Mathematics 310a

The Principle of Mathematical Induction and the Well Ordering Principle

In dealing with the natural numbers $\mathbb{N}$ we must postulate some form of the **principle of mathematical induction** or the **well ordering principle**. We state two forms of the principle of mathematical induction and then state the well ordering principle and prove that all three are equivalent.

Postulate 1 (The Principle of Mathematical Induction, $\mathbf{MI_1}$). *Let $S \subseteq \mathbb{N}$. Suppose*

$$1 \in S \tag{1}$$

and

$$n + 1 \in S \textit{ whenever } n \in S. \tag{2}$$

Then $S = \mathbb{N}$.

Postulate 2 (The Principle of Mathematical Induction, $\mathbf{MI_2}$). *Let $S \subseteq \mathbb{N}$. Suppose*

$$1 \in S \tag{3}$$

and

$$n + 1 \in S \textit{ whenever } 1, 2, \ldots, n \in S. \tag{4}$$

Then $S = \mathbb{N}$.

Postulate 3 (The Well Ordering Principle, W). *Suppose S is a non-empty subset of $\mathbb{N}$. Then S contains a smallest element.*

We show that these three postulates are equivalent by establishing the chain of implications, $\mathbf{MI_1} \Rightarrow \mathbf{MI_2} \Rightarrow \mathbf{W} \Rightarrow \mathbf{MI_1}$.

- $\mathbf{MI_1} \Rightarrow \mathbf{MI_2}$

  We assume $\mathbf{MI_1}$ and must prove $\mathbf{MI_2}$. Let S be any set of positive integers satisfying the hypotheses (3) and (4). We must show that $S = \mathbb{N}$ using $\mathbf{MI_1}$. Consider the statement

  $$P(n): \text{ The integers 1 to } n \text{ are in } S.$$

  Let

  $$S' = \{n \in \mathbb{N} | P(n) \text{ is true.}\}$$

$P(1)$ is true by hypothesis, so $1 \in S'$. Assume $P(k)$ is true where k is a fixed but arbitrary positive integer. That is assume $k \in S'$, i.e. 1 to k inclusive are in S. Hence by (4), we have $k+1 \in S$. That is to say we have 1 to $k+1$ inclusive are in S. So $P(k+1)$ is true which means $k+1 \in S'$. Now we have $1 \in S'$ and $k+1 \in S'$ whenever $k \in S'$ so that (1) and (2) are satisfied with S replaced by S'. $\mathbf{MI_1}$ gives $S' = \mathbb{N}$. But this means that $S = \mathbb{N}$ which is what we had to prove.

- $\mathbf{MI_2} \Rightarrow \mathbf{W}$

  Let S be a nonempty subset of $\mathbb{N}$. We must show that S has a smallest element. Assume that S has no least element and consider the statement

  $$P(n): \ n \text{ is not an element of } S.$$

  Now $P(1)$ is true, because 1 is the smallest element of $\mathbb{N}$ so that if $1 \in S$ then S would have a smallest element contradicting our assumption. Assume that $P(1)$, $P(2)$,...,$P(k)$ are all true. But now we see that $P(k+1)$ is true, otherwise $k+1$ is the smallest element of S and S, by hypothesis, has no smallest element. $\mathbf{MI_2}$ now implies that $P(n)$ is true for all $n \in \mathbb{N}$. But this means that $S = \emptyset$. This contradiction yields the desired result.

- $\mathbf{W} \Rightarrow \mathbf{MI_1}$

  Let S be a set of positive integers satisfying (1) and (2). Assuming $\mathbf{W}$, we must show that $S = \mathbb{N}$. Suppose that $S \neq \mathbb{N}$. Then $\mathbf{W}$ shows that S^c has a smallest element. Clearly, this smallest element cannot be less than 1 and, since, by hypothesis, $1 \in S$, nor can this smallest element be 1. Thus this smallest element has the form $k+1$ where k is a positive integer. Now this means that $k \in S$ but $k+1 \notin S$. This contradicts the (2). Thus $S^c = \emptyset$. Hence $S = \mathbb{N}$.