THE UNIVERSITY OF WESTERN ONTARIO
London Ontario

Applied Mathematics Ph.D. Comprehensive Examination

25 May 2023
Part II: 9:00 am - 12:00 pm

Instructions: The comprehensive exam consists of two parts. This is Part II, for which a minimum of 60% is required to pass. Questions in (A) and (B) are for all candidates. Questions in (C)-(E) are area dependent, and you may choose (4 of 6) to be graded.

You may use a calculator, pen, and pencil. NO other aids are allowed. Your calculator must NOT be capable of wireless communication or capable of storing and displaying large text files. NO other aids are allowed.

(A) Numerical Analysis

NA1. For a square matrix A, the undergraduate way to calculate eigenvalues and eigenvectors is to solve $\det(A-\lambda I)=0$ and hence solve $(A-\lambda I)x=0$.

(a) Explain why this method is not useful for large matrices.

(b) Explain how the power method for eigenvalue computation works.

(c) For the matrix

$$A=\begin{bmatrix} 2 & 3 & 1 & 1 \\ 2 & 3 & -3 & -1 \\ 2 & -1 & 1 & -3 \\ 2 & -1 & -3 & 3 \end{bmatrix} ,$$

use the seed vector $v_0 =< 2, 2, -1, 2 >$ to conduct two steps of the power method, and hence estimate one eigenvalue of the matrix.

(d) Another algorithm for eigenvalue calculation is the QR method. Conduct one step of this method for the given matrix.

NA2. Consider the following initial-value problem for $y(t)$.

$$\frac{dy(t)}{dt} = f(t, y(t)) ,$$
$$y(0) = y_0 .$$

Consider the following numerical method for integrating the o.d.e. Given a solution at $t = t_n$, namely a value for $y_n = y(t_n)$, the solution at $t = t_{n+1}$ is given by the scheme below for taking a step h.

$$\begin{aligned} t_{n+1} &= t_n + h , \\ s_1 &= f(t_n, y_n) , \\ s_2 &= f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}h\, s_1) , \\ y_{n+1} &= y_n + hs_2 . \end{aligned}$$

Determine the accuracy of the solution step.

(B) Partial Differential Equations

PDE1. Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and let ∇^2 be the 2-D Laplacian operator. Consider the BV-problem for the Poison equation:

$$\begin{cases} \nabla^2 u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ Bu(\mathbf{x}) = 0, & x \in \partial\Omega. \end{cases}$$

where the boundary condition is on $Bu = \alpha u + \beta \nabla u \cdot \mathbf{n}$ ($\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$) which includes both Neumann type ($\alpha = 0$ but $\beta \neq 0$) and Dirichlete type (($\alpha \neq 0$ but $\beta = 0$) as special cases.

(a) State the Green's formula for the operator ∇^2 with respect to Ω and $\partial\Omega$.
(b) State the definition of the Green's function for this problem.
(c) State the Fredholm Alternative Theorem for this prblem.
(d) When $\alpha = 0$ but $\beta \neq 0$, find the condition(s) on $f(\mathbf{x})$ under which, this BV problem has a solution. Express the solution in terms of the Green's function and $f(\mathbf{x})$.
(e) When $\beta = 0$ but $\alpha \neq 0$, find the condition(s) on $f(\mathbf{x})$ under which, this BV problem has a solution. Express the solution in terms of the Green's function and $f(\mathbf{x})$.

PDE2. Consider the scalar reaction diffusion equation

$$u_t(t, x) = D u_{xx}(t, x) + u g(u(t, x)), \quad t > 0, \quad x \in (0, L). \tag{3.1}$$

with $u(t, x)$ denoting the population of a species at time $t > 0$ and location $x \in (0, L)$. Here $D \geq 0$ is the diffusion rate and $L > 0$ represents the size of the habitat for the species.

(a) Assume that there is a constant $k > 0$ such that $g(k) = 0$ and $g'(k) < 0$. Show that if the homogeneous Neumann boundary condition N-BC: $u_x(t, 0) = 0 = u_x(t, k)$ is imposed, then $u = k$ remains stable for any value of the diffusion rate $D > 0$; that is, Turin instability cannot occur.
(b) Now if the homogeneous Dirichlet boundary condition D-NC: $u(t, 0) = 0 = u(t, k)$ is imposed for (3.1), show that large diffusion rate can drive the otherwise persistent species (implied by $g(0) > 0$) to extinction.
(c) Continue to impose the homogeneous Dirichlet boundary condition D-NC: $u(t, 0) = 0 = u(t, k)$ and assume $g(0) > 0$ for (3.1) as in (b). Show that for fixed $D > 0$, there is a critical habitat size L^*, such that when $L < L^*$ the population will go to extinction ($u(t, x) \to 0$ as $t \to \infty$); while when $L > L^*$, the population will persist (there will be stable mode(s) near $u = 0$).

(C) Neural Networks

NN1. The leaky integrate-and-fire model is defined by the equation:

$$\tau_m \dot{v} = -v + R_m I_e \,. \tag{1}$$

When $v \geq v_{th}$, the reset condition is $v \to 0$ mV. Starting at time $t = 0$, a current I_e (s.t. $R_m I_e > v_{th}$) is applied to the model neuron. (i) Solve for t_{isi}, the time of the next action potential. (ii) Use this expression to write the interspike-interval firing rate of the neuron. (iii) Find an approximation for this expression and study how the firing rate grows with increasing I_e.

NN2. Spike-time dependent plasticity (STDP) is the process by which synapses between neurons change strength based on the timing of spikes between input (pre-synaptic) and output (post-synaptic) neurons. Considering the spike times between an input neuron (t_1) and an output neuron (t_2) and their difference $s = t_2 - t_1$, the STDP rule is defined by the piecewise exponential function on s:

$$f(s) = \begin{cases} \alpha_+ e^{\frac{-s}{\tau_+}}, & s \geq 0 \\ -\alpha_- e^{\frac{s}{\tau_-}}, & s < 0 \end{cases} \tag{2}$$

Consider an input population oscillating with time-varying rate

$$r(t) = \frac{r_0}{2}\Big[1 - \cos(\nu t)\Big] \tag{3}$$

where ν is the angular frequency of the oscillation. The output neuron spikes at times

$$S(t) = \sum_n \delta\left(t - \frac{2\pi n + \phi}{\nu}\right) \tag{4}$$

where ϕ is the phase of the output spike relative to the input.

(i) Derive an expression for the expected weight change over time in terms of the learning rule $f(s)$ and the correlation function $C(s)$ between spikes of an input population oscillating synchronously and the output neuron spiking at a phase ϕ relative to the input. (ii) Sketch a plot of the expression for $\frac{dw}{dt}$ as a function of the output spike phase ϕ. (iii) Explain the key features of this plot and how the phase of an integrate-and-fire neuron (whose phase response curve is strictly positive) will change with oscillating inputs and synapses exhibiting STDP.

(D) Mathematical Biology

MB1. (*Site frequency spectrum*) The Fokker-Planck equation for the diffusion limit of the Wright-Fisher model can be written as

$$\frac{\partial \phi(p,t)}{\partial t} = \frac{1}{2N}\frac{\partial^2}{\partial p^2}\left[p(1-p)\phi(p,t)\right] - \frac{\partial}{\partial p}\left[M(p)\phi(p,t)\right]$$

We assume only two genetic variants are possible: variant A (frequency p), and variant B (frequency $1-p$). N is (constant) population size and $M(p)$ is a function describing the expected directional change in frequency.

(a) Mutations occur at rate μ per individual per generation, causing whichever variant was present in an individual to be switched to the alternate in the mutant offspring. Derive an expression for $M(p)$ implied by this mutation model.

(b) Deduce the value of the probability flux for the steady-state distribution $\phi(p)$ (hint - use symmetry).

(c) Solve for $\phi(p)$ (note: you do not need to evaluate the constant of proportionality needed to normalize $\phi(p)$).

(d) The shape of $\phi(p)$ is determined entirely by the quantity $2N\mu$. Explain how $\phi(p)$ changes with $2N\mu$, and interpret the significance of this quantity referring to the concept of random genetic drift.

MB2. (*Coexistence*) A plant community consists of two types of plant $i = 1, 2$. These plants live in a cyclical environment that alternates between wet years and dry years. A fraction $0 < f < 1$ of the plants from each type survive each year. The change in plant abundance is given by

$$n_i(t+1) = fn_i(t) + (1-f)\frac{w_i}{\overline{w}}n_i \tag{5}$$

where n_i is abundance, w_i is a type-specific constant that depends on the state of the environment and $\overline{w}$ is the average value of w in the community.

(a) Assume $w_1 = 1 + a$, $w_2 = 1$ in a wet year and $w_1 = 1$, $w_2 = 1 + b$ in a dry year, where $a, b > 0$. In a few sentences give a biological interpretation of what this assumption means.

(b) Use invasion analysis to show that the two types can coexist even if $a \neq b$.

(E) Disease Modelling

DM1. Consider the following set of differential equations for the spread of an infectious disease, where S is the density of susceptible individuals and I is the density of infectious individuals. K and p are positive parameters. (Note that this system is not a realistic model.)

$$\frac{dS}{dt} = \left(\frac{K^2}{4}\right) - \left(S - \frac{K}{2}\right)^2 - I \tag{6}$$

$$\frac{dI}{dt} = Ip - I^2 \tag{7}$$

(a) Find the nullclines of this system.

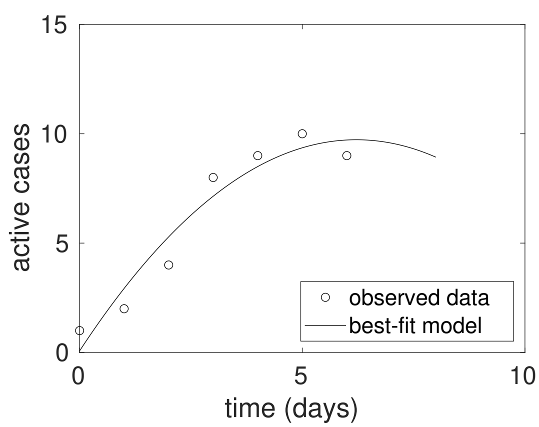
(b) Assume $p = K^2/2$. Sketch the nullclines on a phase-plane plot. Indicate which are S- and which are I-nullclines.

(c) Find all of the equilibria on the phase-plane sketch above, give their values and describe each one in words.

(d) We will now relax the assumption that $p = K^2/2$. Find the value of p such that there is exactly one endemic equilibrium in this model.

(e) Give the definition of R_0 and provide an argument, from first principles, for what the expression for R_0 would be in this model, in terms of the parameters.

DM2. A single case of a novel infectious disease is known to have arrived in a region on Monday, October 4, "day 0". The numbers of active cases reported in the region each day following day 0 are plotted in the graph below (dots). The regional public health team has fit these data to a model $I = at^2 + bt + c$ where t is time in days, $I(t)$ is the number of active cases on day t, and a, b and c are parameters. The best fit parameters are $a = -0.25$, $b = 3$ and $c = .07$, and the best-fit curve is also plotted by a solid line on the graph.



You may use point form comments, sketches and/or equations for this question.

(a) Comment on this data fitting. You may discuss strengths or weaknesses, and why this is a good or a bad approach to fit to the data.

(b) The public health team would like to predict when this wave of the epidemic will be over. Is it reasonable to predict this based on this fit?

- If your answer is "yes", explain why and predict when we will have zero cases.
- If your answer is "no", explain why not, and suggest an alternative approach for data fitting.