The UNIVERSITY *of* WESTERN ONTARIO

Applied Mathematics Ph.D. Comprehensive Examination
Part 2
Date: 7 June 2022

Answer ?? of the following questions

Mathematical Biology

1. The following system of differential equations describes the densities, x_1 and x_2, of two species:

$$\frac{dx_1}{dt} = r_1 x_1 \tfrac{K_1 - x_1 - \beta_{12} x_2}{K_1} \ ,$$
$$\frac{dx_2}{dt} = r_2 x_2 \tfrac{K_2 - x_2 - \beta_{21} x_1}{K_2} \ ,$$

where all parameters are positive.

   (a) *Model interpretation:* Explain the meaning of the terms involving β, that is, describe the biological phenomenon being modelled here (explain simply to a non-mathematician).

   (b) Find all the equilibria of this system.

   (c) Take $r_1 = r_2 = 1$, $K_1 = 8$, $K_2 = 10$, $\beta_{12} = 4$ and $\beta_{21} = 1$. Sketch the nullclines for this system. Indicate any equilibria on the sketch.

   (d) Add arrows indicating the direction of flow in each region of the nullcline plot. Predict the longterm outcome for these populations.

   (e) Make a second nullcline sketch, including the direction of flow, taking all parameters the same as before except $\beta_{21} = 2$. Predict the longterm outcome(s) and explain this result in words to a non-mathematician.

2. Consider the following discrete time disease model with non-zero vaccination. Let S_t be the number of susceptible individuals, I_t the number of infectious individuals, and R_t the number of recovered individuals in week t. A fraction p of susceptibles are vaccinated each week, with $0 < p < 1$. The infection rate is α. B represents a constant number of births and deaths each week. All parameters are positive. This yields:

$$\begin{aligned} S_{t+1} &= (1-p)S_t - \alpha I_t S_t + B \\ I_{t+1} &= \alpha I_t S_t \\ R_{t+1} &= R_t + I_t - B + pS_t \end{aligned}$$

   (a) *Model interpretation*: note that R_{t+1} includes the term I_t. What does this mean about the recovery time for this disease?

   (b) *Model interpretation*: What is the total population size in week $t+1$ and what does that mean (in words)?

(c) Note that the dynamics of this system can be analysed by considering S and I only, since R is decoupled. For the remaining parts of this question, analyse the reduced system consisting of only the two equations for S and I. Find and describe in words the two equilibria of this system.

(d) Evaluate the stability of both equilibria. **Assume** $\alpha B < 1$**.** Recall the Jury conditions for $n = 2$ which are

$$|\mathrm{Tr}(J)| < 1 + \det(J) < 2 \; .$$

(e) Treat p as a bifurcation parameter and draw the bifurcation diagram in terms of I.

(f) Classify (give the type of) the bifurcation in the diagram.

Dynamical Systems

3. Consider the following switching system,

$$\text{Right system:} \quad \begin{cases} \dot{x} = x + y, \\ \dot{y} = -x + y, \end{cases} \quad \text{for } x \geq 0; \qquad \text{Left system:} \quad \begin{cases} \dot{x} = -x + y, \\ \dot{y} = -x - y, \end{cases} \quad \text{for } x < 0.$$

(a) Show that the origin $(0, 0)$ is a **focus** for each of the right and left systems, and indicate its stability.

(b) Find the general solution for each of the right and left systems.

(c) Prove that the origin $(0, 0)$ is a **center** of the switching system.

(d) Sketch the phase portrait of the switching system.

4. Consider the Lorenz system,

$$\begin{cases} \dot{x} = a\,(y - x), \\ \dot{y} = -\,xz - y, \\ \dot{z} = xy - z - b, \end{cases}$$

where a and b are real positive parameters.

(a) Find all equilibrium solutions.

(b) Use a linear analysis to find the condition under which the equilibrium $(0, 0, -b)$ is asymptotically stable.

(c) Use a Lyapunov function to prove that the equilibrium $(0, 0, -b)$ is globally asymptotically stable under the condition obtained in Part (b).

(d) Find the stability of the other equilibria, and drive the condition under which Hopf bifurcation occurs from the equilibria.

Partial Differential Equations

5. Let $\nabla u = (\partial u/\partial x, \partial u/\partial y)$ where $(x, y) \in \Omega \subset \mathbb{R}^2$ and Ω is a simply connected bounded set with smooth boundary $\partial\Omega$. Let U be the set of continuously differentiable functions that vanish on the boundary of Ω: $U = \{u \in C^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$ and suppose U has inner product
$\langle u, \widetilde{u} \rangle = \int\int_\Omega u\, \widetilde{u}\, dxdy$. Consider the operator ∇ acting on the space U defined by $\nabla : U \to V$ with $u \to \mathbf{v} = \nabla u \in V \subset \mathbb{R}^2$. Here V has inner product $\langle\langle \mathbf{v}, \widetilde{\mathbf{v}} \rangle\rangle = \int\int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}}\, dxdy$.

   (a) Show that $\nabla^* : V \to U$ is given by $\nabla^*(\mathbf{v}) = -\nabla \cdot \mathbf{v}$. Hint: you can assume that

$$\int\int_\Omega \nabla u \cdot \mathbf{v}\, dxdy = \oint_{\partial\Omega} u(\mathbf{v} \cdot \mathbf{n})\, ds - \int\int_\Omega u(\nabla \cdot \mathbf{v})\, dxdy$$

   (b) Why is $S = -\nabla \cdot \nabla$ self-adjoint? You can assume that $(L \circ M)^* = M^* \circ L^*$.

   (c) Show that $S > 0$.

   (d) What can be said about the eigenvalues λ and eigenfunctions u of $S[u] = -\nabla \cdot \nabla u = \lambda u$ without computation?

6. Consider a density $u(x, t) \geq 0$ mass per unit length at position $x \in \mathbb{R}$ and time $t \geq 0$, with $u(x, 0) = u_0(x)$ being a given function of $x \in \mathbb{R}$. Suppose this is modeled by $u_t + \phi_x = f(x, t)$ where $\phi(x, t) = -u_x$.

   (a) The equation $u_t + \phi_x = f(x, t)$ is called a differential form of a conservation (or balance) law. Give a brief interpretation of a possible application and meaning of the terms u, ϕ, $f(x, t)$. What is the corresponding integral form of the model? Which is more fundamental – the integral form or the differential form of the conservation law and why?

   (b) Find the Fourier Transform of $u_t - u_{xx} = 0$, $u(x, 0) = u_0(x)$ and solve the resulting ODE for the Fourier transform of u in the transform space. Please refer to the short Table of Fourier Transforms on the next page.

   (c) Use convolution to simplify the inverse transform to give an integral formula for u. Identify a Green's function in your solution and briefly describe how it evolves with respect to time.

Fourier Transform Table next, followed by more questions.

Concise Table of Fourier Transforms

$f(x)$	$\widehat{f}(k)$
1	$\sqrt{2\pi}\,\delta(k)$
$\delta(x)$	$\dfrac{1}{\sqrt{2\pi}}$
$\sigma(x)$	$\sqrt{\dfrac{\pi}{2}}\,\delta(k) - \dfrac{\mathrm{i}}{\sqrt{2\pi}\,k}$
$\operatorname{sign} x$	$-\,\mathrm{i}\,\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{k}$
$\sigma(x+a) - \sigma(x-a)$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin a\,k}{k}$
$e^{-a\,x}\,\sigma(x)$	$\dfrac{1}{\sqrt{2\pi}\,(a + \mathrm{i}\,k)}$
$e^{a\,x}\,(1 - \sigma(x))$	$\dfrac{1}{\sqrt{2\pi}\,(a - \mathrm{i}\,k)}$
$e^{-a\,\lvert x\rvert}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{k^2 + a^2}$
$e^{-a\,x^2}$	$\dfrac{e^{-k^2/(4\,a)}}{\sqrt{2\,a}}$
$\tan^{-1} x$	$-\,\mathrm{i}\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-\lvert k\rvert}}{k}$
$f(c\,x+d)$	$\dfrac{e^{\,\mathrm{i}\,k\,d/c}}{\lvert c\rvert}\,\widehat{f}\left(\dfrac{k}{c}\right)$
$\overline{f(x)}$	$\overline{\widehat{f}(-k)}$
$\widehat{f}(x)$	$f(-k)$
$f'(x)$	$\mathrm{i}\,k\,\widehat{f}(k)$
$x\,f(x)$	$\mathrm{i}\,\widehat{f}\,'(k)$
$f * g(x)$	$\sqrt{2\pi}\,\widehat{f}(k)\,\widehat{g}(k)$

Note: The parameters a, c, d are real, with $a > 0$ and $c \neq 0$.

7. Consider the PDE
$$u_t = u_{xx} + K(u) \tag{1}$$

(a) Find the second order ODE for $y(z)$ for traveling wave solutions (TWS) of form $u(x,t) = y(z)$ where $z = x - ct$. What obvious symmetries does (1) have and how are they related to the form of the TWS?

(b) Write the ODE for $y(z)$ as a system of first order ODE for $y(z)$ and $v(z) = y'(z)$.

(c) A monontonic TWS is one in which y is a bounded monotonic function of z so that $\lim_{z\to\pm\infty} y(z) = \pm a$ for some finite values a^-, a^+ with $a^- \neq a^+$. Show that for a monotonic TWS to exist $(a^-, 0)$ and $(a^+, 0)$ must be equilibrium points of the system of ODE. Note it is not possible to find all the equilibrium points completely explicitly here (they depend on K).

(d) Describe what further steps you would take, once given K explicitly, for the analysis and description of such monotonic TWS. [Max 25 words: point form OK].

Numerical Analysis Questions

8. Consider the PDE $u_t = u_{xx} + e^{-t}u^3$ ($\heartsuit$) with initial condition $u(x,0) = x(1-x)$ for $0 < x < 1$ and boundary conditions $u(0,t) = 0$, $u(1,t) = 0$ for $t \geq 0$.

(a) Set up a typical cell for an explicit forward finite difference discretization of this IBVP. Give the formula for the approximate value of $u(x, t + \Delta t)$ in terms of neighboring approximate grid values of u. Briefly also discuss how boundary and initial conditions would be accommodated in the application of this scheme.

(b) Show that as $t \to \infty$ the PDE approaches the linear Heat PDE. The forward finite difference scheme for the limiting linear Heat PDE is stable provided $\frac{\Delta t}{(\Delta x)^2} < \frac{1}{2}$. Discuss the implication of this criterion: its impact on accuracy, efficiency and how the instability can be addressed. Make a brief comment on implications for ($\heartsuit$).

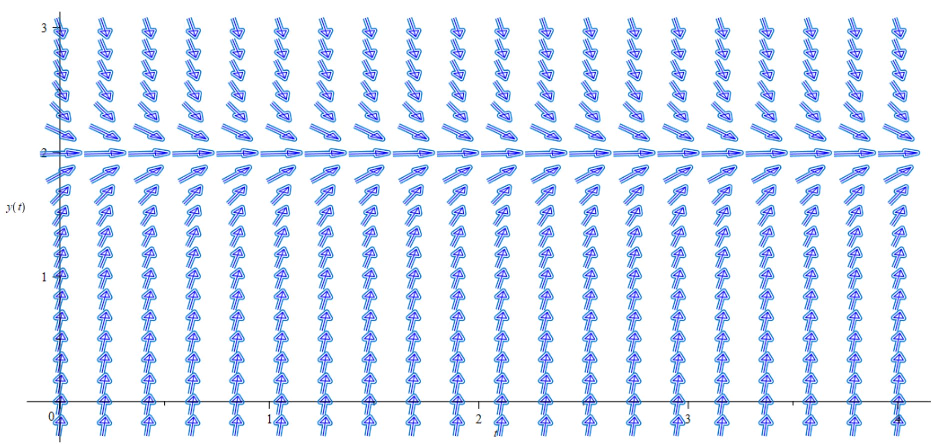
See next page for more questions.

9. (a) Consider differential equations of form $y' = f(y)$, where f is analytic and $y(0) = y_0$. For a given explicit numerical ODE method how can the order p of the method's $O(h^p)$ local error be determined.

(b) Consider the IVP $y' = 5\,(2 - y)$, where $y(0) = 1$. Show that the exact solution of this IVP is given by $y = 2 - e^{-5t}$. See the Figure of the IVP's direction field. Give the Euler iteration formula for this IVP, expressing it in the fixed point form $y_{j+1} = G(y_j)$. Determine $\hat{h}$ such that for fixed stepsize $h < \hat{h}$ the Euler iteration will yield $y_j \to 2$ as $j \to \infty$.

(c) The Euler iteration generally performs poorly on this IVP over $[0,\,4]$ even when $h < \hat{h}$. Better alternatives are provided by Backwards implicit methods. Give the Backwards Euler method for the IVP in (b), writing it in the form $y_{j+1} = B(y_j)$. Show that there is no restriction on h for this method.

(d) What geometrical aspect of the IVP (see the Figure) above causes difficulties for the Euler method (and indeed other explicit methods)? In terms of this aspect why can backwards methods offer improved performance?



Computer Algebra

10. (a) Given two polynomials

$$p(x) = \sum_{k=0}^{n} p_k x^k \ ,$$
$$q(x) = \sum_{k=0}^{m} q_k x^k \ ,$$

describe how the *Sylvester matrix* is formed, and how it is used to decide whether $p(x)$ and $q(x)$ have a common root.

(b) For the following polynomials

$$p(x) = 6x^2 + 13x - 5 \ ,$$
$$q(x) = 6x^2 + 17x + 5 \ ,$$

use the Sylvester matrix to decide whether there is a common root. [Obviously, solving the polynomials using the quadratic formula gets no credit.]

(c) Given bivariate polynomials $P(x, y)$ and $Q(x, y)$, describe how the Sylvester matrix can be used to solve for x and y.

(d) Consider the polynomials

$$P(x, y) = 12x^2y + 13x - 10y \ ,$$
$$Q(x, y) = 24y^2x^2 + 34xy + 5 \ .$$

Using the Sylvester matrix, eliminate x and obtain an equation only in y.

(e) Comment on the solutions obtained by this method.

NeuroScience

11. Spike-time dependent plasticity (STDP) is the process by which synapses between neurons change strength based on the timing of spikes between input (pre-synaptic) and output (post-synaptic) neurons. Considering the spike times (t_1 and t_2) between two neurons connected by a synapse and their difference $s = t_2 - t_1$, the STDP rule is defined by the piecewise exponential function on s:

$$f(s) = \begin{cases} \alpha_+ e^{\frac{-s}{\tau_+}}, & s \geq 0 \\ -\alpha_- e^{\frac{s}{\tau_-}}, & s < 0 \end{cases} \tag{2}$$

Using this STDP rule, (i) provide an example of a set of spikes that will produce (a) no weight change, (b) positive weight change, and (c) negative weight change. (ii) What will the effect of the STDP rule be for an input population that oscillates synchronously and synapses onto a neuron that spikes (a) on the rising phase of the input oscillation and (b) on the falling phase of the input oscillation? Explain why this is the case.

12. To calculate the expected weight change over time, we can consider the integro-differential equation:

$$\frac{dw}{dt} = w_{max} \int_{-\infty}^{\infty} f(s)C(s)ds \tag{3}$$

where $f(s)$ is the STDP rule and $C(s)$ is the correlation function between the input to a neuron and its spiking output. Using Equation (3), (i) derive an expression for the expected weight change over time in terms of the learning rule $f(s)$ and the correlation function $C(s)$ between spikes of an input population oscillating synchronously and an output neuron that emits a single spike at a phase ϕ relative to the input. The input population oscillates with time-varying rate

$$r(t) = \frac{r_0}{2}\left[1 - \cos(\nu t)\right] \tag{4}$$

where ν is the angular frequency of the oscillation. The output neuron spikes at times

$$S(t) = \sum_n \delta\left(t - \frac{2\pi n + \phi}{\nu}\right) \tag{5}$$

where ϕ is the phase of the output spike relative to the input oscillation. (ii) Sketch a plot of the expression for $\frac{dw}{dt}$ as a function of the output spike phase ϕ. (iii) Explain the key features of this plot and how the phase of an integrate-and-fire neuron (whose phase response curve is strictly positive) will change with oscillating inputs and synapses exhibiting STDP.