Ph.D. Comprehensive Examination — Part II

Wednesday June 10, 2020
2:30 - 5:30 pm

The comprehensive exam consists of two parts. Part I contains mandatory problems and covers basic materials, while Part II covers slightly more advanced materials at graduate course level.

This is **Part II**, for which a minimum of 60% is required to pass. Questions in (A) and (B) are **for all candidates**, while those in (C)-(G) are area dependent, and you just need to answer questions in one of these areas. You may use a calculator, but your calculator **must NOT** be capable of wireless communications or capable of storing and displaying large text files. **NO other aids** are allowed.

A. Numerical Methods. Do **both** A-1 and A-2.

A-1. Suppose that 100 values of a function $y = f(x)$ are given at equally spaced points in the interval [0,1].

(a) If you were to create an interpolating polynomial $p(x)$ through these points what properties define this polynomial (e.g. what properties does the polynomial have regardless of how it is constructed, what is the order of the polynomial, what can you expect for how well the polynomial approximates the function $f(x)$). This polynomial could be constructed using a number of different forms such as the Lagrange form, the Newton form, or the Barycentric form. Outline what procedure is involved in constructing one of these forms. (This should not be longer than a few lines. The big picture, not the details is what is asked for.)

(b) An alternative is to construct a natural cubic interpolatory spline $S(x)$. Outline what procedure is involved in computing S(x). You do not have to give specific equations, but you have to describe what basic form they have, and what is involved in solving them. (This should not be longer than a few lines. The big picture, not the details is what is asked for.)

(c) If $p(x)$ and $S(x)$ were to be computed and then used to approximate f(x) at 1000 points in [0,1], which of these two methods would be more efficient? Which one would be more accurate? Justify your answers.

(d) Would anything change if you could select the points yourself instead of using the given 100 values?

A-2. The midpoint method is given as

$$w_{n+1} = w_n + hf\left(t_n + \frac{h}{2}, w_n + \frac{h}{2}f(t_n, w_n)\right).$$

(a) Derive its local truncation error.
(b) Describe its region of absolute stability.

(c) Is the method A-stable? Justify your answers. An alternative midpoint method is

$$w_{n+1} = w_n + hf\left(t_n + \frac{h}{2}, w_n + \frac{w_n + w_{n+1}}{2}\right).$$

(d) Describe its region of absolute stability.
(e) Is the method A-stable?
(f) Based upon this analysis, should one expect one of these methods to be preferable over the other when solving stiff problems?
(g) Note that the alternative midpoint method is implicit. Describe Newton's method for this context, and explain how it would be used. Based upon the general properties of Newtons method, how many iterations do you think would typically be required? Justify your answers.

B. Partial Differential Equations. Do __two__ out of the four questions.

B-1. The Swift-Hohenberg equation can be written as

$$\frac{\partial \phi(x,t)}{\partial t} = \varepsilon \phi(x,t) - \left(\frac{\partial^2}{\partial x^2} + q_c^2\right)^2 \phi(x,t) - \phi^3(x,t),$$

where ε is the control parameter measuring the distance from instability, $\phi(x,t)$ the order parameter and q_c the critical wave number. Derive the Swift-Hohenberg equation and the corresponding dispersion relation using *symmetry* arguments. Using the dispersion relation, sketch the growth rate for values $\varepsilon = 0$, $\varepsilon < 0$ and $\varepsilon > 0$.

B-2. Using operator

$$\mathcal{L} = -\left[\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + r(x)\right]$$

to derive the jump condition for Green's function $G(x, x')$.

B-3. Write the PDE $u_t(t,x) + 2u(t,x)u_x(t,x) + [\sin u(t,x)]_x = 0, \;\; t \geq 0, \;\; x \in [a,b]$, in the form of conservation law, explicitly identifying the flux ϕ. Give some boundary condition(s) at $x = a$ and $x = b$ so that for a solution $u(t,x)$ to this PDE satisfying the boundary condition(s), the quantity $\int_a^b u(t,x)\,dx$ is independent of time t.

B-4. Solve the initial value problem:

$$\begin{cases} u_t(t,x) + cu_x(t,x) = x\,u(t,x), \;\; x \in \mathbb{R}, \;\; t > 0, \\ u(0,x) = u_0(x), \;\; x \in \mathbb{R}. \end{cases}$$

C. Physics. Do **two** out of the four questions.

C-1. Classical Mechanics

(a) Show that the transformation from canonical conjugate variables p_n, q_n to the complex variables P_n, Q_n with $n = 1, 2, 3 \cdots$

$$Q_n = \frac{1}{\sqrt{2\lambda_n}}(p_n - i\lambda_n q_n)$$

$$P_n = iQ_n^*$$

(λ_n real) is a canonical transformation.

(b) Apply such a transformation to the classical Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2$$

and find the expression for the choice $\lambda = m\omega$.

(c) Assume a classical Hamiltonian has the form

$$H(P,Q) = -i\omega PQ$$

Write down the equations of motion for Q and P.

C-2. Theomodynamics/Statistial Physics

(a) Define the partition function for a system with states of energy E_n.

(b) Express the thermodynamic internal energy of the system in terms of the partition function and its temperature derivative.

(c) Find the partition function is the system consists of a single harmonic oscillator for which

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

and find the thermodynamic energy as a function of the temperature.

C-3. Quantum Mechanics

(a) Using $\vec{p} = -i\hbar\vec{\nabla}$, as in the coordinate-space (Schrödinger) representation of quantum mechanics, show that

$$[x, p_x] = [y, p_y] = [y, p_y] = i\hbar$$

[Hint: Consider $[x, p_x]\psi(x, y, z)$]

(b) If $\vec{L} = \vec{r} \times \vec{p}$, use the above results to show that

$$[L_x, L_y] = i\hbar L_z$$

(c) Generalize your results of parts (a) and (b) to show that

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$$

(d) Define $L_+ \equiv L_x + iL_y$, $L_- \equiv L_x - iL_y$ and show that

$$[L_z, L_+] = \hbar L_+$$

$$[L_z, L_-] = -\hbar L_-$$

(e) If $|m\rangle$ is an eigenstate of L_z with eigenvalue $m\hbar$, $L_z|m\rangle = m\hbar|m\rangle$, use the commutation relations of part (d) to show that $L_+|m\rangle$ is proportional to the state $|m+1\rangle$ and that $L_-|m\rangle$ is proportional to the state $|m-1\rangle$.

(f) Derive the relation

$$L_+L_- = \vec{L}^2 - L_z^2 + \hbar L_z$$

C-4. Electricity and Magnetism

(a) Given that the magnetic field $\vec{B}$ and the electric field $\vec{E}$ are related to the vector potential $\vec{A}$ and the scalar potential ϕ by

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t\vec{A}$$

derive the general form of local gauge transformations for $\vec{A}(\vec{r}, t)$ and $\phi(\vec{r}, t)$. Your answer should be expressed in terms of some arbitrary scalar function $\xi(\vec{r}, t)$.

(b) If one incorporates the Lorentz gauge condition

$$\vec{\nabla} \cdot \vec{A} + \frac{\epsilon\mu}{c}\,\partial_t\phi = 0$$

show that this function ξ satisfies a homogeneous wave equation.

(c) Using Eqs. (a) and (b) and Maxwell's equation

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi\rho}{\epsilon}$$

derive the inhomogeneous wave equation satisfied by $\phi(\vec{r}, t)$.

(d) Using Eqs. (a) and (b) and Maxwell's equation

$$\vec{\nabla} \times \vec{B} - \frac{\epsilon\mu}{c}\,\partial_t\vec{E} = \frac{4\pi\mu}{c}\vec{J}$$

derive the inhomogeneous wave equation satisfied by $\vec{A}(\vec{r}, t)$. [Hint: $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2\vec{A}$.]

D. Material Sciences.

D-1. Surface growth is a rich phenomenon and a large number of models has been developed to describe it. Among the most common ones are the Edwards-Wilkinson (EW), the Kardar-Parisi-Zhang (KPZ) and the MBE model.

(a) Describe the different models, provide the equations if possible.

(b) The highest order gradient term in both EW and KPZ model is of the second order. Show what kind of a difference would the addition of a fourth order (∇^4) term make at the hydrodynamic limit. You can assume that the surface is self-affine.

D-2. Consider a two dimensional solid. Its strain matrix can be given as

$$u_{ij}(\vec{x}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i),$$

where u_i is the lattice distortion field. The elastic Hamiltonian for such a triangular lattice can be given as

$$\beta\mathcal{H} = \frac{1}{2}\int d^2\vec{x}\,(2\mu u_{ij}u_{ij} + \lambda u_{ii}u_{jj})\,,$$

where μ and λ are elastic Lamé coefficients.

(a) Show that the system is translationally invariant.

(b) Find the normal modes (fluctuations) of the transverse and longitudinal modes.

E. Neuron Sciences.

E-1. The leaky integrate-and-fire model is defined by the equation:

$$\tau_m \dot{v} = -v + R_m I_e\,. \tag{1}$$

When $v \geq v_{th}$, the reset condition is $v \to 0$ mV. Starting at time $t = 0$, a current I_e (s.t. $R_m I_e > v_{th}$) is applied to the model neuron.

(a) Solve for t_{isi}, the time of the next action potential.

(b) Use this expression to write the interspike-interval firing rate of the neuron.

(c) Find an approximation for this expression and study how the firing rate grows with increasing I_e.

E-2. Spike-time dependent plasticity (STDP) is the process by which synapses between neurons change strength based on the timing of spikes between input (pre-synaptic) and output (post-synaptic) neurons. Considering the spike times (t_1 and t_2) between two neurons connected by a synapse and their difference $s = t_2 - t_1$, the STDP rule is defined by the piecewise exponential function on s:

$$f(s) = \begin{cases} \alpha_+ e^{\frac{-s}{\tau_+}}, & s \geq 0 \\ -\alpha_- e^{\frac{s}{\tau_-}}, & s < 0 \end{cases} \tag{2}$$

(a) Derive an expression for the expected weight change over time in terms of the learning rule $f(s)$ and the correlation function $C(s)$ between spikes of input and output neurons.

(b) Solve the expression for the expected weight change with a large input population oscillating synchronously with temporal frequency ν and an output neuron spiking at a constant phase ϕ relative to the input.

(c) Study how the solution changes for different values of α_+, α_-, and ν.

G. Dynamical Systems.

G-1. Consider the given differential system,

$$\dot{x} = x\,(1-x) - \frac{2\,x\,y}{1+2\,x}$$
$$\dot{y} = \frac{8\,x\,y}{1+2\,x} - y,$$

restricted to the first quadrant of the x-y plane.

(a) Show that the system has a positive equilibrium and it is unstable.

(b) Use Poincaré-Bemdixson theory to prove that there exists at least one closed orbit (limit cycle).

G-2. Consider the nonlinear system,

$$\dot{x} = 2x - y - 3\,x\,(x^2+y^2) + x\,(x^2+y^2)^2$$
$$\dot{y} = x + 2y - 3\,y\,(x^2+y^2) + y\,(x^2+y^2)^2$$

(a) Show that the origin $(0,0)$ is the unique equilibrium of the system and is unstable.

(b) Find all possible limit cycles.

(c) Determine stability of limit cycles found in part (b).