts. Part I contains mandatory problems and covers basic material, while Part II covers slightly more advanced materials at graduate course level.

This is **Part I**, for which **80% is required to pass**. You may use a calculator, pen, and pencil. **NO other aids** are allowed. Your calculator **must NOT** be capable of wireless communication or capable of storing and displaying large text files.

PART I. Do **all** of the following questions.

Calculus:

C1. Evaluat the limit

$$\lim_{x\to 0} \frac{1}{x^4} \int_{x^2}^{0} \tan t \, dt.$$

C2. Find the function that is represented by the power series $\sum_{n=1}^{\infty} nx^n$ on $(-1, 1)$.

C3. Find condition(s) on r and d such that the equation $bxe^{-x} - dx = 0$ has a *positive solution* . Under the condition(s) you find, is the positive solution unique?

C4. Find the local maximum/minimum and saddle points (if any of the function $f(x, y) = x^4 + y^4 - 4xy + 1$.

C5. Evaluate the $\iint_{\Omega}(3x + 4y^2)dA$ where Ω is the region in the *first quadrant* of the $x - y$ plane bounded by the curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

C6. Evaluate the indefinite integral $\int x \ln x \, dx$.

Linear Algebra:

L1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

(i) Find an invertible matrix S such than $S^{-1}AS$ is a diagonal matrix.

(ii) Can you also find a matrix S such than $S^{-1}BS$ is a diagonal matrix ? If yes, find such an S, and if no, explain why.

L2. Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Find conditions on real numbers a, b and c such that $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. is orthogonal to both v_1 and v_2 with respect to the standard Euclidian product in $\mathcal{R}^3$.

L3. Let the matrix M be defined by

$$M = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}.$$

where a, b, c are real numbers.

(i) Evaluate the determinant $\det M$

(ii) Using the result in (i) to find the conditions on a, b, c under which M is non-singular (invertible).

Ordinary Differential Equations:

O1. Find the general solution to the Euler type equation $t^2u''(t) - tu'(t) - 3u = 0$ for $t > 0$.

O2. Solve the initial value problem $u'(t) = te^{-\sin t} - u\cos t,\;\; u(0) = 1$.

O3. Find the general solutions of the second order linear ODEs

$$u''(t) + 4u(t) = 3\cos t \tag{1}$$

and

$$u''(t) + 4u(t) = 3\cos(2t). \tag{2}$$

Comment/explain the difference between solutions to these two equations.

Numerical Methods:

Explicitly show how you obtain your numerical answers in the following.

N1. Write out the Taylor series for $(e^x - 1)/x$ to order n about the origin, *including the form of the remainder.* Using the expression for the remainder, and given that you pick a point within the radius of convergence of the series, derive a condition to determine a number of terms of the series that will guarantee an error less than some given tolerance tol. You do not need to solve for the number of terms, just find a condition that you could test after computing n terms to see if the error is less than tol.

N2. How many points are necessary to construct an interpolating polynomial of degree 2? Choose the *best* points from the table below to construct an interpolating polynomial **of degree 2** for approximating the value of y when $x = 1$. Justify your choice of points. What value do you predict for y at $x = 1$?

x	y
0.000	1.0
0.231	1.1
0.528	1.2
0.897	1.3
1.344	1.4
1.875	1.5
2.496	1.6

N3. Explain how best to numerically evaluate $z = \sqrt{x^2 + y^2}$ if either x or y is large.

N4. Determine c_1, c_2, x_1, and x_2 so that the integration formula

$$\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

gives the exact result whenever $f(x)$ is a polynomial of degree 3 or less (writing down a correct answer without derivation or demonstration that it works will not get you any points for this question).