Ph.D. Comprehensive Examination — Part I

9:00-12:00, Wednesday June 10, 2020

The comprehensive exam consists of two parts. Part I contains mandatory problems and covers basic material, while Part II covers slightly more advanced materials at graduate course level.

This is **Part I**, for which a minimum of **80% is required to pass**. You may use a calculator, but your calculator **must NOT** be capable of wireless communication or capable of storing and displaying large text files. **NO other aids** are allowed.

Do **all** of the following questions.

Calculus:

C1. We know that the derivative of $\sin x$ is $\cos x$. Observe that (i) $\sin x$ is odd and $\cos x$ is even; (ii) both $\sin x$ and $\cos x$ are $2\pi-$periodic functions; and (iii) $|\sin x| \leq 1$ and $|\cos x| \leq 1$ both hold. The above observations for $\sin x$ and $\cos x = (\sin x)'$ may make one wonder if they can be generalized, leading to the following conjectures:

Assume that $f(x)$ is continuously differentiable in $\mathbb{R}$.

(I) If $f(x)$ is odd, then $f'(x)$ is even.

(II) If $f(x)$ is p-periodic (i.e., $f(x+p) = f(x)$ for all $x \in \mathbb{R}$, so is $f'(x)$.

(III) If $|f(x)| \leq M$ for all $x \in \mathbb{R}$, then there also holds $|f'(x)| \leq M$.

Prove or disprove (by counterexample) the above (I)-(III). The answer to each of them should be just one line or two, so if you run into complicated arguments, you must be in a wrong track.

C2. Let $f(x) = x^2 e^{-x}$. Sketch the graph of this function, clearly identifying those important features: monotonicity and maximum/minimum points (if any), cancavity and inflection points (if any), asymptotes (if any).

C3. Consider the linear system $Ax = b$ where $A = (a_{ij})_{m\times n}$ is an $m \times n$ real matrix, $x = (x_1, \cdots, x_n)^T$ is the unknown vector, and $b = (b_1, \cdots, b_n)^T$. A least square solution to $Ax = b$ is a vector x^* in $\mathbb{R}^n$ at which the function $f(x) = ||Ax - b||^2$ is minimized, where $|| \cdot ||$ is the Euclidean norm.

(a) Derive the equations that determine x^* and express them in terms of the coefficient matrix A and the constant vector b. You need to show your derivation, not simply writing down the equation).

(b) Give condition(s) on the matrix A that ensures/ensure there exists a unique least square solution for $Ax = b$.

Your will be required to revisit this problem by linear algebra approach (geometric approach) in Part L below.

C4. In probability and statistics, there is the important identity

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$$

Prove this identity by relating it to the improper double integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx\, dy$$

C5. The gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

(a) For nonnegative integer n, find $\Gamma(n+1)$; (b) Find $\Gamma(1/2)$.

Linear Algebra:

L1. Using the properties of eigenvectors of a symmetric matrix associated to different eigenvalues, construct a 2×2 matrix A which has an eigenvalue 5 and an associated eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and for which no entry is zero.

L2. Let $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$

(a) Find all eigenvalues and the associated eigenvectors of A;

(b) Is A diagonalizable? If no, explain why; if yes, diagonalize it.

(c) Find an orthogonal basis for the kernel of A, Ker (A).

(d) Find a basis for the range of A, range (A).

L3. Consider the linear system $Ax = b$ again where $A = (a_{ij})_{m\times n}$ is an $m \times n$ real matrix, $x = (x_1, \cdots, x_n)^T$ is the unknown vector, and $b = (b_1, \cdots, b_n)^T$.

(a) State a condition on b in terms of the column space of A (i.e., subspace spanned by the column vectors of A, also called range of A, denoted by range (A)) under which the system $Ax = b$ has a solution.

(b) A least square solution to $Ax = b$ is a vector x^* in $\mathbb{R}^n$ at which the function $f(x) = ||Ax - b||^2$ is minimized. Give an alternative description for x^* in terms of the column space range(A) and vector b with respect to the Euclidean orthogonality and distance.

(c) Based on (a)-(b) and using the notions of orthogonal projection and complement, derive the equation that determines x^*, expressed in terms of the coefficient matrix A and the constant vector b (which is also referred to as the normal system of $Ax = b$). You need to show your derivation, not simply writing down the equation).

Ordinary Differential Equations:

O1. Find the general solution to $\frac{dx}{dt} = rx(x-1)(x+1)$ in implicit form.

O2. Use an appropriate substitution to solve the initial value problem $\frac{dx}{dt} = (t+x)^2, \;\; x(0) = 1$.

O3. Find the general solutions of the second order linear ODEs $u''(t) + 4u(t) = 3\sin(2t) + e^{-2t}$

Numerical Methods:

Explicitly show how you obtain your numerical answers in the following.

N1. Assuming you are explaining to your non-mathematically inclined acquaintances:

- (a) What does it mean for a problem to be well-posed?
- (b) What does it mean for an algorithm to be stable ?

N2. Do 3 iterations of Newton's method starting with $x_0 = 2$ to solve $\sqrt{x} = 2\cos x$.

- (a) What is the backward error for your third iterate?
- (b) Estimate the forward error for your second iterate. Explain why you do or do not have any trust in this estimate.

N3. Explain how best (give reasoning) to numerically evaluate the length of a vector in 3-dimensions when one of the components is either very large or if the components vary significantly in magnitude.

N4. Given the following data for a function $f(x)$ that you may assume is reasonably smooth with well-behaved derivatives:

x	f(x)
0.000	0.0000
0.500	0.5205
1.000	0.8427

- (a) Find an interpolating polynomial through the data.
- (b) Find an approximation to the derivative at $x = 0.5$ and $x = 0.75$.
- (c) Find an approximation to $\int_0^1 f(x)dx$.

Note: Better approximations will lead to higher marks.