Ph.D. Comprehensive Examination — Part II

Wednesday June 9, 2021, 9:00am - 12:00 noon (Toronto Time)

The comprehensive exam consists of two parts. Part I contains mandatory problems and covers basic materials, while Part II covers slightly more advanced materials at graduate course level.

This is **Part II**, for which a minimum of 60% is required to pass. Questions in (A) and (B) are **for all candidates**, while those in (C)-(F) are area dependent, and you can choose to do **two (2) questions from (C)-(F)**. You may use a calculator, but your calculator **must NOT** be capable of wireless communications or capable of storing and displaying large text files. **NO other aids** are allowed.

A. Numerical Methods. Do both A-1 and A-2.

A-1. Consider the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} .$$

(i) Calculate the eigenvalues and eigenvectors of A by any method.

(ii) Illustrate the power method for obtaining numerical estimates of the largest eigenvalue by using the following initial vector:

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Calculate 3 iterations of the power method. How accurate is the estimate?

(iii) How can the power method be used to find the smaller eigenvalue, using the same x ?

(iv) Use the connection between the singular values of A and the eigenvalues of A^TA to calculate the singular values of A. (Hint: unlike the eigenvalues, the singular values are not integers.)

A-2. Consider the ODE

$$y' = f(t, y(t)) .$$

Let (y_n, t_n) and (y_{n+1}, t_{n+1}) be two points in the solution of the equation. That is, assume $y(t_n) = y_n$ and $y(t_{n+1}) = y_{n+1}$. We let $t_{n+1} = t_n + h$. Moler's book claims that y_{n+1} can be estimated from the computational scheme

$$s_1 = f(t_n, y_n) , \tag{1}$$

$$s_2 = f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hs_1) \tag{2}$$

$$y_{n+1} = y_n + hs_2 \tag{3}$$

$$t_{n+1} = t_n + h \tag{4}$$

(i) Write out 3 terms of the Taylor expansion of $y(t_{n+1})$ in terms of $y(t_n)$ and powers of h.

(ii) Expand s_2 as a Taylor series in powers of h.

(iii) Verify that Moler's assertion is correct.

B. Partial Differential Equations. Do **two** out of the four questions.

B-1. Suppose that a pollutant has density $u(x,t)$ mass units per unit length at position $x \in \mathbb{R}$ and time $t \geq 0$ in an infinite one dimensional model of a river. The flux of pollutant passing $x \in \mathbb{R}$ is given by $\phi(u(x,t))$ mass units per unit time.

(i) Suppose the contaminant is neither created or destroyed in the river. Explain why

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \phi(u(a,t)) - \phi(u(b,t)) \text{ for all } [a,b], a \leq b \quad (\clubsuit)$$

(ii) Explain why $\int_a^b (u_t + \phi_u u_x)\,dx = 0$ for all $a \leq b$. Explain why assuming the continuity of the integrand of this expression implies that $u_t + \phi_u u_x = 0$.

(iii) Assuming that $\phi = u^2/2$, find $u(x,t)$ for $t > 0$ for the initial value problem $u_t + \phi_x = 0$ where $u(x,0) = 2$, for $x < 0$ and $u(x,0) = 1$, for $x \geq 0$. [Hint: the formula for a shock path is: $[u]_{s(t)}s'(t) = [\phi]_{s(t)}$ where $[v]_{s(t)} = \lim_{x\to s(t)^-} v(x,t) - \lim_{x\to s(t)^+} v(x,t)$.]

(iv) Either answer (iv) or (v) below. Justify the shock path formula given in (iii). [Hint: Start with ($\clubsuit$) and consider $[a,b]$ containing $s(t)$].

(v) Either answer (iv) above or (v). A theme throughout the AM 9505 course has been the interplay between integral and differential equations. Write about this theme and its importance [max 50 words, point form OK].

B-2. Consider the evolution equation for $u(x,t)$ given by $u_t = -S[u] = -u_{xxxx}$ with initial condition $u(x,0) = u_0(x)$ for $x \in \Omega$.

(i) Set $u = e^{-\lambda t}v(x)$ and obtain the equation $S[v] = \lambda v$ for $v(x)$ in terms of $S = D^4$ where $D = \frac{d}{dx}$.

(ii) Suppose that $x \in \Omega = [0,1]$ and u satisfies the boundary conditions $u(0,t) = 0 = u(1,t)$ and $u_{xx}(0,t) = 0 = u_{xx}(1,t)$. Suppose $D : V \to W$ where V and W have $L^2[0,1]$ inner products $\langle \cdot,\cdot \rangle$ and $\langle\langle \cdot,\cdot \rangle\rangle$ respectively. Show that $D^* = -D$ assuming the $L^2[0,1]$ inner product on functions $v(x)$ satisfying the same boundary conditions $v(0) = 0 = v(1)$ and $v''(0) = 0 = v''(1)$.

(iii) Show that $S^* = (D^4)^* = S$.

(iv) Show that $S > 0$. What does this imply about the eigenvalues of $S[v] = \lambda v$ and the solutions of the IVP as $t \to \infty$?

B-3. The (Dirac) delta function ($\delta(x)$) is commonly used in finding solutions to various problems. Prove the following relation:

$$\int_{-\infty}^{\infty} \delta'(x)f(x)dx = -f'(0)$$

B-4. In three dimensions, the Green's function for the Poisson equation can be given as

$$G(\vec{r},0) = \frac{1}{4\pi}\frac{1}{r},$$

where r is the radial coordinate. Show that this is valid at $r = 0$.

Choose two (2) questions from the areas C-F below. If you attempt more than two, only the first two will be marked.

C. Dynamical Systems

C-1. Apply Index Theory and Dulac's Criterion to prove that the differential system,

$$\begin{cases} \dot{x} = x^2\,(6 - x - y) \;\; \equiv f_1(x,y), \\ \dot{y} = y\,(4x - x^2 - 3) \equiv f_2(x,y), \end{cases}$$

has no closed orbits.

C-2. Consider the following differential system,

$$\begin{cases} \dot{x} = x - 2\,\sin y \\ \dot{y} = -\,2\,y + 2\,\sin x, \end{cases} \qquad -\frac{\pi}{2} \le x \le \pi, \quad -\frac{\pi}{2} \le y \le \pi.$$

(i) Show that the equilibrium $(0,0)$ is a stable focus.

(ii) Prove that there exists a unique non-zero equilibrium (x^*, y^*), where $y^* = \sin x^*$.

[**Hint**: $\frac{\pi}{2} < x^* < \pi$, and note that $\sin 1 \approx 0.8415$.]

(iii) Show that the non-zero equilibrium (x^*, y^*) is a saddle.

D. Material Sciences

D-1. A simple model of a polymer assumes its configuration can be modelled as a random walk in 3 dimensions. Let's assume the polymer is a simple random walk with each stop of length a and a total of N steps.

(i) What is the expectation value for the end to end distance $\mathbf{R}$ (denoted $\langle \mathbf{R} \rangle$) for such a walk?

(ii) What is the expectation value for the square of the end-to-end distance, $\langle \mathbf{R}^2 \rangle$?

(iii) Given that the assumptions of the central limit theorem are valid, what should the probability distribution for end-to-end distance be for large N?

(iv) Random walks are a type of fractal object (They exhibit self-similarity). Self similar objects can be characterized by their fractal dimension, defined as the scaling exponent D between the mass of the object M and the characteristic size of the object R:

$$M \sim R^D.$$

What is the fractal dimension of our polymer modelled as a random walk?

D-2. Let a free energy functional $E[\rho]$ be given by:

$$E = \int_0^a dx \left\{ \rho(x)^4 + \frac{1}{2} K (\partial_x \rho)^2 + \epsilon \rho(x) \cos(2\pi x/a) \right\}.$$

(i) What is the Euler-Lagrange equation for $\rho(x)$ to minimize this functional.

(ii) How do things change if the mean density is constrained to satisfy $\langle \rho \rangle = \rho_0$.

(iii) Find an approximation for $\rho(x)$ if the mean density is constrained to satisfy $\langle \rho \rangle = \rho_0$ and assuming ϵ is very small.

E. Mathematical Biology.

E-1. The following system of differential equations describes the population density of naïve immune cells, X and activated immune cells Y, at time T. Naïve cells can become activated through signals produced by activated cells. Activated cells are relatively short-lived and cannot reproduce.

$$\frac{dX}{dT} = rX - sX^2 - pXY \tag{5}$$

$$\frac{dY}{dT} = pXY - qY \tag{6}$$

Here r, s, p and q are positive parameters. **For all parts of this question, assume $rp > sq$.**

(i) *Model interpretation:* Give the biological interpretation of the term sX^2 in this model (explain to a non-mathematician). What is the expected lifetime of an activated cell? Do naïve cells reproduce themselves in this model (e.g. cellular fission) or are they produced at a constant rate by the body (e.g. produced by stem cells in the thymus)?

(ii) Non-dimensionalize this system of equations by setting $t = rT$, $x = pX/q$, and $y = pY/r$. Define new (grouped) parameters as appropriate.

(iii) Find the equilibria of the non-dimensionalized system.

(iv) Plot the nullclines of the non-dimensionalized system and show the direction of flow in each region separated by the nullclines (considering the positive quadrant $x \geq 0$, $y \geq 0$ only).

(v) Use the Jacobian to evaluate the stability of each of the equilibria.

(vi) Provide a one-sentence prediction of the long-term behaviour(s) of this system (for a non-mathematician).

E-2. Let x_t be the number of annual plants in a population in year t. The plants produce seeds once a year. A fraction of these seeds survive the winter and produce plants in the next year. Because of overcrowding, the number of suriving offspring is reduced when the plant population is large. In particular, **each plant in year** t is able to produce a total of $(1 + r - x_t)$ seeds that will survive to become plants in year $t + 1$. Note that $r \in (-\infty, \infty)$.

(i) Write the difference equation describing the dynamics of this plant population, that is, write an expression for x_{t+1}.

(ii) Find any equilibrium population sizes.

(iii) Evaluate the stability of the equilibria.

(iv) Treat r as a bifurcation parameter and draw the bifurcation diagram for this system.

(v) Classify (give the type of) any bifurcation(s) in the diagram.

(vi) Explain the biological meaning of your results.

F. Neuron Sciences.

F-1: Spike-time dependent plasticity (STDP) is the process by which synapses between neurons change strength based on the timing of spikes between input (pre-synaptic) and output (post-synaptic) neurons. Considering the spike times (t_1 and t_2) between two neurons connected by a synapse and their difference $s = t_2 - t_1$, the STDP rule is defined by the piecewise exponential function on s:

$$f(s) = \begin{cases} \alpha_+ e^{\frac{-s}{\tau_+}}, & s \geq 0 \\ -\alpha_- e^{\frac{s}{\tau_-}}, & s < 0 \end{cases} \tag{7}$$

Using this STDP rule, **(i) provide an example** of a set of spikes that will produce (a) no weight change, (b) positive weight change, and (c) negative weight change. **(ii) What will the effect** of the STDP rule be for an input population that oscillates synchronously and synapses onto a neuron that spikes (a) on the rising phase of the input oscillation and (b) on the falling phase of the input oscillation? Explain why this is the case.

F-2: To calculate the expected weight change over time, we can consider the integro-differential equation:

$$\frac{dw}{dt} = w_{max} \int_{-\infty}^{\infty} f(s)C(s)ds \tag{8}$$

where $f(s)$ is the STDP rule and $C(s)$ is the correlation function between the input to a neuron and its spiking output. Using Equation (8), **(i) derive an expression** for the expected weight change over time in terms of the learning rule $f(s)$ and the correlation function $C(s)$ between spikes of an input population oscillating synchronously and an output neuron that spikes at a constant phase ϕ relative to the input. The input population oscillates with time-varying rate

$$r(t) = \frac{r_0}{2}[1 - \cos(\nu t)] \tag{9}$$

where ν is the angular frequency of the oscillation. The output neuron spikes at times

$$S(t) = \sum_n \delta\left(t - \frac{2\pi n + \phi}{\nu}\right) \tag{10}$$

where ϕ is the phase of the output spike relative to the input oscillation. Taking $\alpha_+ = \alpha_-$, $\tau_+ = \tau_-$ in (7), and $w_{max} = 1$ in (8), **(ii) use this expression to find the points** where weight change stops. Assuming an integrate and fire neuron (whose phase response curve is strictly positive), **(iii) predict which of these points** will be stable and explain why this is the case.