HOMOLOGY OF SUBGROUPS OF RIGHT-ANGLED ARTIN GROUPS

GRAHAM DENHAM

Abstract. We describe the (co)homology of a certain family of normal subgroups of right-angled Artin groups that contain the commutator subgroup, as modules over the group algebra of the quotient. We do so in terms of (skew) commutative algebra of squarefree monomial ideals. In particular, the Krull dimension of homology under the monodromy action can be computed for non-FP subgroups. On the other hand, we find the cohomology ring of the subgroup is a quotient of an exterior Stanley-Reisner ring by a regular sequence in the case for which it is finite-dimensional.

1. Introduction

Let Γ be a simple graph on n vertices $V = \{v_1, v_2, \dots, v_n\}$. The right-angled Artin group (or graph group) G_Γ is the group with generators V and relations $v_i v_j = v_j v_i$ for each edge $v_i v_j$ in Γ. Charney and Davis [7] showed that such groups admit a finite classifying space, a subcomplex of the n-torus first introduced in [15]. It follows that the cohomology ring of G_Γ has a combinatorially explicit description as an exterior Stanley-Reisner ring.

Bestvina and Brady [2] make use of a particular subgroup of G_Γ with infinite cyclic quotient as examples to distinguish finiteness properties: the kernel of the map to $\mathbb{Z}$ sending each v_i to the generator 1 is finitely generated if and only if Γ is connected and finitely presented if and only if Γ is simply connected. Moreover, they show that this subgroup is FP_k if and only if the flag complex K_Γ of Γ is homologically k-connected. The cohomology ring of this subgroup is computed in [16, 21] when it is finite-dimensional, by relating it to the simplicial topology of the flag complex K_Γ.

More generally, for an integer m let $\rho\colon G_\Gamma \to \mathbb{Z}^m$ be a surjective group homomorphism with the property that $\rho(v_i)$, for each i, is a generator of the abelian group $\mathbb{Z}^m$. We shall call such a map ρ a *coordinate homomorphism*, and denote its kernel by $N_\rho = N_{\Gamma,\rho}$, the *coordinate subgroup.* From Meier, Meinert and VanWyk's calculation of the Bieri-Neumann-Strebel invariants of right-angled Artin groups [17], one sees that the homology groups of N_ρ are not finitely generated except under very restrictive hypotheses.

One observation made here is that, nevertheless, $H_\bullet(N_\rho, \mathbb{Z})$ is a finitely-generated module over the group ring $\mathbb{Z}[\mathbb{Z}^m]$, and so is amenable to description in terms of the graph Γ via combinatorial commutative algebra. Accordingly, we can describe this module in terms of the exterior Stanley-Reisner ring of the clique complex of the graph Γ. For example, one can determine the Krull dimension of each $\mathbb{Z}[\mathbb{Z}^m]$-module $H_p(N_\rho, \mathbb{Z})$ explicitly (dimension zero being equivalent to finite generation over $\mathbb{Z}$.)

2000 *Mathematics Subject Classification.* Primary 20F36, Secondary 13F55, 20J05.

Key words and phrases. Right-angled Artin groups, monomial ideals, homological finiteness properties.

Partially supported by a grant from NSERC of Canada.

In §4.2, we recall the notion due to Aramova, Avramov and Herzog [1] of regular sequences for skew-commutative algebras. We find that a subgroup N_ρ has finitely-generated homology if and only if the image of the generators under the inflation map

$$H^1(\mathbb{Z}^m, \mathbb{Z}) \to H^1(G_\Gamma, \mathbb{Z})$$

forms a regular sequence in the ring $H^\bullet(G_\Gamma, \mathbb{Z})$. This leads to a description of the cohomology ring of N_ρ in the finitely-generated case (Theorem 4) that generalizes a result of Leary and Saadetoğlu [16].

Under the additional hypothesis that the complex K_Γ is Cohen-Macaulay, these results can be made more explicit via Bernstein-Gelfand-Gelfand duality (Section 5). In particular, we find that $H^q(G_\Gamma, \mathbb{Z}[G_\Gamma^{ab}])$ is zero except for $q = d + 1$ if and only if K_Γ is Cohen-Macaulay of dimension d. This is an abelian analogue of Brady and Meier's characterization in [3] of Γ for which G_Γ is a duality group; in this case, we are also able to describe the dualizing module explicitly.

2. Classifying spaces

The construction used here is a generalization of constructions that appear independently in the work of various authors. In the context of right-angled Artin groups, the idea originates with Charney and Davis [7]. The language of partial product complexes is convenient, however; details and further references may be found in [9]. (These are also known as generalized moment-angled complexes; see [26, 4].)

2.1. Partial product complexes.

Definition 2.1. Let X be a space, and $A \subset X$ a non-empty subspace. Given a simplicial complex K on vertex set $[n] = \{1, 2, \ldots, n\}$, define $\mathcal{Z}_K(X, A)$ to be the following subspace of the Cartesian product $X^{\times n}$:

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma, \tag{1}$$

where $(X, A)^\sigma = \{x \in X^{\times n} \mid x_i \in A \text{ if } i \notin \sigma\}$.

If X is a pointed space, let $\mathcal{Z}_K(X) = \mathcal{Z}_K(X, *)$. For example, $\mathcal{Z}_K(S^1)$ is a subcomplex of the n-torus $(S^1)^{\times n}$.

2.2. **Right-angled Artin groups.** If Γ is a graph with vertices $V(\Gamma) = \{v_1, \ldots, v_n\}$ and edges $E(\Gamma)$, recall the *right-angled Artin group* G_Γ is defined by the presentation

$$G_\Gamma = \langle v_1, \ldots, v_n \mid v_i v_j = v_j v_i \text{ for each } v_i v_j \in E(\Gamma)\rangle\,; \tag{2}$$

see [6] for a survey. If Γ is a graph, its *clique complex* K_Γ is the simplicial complex with vertices $V(\Gamma)$ and simplices σ, for all $\sigma \subseteq V(\Gamma)$ with the property that each pair of vertices of σ are connected by an edge. If a simplicial complex K is the clique complex of its 1-skeleton, K is called a flag complex. Then the construction of Definition 2.1 recovers the construction of [7]:

Proposition 2.2 ([7]). *Let K be a simplicial complex and let $\Gamma = K^{(1)}$, its 1-skeleton. Then $G_\Gamma \cong \pi_1(\mathcal{Z}_K(S^1), *)$. If, further, K is a flag complex, then $\mathcal{Z}_K(S^1)$ is an Eilenberg-Maclane space for G_Γ.*

For example, $\pi_1(\mathcal{Z}_K(S^1))$ is abelian if and only if the 1-skeleton of K is a complete graph. The clique complex of a complete graph is simply a full simplex $K = \Delta^{n-1}$ on n vertices, in which case $\mathcal{Z}_K(S^1)$ is the n-torus $(S^1)^{\times n}$.

Since the main tool used here is the space $\mathcal{Z}_K(S^1)$ (rather than the group G_Γ), it will be natural to consider the homology of spaces, rather than groups; as the Proposition indicates, we will recover the case of groups by specializing to those K which are flag complexes. We shall correspondingly regard simplicial complexes as our primary objects, rather than graphs.

2.3. **Coordinate homomorphisms.** Here we identify the coordinate subgroups, defined in the Introduction, in terms of our topological construction.

Such subgroups are a special case of a more general construction. If $f\colon K \to L$ is a map of simplicial complexes sending vertices $[n]$ to vertices $[m]$, there is a natural map $\mathcal{Z}_f : \mathcal{Z}_K(S^1) \to \mathcal{Z}_L(S^1)$ obtained by restricting a map $\overline{f}\colon (S^1)^{\times n} \to (S^1)^{\times m}$. Here, $\overline{f}(x)_j = \prod_{i\colon f(i)=j} x_i$; see [9, Lemma 2.2.2] for details. In terms of right-angled Artin groups:

Proposition 2.3. *If $f\colon K \to L$ is a map of simplicial complexes, the induced map of fundamental groups $\mathcal{Z}_f^\sharp\colon G_{K^{(1)}} \to G_{L^{(1)}}$ sends the ith generator of $G_{K^{(1)}}$ to the $f(i)$th generator in $G_{L^{(1)}}$.*

In particular, if $\Gamma = K^{(1)}$ has n vertices, the abelianization of G_Γ is $\mathbb{Z}^n$, and the natural map $G_\Gamma \to \mathbb{Z}^n$ is obtained by choosing $L = \Delta^{n-1}$, the full simplex on n vertices. More generally, we consider the following case:

Definition 2.4. Let K be a simplicial complex on n vertices, and let $f\colon [n] \to [m]$ be a surjective function on sets. Abusing notation, regard f as the induced map of simplicial complexes $f\colon K \to \Delta^{m-1}$. We will call f a coordinate map. Letting $\rho = \mathcal{Z}_f^\sharp$, we obtain a short exact sequence of groups

$$\begin{tikzcd} 1 \arrow[r] & N_\rho \arrow[r] & G_{K^{(1)}} \arrow[r, "\rho"] & \mathbb{Z}^m \arrow[r] & 0. \end{tikzcd}$$

Conversely, any coordinate subgroup N_ρ arises in this way. However, we will write N_f in place of $N_{\mathcal{Z}_f^\sharp}$.

Example 1. At the one extreme, we may take f to be the identity map. Then the homomorphism $\mathcal{Z}_f^\sharp\colon G_{K^{(1)}} \to \mathbb{Z}^n$ is the abelianization, and its kernel N_f is the commutator subgroup of the right-angled Artin group.

Example 2. On the other hand, the (unique) map $f\colon [n] \to [1]$ induces a homomorphism $\mathcal{Z}_f^\sharp\colon G_{K^{(1)}} \to \mathbb{Z}$ sending each v_i to 1. The kernel N_f is the Bestvina-Brady group considered in [2, 5, 17, 18, 16, 21]: following convention, we will denote it by H_Γ, where $\Gamma = K^{(1)}$.

2.4. **Abelian covers.** Let $\pi : \mathbb{R} \to S^1$ be the universal cover of S^1, sending $\mathbb{Z}$ to the basepoint $*$. By [9, Lemma 2.9], the map of pairs $\pi : (\mathbb{R}, \mathbb{Z}) \to (S^1, *)$ induces a fibration

$$\mathbb{Z}^n \to \mathcal{Z}_K(\mathbb{R}, \mathbb{Z}) \overset{\mathcal{Z}_\pi}{\to} \mathcal{Z}_K(S^1), \tag{3}$$

which is in fact the universal abelian cover of $\mathcal{Z}_K(S^1)$. Then if $f : K \to [m]$ is a coordinate homomorphism, the subgroup N_f is the fundamental group of the fibred coproduct $\mathcal{Z}_K(\mathbb{R},\mathbb{Z}) \times_{\mathbb{Z}^n} \mathbb{Z}^m$, where $\mathbb{Z}^n$ acts on $\mathcal{Z}_K(\mathbb{R},\mathbb{Z})$ by deck transformations, and on $\mathbb{Z}^m$ by the induced map $\overline{f}^\sharp$.

2.5. **CW-complexes.** There are combinatorially explicit cell structures for each complex. First, the standard cell structure on the torus $(S^1)^{\times n}$ restricts to give a cell structure for $\mathcal{Z}_K(S^1)$ (see [15]). From the definition (1), the cells are naturally labelled by the simplices of K. For each simplex $I \in K$ with k vertices, let ε_I denote the corresponding k-cell. Note that the complex is minimal in the sense that the attaching maps are zero.

We need the following notation. Let $E = \mathbb{Z}\langle e_1, \dots, e_n\rangle$ denote the exterior algebra, which we regard as a graded-commutative Hopf algebra with generators in degree 1. Let $\{\varepsilon_i\}$ denote the $\mathbb{Z}$-dual basis to the generators, so that $E^* = \mathbb{Z}\langle \varepsilon_1, \dots, \varepsilon_n\rangle$ is also an exterior algebra.

Now let $\mathbb{Z}\langle K\rangle = E/J_K$, the exterior Stanley-Reisner ring of K, where J_K is the ideal generated by monomials indexed by nonfaces of K. Then $\mathbb{Z}\langle K\rangle^*$ is a sub-coalgebra of E^*, spanned by monomials ε_I, where we set $\varepsilon_I := \varepsilon_{i_1} \cdots \varepsilon_{i_k}$ if $I = \{i_1, \dots, i_k\}$ is a simplex of K. We identify $\mathbb{Z}\langle K\rangle^*$ with the cellular chain complex of $\mathcal{Z}_K(S^1)$ (with zero differential). The reason for this notation is the following foundational result.

Theorem 1 ([15]). *Let K be a simplicial complex. Then $H^\bullet(\mathcal{Z}_K(S^1),\mathbb{Z}) \cong \mathbb{Z}\langle K\rangle$ as graded rings.*

Second, we will use a similar cell structure for the complex $\mathcal{Z}_K(\mathbb{R},\mathbb{Z})$. In the flag-complex setting, it should be noted that the construction below is simply the (abelianized) Salvetti complex for the right-angled Artin group, as explained and generalized by Charney and Davis [7].

To start, label the zero cells of $\mathbb{R}$ by $\{x^i : i \in \mathbb{Z}\}$, and the 1-cells by $\{\varepsilon \cdot x^i : i \in \mathbb{Z}\}$ so that $\partial(\varepsilon \cdot x^i) = x^{i+1} - x^i$ for each i. Extending this to the product structure on $\mathbb{R}^n$ and restricting to the subcomplex $\mathcal{Z}_K(\mathbb{R},\mathbb{Z})$ gives a complex with k-cells labelled in a natural way by

$$\{\varepsilon_I \cdot x^\alpha \colon I \in K, |I| = k, \text{ and } \alpha \in \mathbb{Z}^n\},$$

where we regard x^α equivalently as an element of $\mathbb{Z}^n$ (written multiplicatively) or a Laurent monomial $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. It is straightforward to check that the differential is given by

$$\partial(\varepsilon_I x^\alpha) = \sum_{i\in I} (-1)^{\sigma(I,i)} \varepsilon_{I-\{i\}} \cdot x^{\alpha+\chi_i} - \varepsilon_{I-\{i\}} \cdot x^\alpha, \tag{4}$$

where $\sigma(I,i)$ is the number of elements preceding i in I (in the standard order) and χ_i is the ith coordinate vector in $\mathbb{Z}^n$. Make the identification

$$C_\bullet^{\text{cell}}(\mathcal{Z}_K(\mathbb{R},\mathbb{Z})) \cong \mathbb{Z}\langle K\rangle^* \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^n]. \tag{5}$$

Let v_i denote the ith standard generator of $\pi_1(\mathcal{Z}_K(S^1))$ from (2). Then, from the construction:

Proposition 2.5. *For any K,*

(1) *The universal abelian cover* $\mathcal{Z}_\pi$ *of* (3) *is cellular and satisfies* $\mathcal{Z}_\pi(\varepsilon_I x^\alpha) = \varepsilon_I$ *for all choices of* I *and* α.
(2) *The cell structure* (5) *is* $\mathbb{Z}^n$*-equivariant, and* v_i *acts by multiplication by* x_i *for* $1 \le i \le n$*;*
(3) *The differential* (4) *is induced by* $\partial(\varepsilon_i) = x_i - 1$ *for each* $1 \le i \le n$*, together with* $\mathbb{Z}[\mathbb{Z}^n]$*-linearity and the Leibniz rule on* $\mathbb{Z}\langle K\rangle^*$*;*

3. (Co)homology of abelian covers

In this section, we describe the homology of the coordinate subgroups in terms of commutative algebra. If f is a coordinate map (Def. 2.4), then $\mathbb{Z}[\mathbb{Z}^m]$ is a $\pi_1(\mathcal{Z}_K(S^1))$-module via the homomorphism $\mathcal{Z}_f^\sharp$. Then

$$H_\bullet(\mathcal{Z}_K(S^1) \times_{\mathbb{Z}^n} \mathbb{Z}^m, \mathbb{Z}) \cong H_\bullet(\mathcal{Z}_K(S^1), \mathbb{Z}[\mathbb{Z}^m]);$$

in the case where K is a flag complex and $\Gamma = K^{(1)}$, this is simply Shapiro's Lemma:

$$H_\bullet(N_f, \mathbb{Z}) \cong H_\bullet(G_\Gamma, \mathbb{Z}[\mathbb{Z}^m]).$$

3.1. **Linearization.** In this section, fix a coordinate map $f\colon [n] \to [m]$. Let $S = \mathbb{Z}[t_1, \dots, t_n]$, a (commutative) polynomial ring, and let $R = \mathbb{Z}[s_1, \dots, s_m]$. Regard R as a module over S via the ring homomorphism $\widetilde{f}\colon S \to R$ given by letting $\widetilde{f}(t_i) = s_{f(i)}$, for each i. Let $\mathfrak{m}$ denote the maximal ideal of S generated by $\{t_1+1, t_2+1, \dots, t_n+1\}$. We will abuse notation and also write $\mathfrak{m}$ for its image $\widetilde{f}(\mathfrak{m})$ in R. Then the Laurent polynomials are obtained by localization: $\mathbb{Z}[\mathbb{Z}^n] \cong \mathfrak{m}^{-1}S$ and $\mathbb{Z}[\mathbb{Z}^m] \cong \mathfrak{m}^{-1}R$.

Proposition 3.1. *For any* K*, we have isomorphisms of complexes of* $\mathbb{Z}[\mathbb{Z}^n]$*-modules:*

$$C_p^{\mathrm{cell}}(\mathcal{Z}_K(\mathbb{R},\mathbb{Z})) \cong \mathfrak{m}^{-1}(\mathbb{Z}\langle K\rangle_p^* \otimes S, \partial), \tag{6}$$

$$C_p^{\mathrm{cell}}(\mathcal{Z}_K(\mathbb{R},\mathbb{Z}) \times_{\mathbb{Z}^n} \mathbb{Z}^m) \cong \mathfrak{m}^{-1}(\mathbb{Z}\langle K\rangle_p^* \otimes R, \overline{\partial}), \text{ and} \tag{7}$$

$$C_c^p(\mathcal{Z}_K(\mathbb{R},\mathbb{Z}) \times_{\mathbb{Z}^n} \mathbb{Z}^m) \cong \mathfrak{m}^{-1}(\mathbb{Z}\langle K\rangle_p \otimes R, \delta), \tag{8}$$

for all $p \ge 0$*, where the complex* (8) *is the cellular cochain complex with compact support. The differential* ∂ *is induced by* $\partial(\varepsilon_i) = t_i$*, for all* i *by* $\mathbb{Z}[\mathbb{Z}^n]$*-linearity and* $\mathbb{Z}\langle K\rangle^*$*-Leibniz, and* $\overline{\partial}$ *by* $\overline{\partial}(\varepsilon_i) = s_{f(i)}$*. The differential* δ *acts by left multiplication by the element* $\sum_{i=1}^n e_i \otimes s_{f(i)}$*.*

Proof. The isomorphism (6) is obtained by localizing (5). To establish (8), we identify $\mathbb{Z}[\mathbb{Z}^n]$ with the submodule of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\mathbb{Z}^n], \mathbb{Z})$ with finite supports. However, now v_i acts (contragrediently) by multiplication by x_i^{-1}, for each i. Putting this together with the dual of Proposition 2.5(3) gives the required isomorphism. In fact, the right-hand side of (8) is a complex of $E \otimes \mathfrak{m}^{-1}S$-modules, since δ commutes with multiplication by elements of E. □

Since localization is exact, it is equivalent (but more convenient) to regard the cellular (co)chain complexes above over the polynomial ring S, and we shall do so in the rest of the paper.

Remark 1. Our restriction to coordinate subgroups in place of arbitrary subgroups with abelian quotient is needed for the isomorphisms of the Proposition above. More generally, the cellular (co)chain complexes are filtered (rather than graded) by powers of the augmentation ideal, and the isomorphisms (6)–(8) are merely isomorphisms of associated graded modules. As Stefan Papadima observes [20], the spectral sequence of the filtration fails to converge strongly even for very simple examples of non-coordinate homomorphisms.

3.2. **Commutative algebra.** Given the setup above, it is natural to interpret group (co)homology in terms of skew-commutative algebra. We follow the grading conventions of [11]; in particular, for a graded module M, let $M(r)_q = M_{r+q}$ for all q. For modules M and N, the notation $\operatorname{Ext}^{pq}(M,N)$ refers to cohomological degree p and polynomial degree q. Here, typically $q \leq -p$, so we write $\operatorname{Ext}^p(M,N)_r = \operatorname{Ext}^{pq}(M,N)$, where $r = -p-q$.

Theorem 2. *For any simplicial complex K and for all $q \geq 0$,*

$$H_q(\mathcal{Z}_K(\mathbb{R},\mathbb{Z}),\mathbb{Z}) \cong \operatorname{Ext}_E(\mathbb{Z}\langle K\rangle,\mathbb{Z})_q$$

as S-modules. Moreover, for all $p \geq 0$,

$$\operatorname{Gr}_p H_\bullet(\mathcal{Z}_K(\mathbb{R},\mathbb{Z}),\mathbb{Z}) \cong \mathfrak{m}^{-1}\operatorname{Ext}^p_E(\mathbb{Z}\langle K\rangle,\mathbb{Z}) \tag{9}$$

where $\operatorname{Gr}_\bullet$ denotes the grading associated to the filtration by powers of the augmentation ideal of $\mathbb{Z}[\mathbb{Z}^n]$.

Before beginning the proof, we note that the (left) S-module structure on $\operatorname{Ext}_E(\mathbb{Z}\langle K\rangle,\mathbb{Z})$ here comes from the identification $S \cong \operatorname{Ext}_E(\mathbb{Z},\mathbb{Z})$ with its natural action: see [24].

Proof. We compute $\operatorname{Ext}_E(\mathbb{Z}\langle K\rangle,\mathbb{Z})$ by the standard injective resolution of $\mathbb{Z}$:

$$0 \to \mathbb{Z} \to I^0 \to I^1 \to \cdots$$

Recall that E is self-injective. Then $\mathbb{Z}$ maps to E^n, the socle of E, and $I^q = E(n+q)\otimes_{\mathbb{Z}} S^q$, with differential induced $E \otimes S$-linearly by multiplication by $\sum_{i=1}^n e_i \otimes t_i$. Then

$$\begin{aligned}\operatorname{Ext}_E(\mathbb{Z}\langle K\rangle,\mathbb{Z}) &= H\operatorname{Hom}_E(\mathbb{Z}\langle K\rangle, I^\bullet)\\ &\cong H\left(\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\langle K\rangle,\mathbb{Z})\otimes S_\bullet\right)\end{aligned}$$

since $\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Z}) = \operatorname{Hom}_E(-,E)(n)$.

Now we may compare this with Proposition 3.1(6) to obtain the cellular homology of $\mathcal{Z}_K(\mathbb{R},\mathbb{Z})$. □

Now fix a simplicial complex K with n vertices, and a coordinate map $f\colon [n] \to [m]$. Let $A = A_f$ denote the kernel of the homomorphism $\widetilde{f}\colon S \to R$. Let $\mathfrak{a} = \mathfrak{a}_f$ denote the ideal of E generated in degree 1 by functionals that vanish on A_1. Let $\operatorname{ann}\mathfrak{a}$ denote its annihilator in E. These ideals have the following properties.

Lemma 3.2. *For any $f\colon [n] \to [m]$,*

1. *the ideal A is generated in degree 1, and A_1 is a free $\mathbb{Z}$-module of rank $n-m$.*
2. *the ideal $\mathfrak{a}_f$ is generated by m elements, for $1 \leq j \leq m$:*

$$h_j := \sum_{i\colon f(i)=j} e_i,$$

(3) *the ideal* $\operatorname{ann}\mathfrak{a}_f$ *is principal, generated by* $h_1h_2\cdots h_m$.

Proof. The first two assertions come from the definitions. The third follows from the fact that $\{h_1,\dots,h_m\}$ are linearly independent in E_1. ☐

In view of Lemma 3.2(3), let $a_f = h_1h_2\cdots h_m$, the generator of $\operatorname{ann}\mathfrak{a}_f$. To avoid complications, in what follows we will work over a coefficient field $\mathbf{k}$. In order to state the next result, recall the following definition. The (combinatorial) Alexander dual of a simplicial complex K on $[n]$ is a complex $K^\star$ on $[n]$. By definition,

$$K^\star = \{\sigma \subseteq [n] \colon [n] - \sigma \notin K\}.$$

Theorem 3. *Let* $\mathbf{k}$ *be a field,* K *a simplicial complex, and* $f\colon [n] \to [m]$ *a coordinate map. Then, for all* $q \geq 0$,

$$H_q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) \cong \operatorname{Ext}_E(\mathbf{k}\langle K\rangle, (a_f))_{n-q} \text{ and} \tag{10}$$

$$H_c^q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) \cong \operatorname{Ext}_E(J_{K^\star}, (a_f))_{n-q}, \tag{11}$$

where $J_{K^\star}$ *is the exterior monomial ideal associated with* $K^\star$ *(see §2.5), and* (a_f) *is the principal ideal defined above.*

Proof. Since A_S is generated in degree 1 and a $\mathbf{k}$-basis for A_S^1 is a regular sequence in S, its Koszul complex is a linear, free resolution of R over S. That is, R is a Koszul module over S, so as left E-modules, $E/\mathfrak{a}_f \cong \operatorname{Ext}_S(R,\mathbf{k})$. Koszul duality is an involution, so $R \cong \operatorname{Ext}_E(E/\mathfrak{a}_f,\mathbf{k})$. This is to say that $E/\mathfrak{a}_f$ has a linear, free resolution

$$0 \longleftarrow E/\mathfrak{a}_f \longleftarrow (E \otimes R^*, d) \tag{12}$$

with a Koszul differential δ given by $\delta(1\otimes s_j^*) = \sum_{i\colon f(i)=j} e_i \otimes 1$, extending E-linearly and by the Leibniz rule on R^*. Now apply $\operatorname{Hom}_E(-,E)$. Since $\operatorname{Hom}_E(E/\mathfrak{a}_f, E)$ is naturally identified with $\operatorname{ann}\mathfrak{a}_f = (a_f)$, this ideal has an injective resolution $E\otimes R^\bullet$ with differential given by multiplication by $\sum_{i=1}^n e_i \otimes s_{f(i)}$. The proof of isomorphism (10) concludes as in Theorem 2, using Proposition 3.1(7).

The isomorphism (11) is analogous: using the fact that

$$J_{K^\star} = \operatorname{ann} J_K = \operatorname{Hom}_E(\mathbf{k}\langle K\rangle, \mathbf{k}),$$

we see that the complex $(\mathbf{k}\langle K\rangle \otimes_{\mathbf{k}} R, \delta)$ of Proposition 3.1(8) actually computes $\operatorname{Ext}_E(J_{K^\star},(a_f))$. ☐

Example 1 (continued)**.** Here, the coordinate map f is an isomorphism, so $\mathfrak{a}_f = E^{\geq 1}$, and $\operatorname{ann}\mathfrak{a} \cong \mathbf{k}(-n)$, the socle of E. Thus Theorem 3 reduces to Theorem 2 (with coefficients in $\mathbf{k}$). If K is the clique complex of a graph Γ, then Theorem 2 says

$$H_q(G_\Gamma, \mathbb{Z}[\mathbb{Z}^n]) \cong \mathfrak{m}^{-1}\operatorname{Ext}_E(\mathbb{Z}\langle K\rangle, \mathbb{Z})_q$$

for all $q \geq 0$, as modules over $\mathbb{Z}[\mathbb{Z}^n]$. In general, the homology of the universal abelian cover of the torus complex is encoded in the minimal resolution of a monomial ideal over an exterior algebra.

Example 2 (continued)**.** In the case of the Bestvina-Brady group, the kernel of the map $\widetilde{f}: S \to \mathbf{k}[s_1]$ is generated by $\{t_i - t_j \colon 1 \leq i < j \leq n\}$. Then the ideal $\mathfrak{a}$ generated by $h_1 = \sum_{i=1}^n e_i$, and $\operatorname{ann}(\mathfrak{a}) = (a)$, where $a = a_f = e_1 \cdots e_n$. So

$$H_q(H_\Gamma, \mathbf{k}) = \mathfrak{m}^{-1}\operatorname{Ext}_E(\mathbf{k}\langle K\rangle, (a))_q$$

4. Applications

A main result from the work of [2, 17] is that the Bestvina-Brady group H_Γ is FP_k iff the flag complex $K = K_\Gamma$ is k-acyclic, yet H_Γ is finitely presented iff K is simply connected. By regarding the homology groups as modules over the group algebra of the abelianization, we can measure (in terms of Krull dimension) how far they are from having finite rank.

4.1. **Dimension calculations.** Returning to Example 1, recall Proposition 2.1 of [1] provides a description of the bigraded Betti numbers of (9): for $A = E$ or $A = S$, let $\beta^A_{pq}(M) = \dim_{\mathbf{k}} \operatorname{Tor}^A_{pq}(M, \mathbf{k})$, for an A-module M, and let $P_M(t,u) = \sum_{p,q} \beta^A_{pq}(M) t^p u^q$. Let $I = I_K$ denote the squarefree monomial ideal of K in S. Then

$$P_{E/J}(t,u) = \sum_{p,q} \beta^S_{pq}(S/I) \frac{t^p u^q}{(1-t)^{p+q}},$$

where

$$\beta^S_{pq}(S/I) = \sum_{\substack{\mathcal{I} \subseteq [n]\colon \\ |\mathcal{I}| = p+q}} \dim_{\mathbf{k}} \widetilde{H}^{q-1}(K_{\mathcal{I}}, \mathbf{k}), \tag{13}$$

by Hochster's formula.

Then Theorem 2 has the following corollaries.

Corollary 4.1. *For $q > 0$, the Krull dimension of $H_q(\mathcal{Z}_K(\mathbb{Z}, \mathbb{R}), \mathbf{k})$ as a $\mathbf{k}[\mathbb{Z}^n]$-module is equal to the size of the largest set $\mathcal{I} \subseteq [n]$ for which $\widetilde{H}^{q-1}(K_{\mathcal{I}}, \mathbf{k}) \neq 0$. (If there is no such set, then $H_q(\mathcal{Z}_K(\mathbb{R}, \mathbb{Z}), \mathbf{k}) = 0$.)*

Proof. It follows from [1, Theorem 4.2, Corollary 3.8] that the dimension of $\operatorname{Ext}_E(\mathbf{k}\langle K\rangle, \mathbf{k})_q$ is $p + q$, where p is the largest integer for which $\beta^S_{pq}(S/I) \neq 0$. Now use formula (13). Since the submodule of a graded S-module annihilated by the ideal $\mathfrak{m}$ is zero, localization preserves dimension, and our conclusion follows by Theorem 2. □

We may also make use of Alexander duality. In order to draw a parallel with the main result of [14], this result is stated in terms of group cohomology.

Corollary 4.2. *Let Γ be a graph not isomorphic to a complete graph, and K its clique complex. Then the dimension of the $\mathbf{k}[\mathbb{Z}^n]$-module $H^q(G_\Gamma, \mathbf{k}[\mathbb{Z}^n])$ equals the largest integer r for which there exists a simplex $\sigma \in K$ with $n-r$ vertices satisfying $\widetilde{H}_{r-q-1}(\operatorname{link}_K(\sigma), \mathbf{k}) \neq 0$. (If there is no such simplex, $H^q(G_\Gamma, \mathbf{k}[\mathbb{Z}^n]) = 0$.)*

Proof. Since $\mathbf{k}\langle K^\star\rangle = E/J_{K^\star}$, it follows from the long exact sequence for Ext_E and Theorem 3 that, for all $q > 0$,

$$\begin{aligned} H^q(G_\Gamma, \mathbf{k}[\mathbb{Z}^n]) &\cong \mathrm{Ext}_E(J_{K^\star}, \mathbf{k})_q \\ &\cong \widetilde{H}_{q-1}(\mathcal{Z}_{K^\star}(S^1), \mathbf{k}[\mathbb{Z}^n]). \end{aligned}$$

Now we use the Alexander dual formulation of Hochster's formula, for which we refer to [19]: if $\sigma \in K$ and $\mathcal{I} = [n] - \sigma$, then for all q,

$$\widetilde{H}^{q-2}(K^\star_{\mathcal{I}}, \mathbf{k}) \cong \widetilde{H}_{|\mathcal{I}|-q-1}(\mathrm{link}_K(\sigma), \mathbf{k}).$$

Then formula (13) becomes

$$\beta^S_{p,q-1}(S/I_{K^\star}) = \sum_{\substack{\sigma\in K\colon \\ |\sigma|=n-(p+q-1)}} \dim_{\mathbf{k}} \widetilde{H}_{p-2}(\mathrm{link}_K(\sigma), \mathbf{k}),$$

from which the result follows as in Corollary 4.1, by letting $r = p + q - 1$. □

Recall the abelianization of G_Γ is $\mathbb{Z}^n$. By comparing with Jensen and Meier's result in [14], we see $H^\bullet(G_\Gamma, \mathbf{k}[G_\Gamma])$ and $H^\bullet(G_\Gamma, \mathbf{k}[G^{ab}_\Gamma])$ depend on the combinatorics of Γ in the same way: we return to this following Corollary 5.3.

4.2. **The rank variety of $\mathbf{k}\langle K\rangle$.** The purpose of this section is to point out that the homological finiteness characterization from [17] and its reinterpretation by Bux and Gonzales [5, Theorem A] can also be extracted from work of Aramova, Avramov, and Herzog [1]. Following their algebraic approach has the additional advantage of explicit formulas (Corollary 4.5), although we require our coefficient ring $\mathbf{k}$ to be a field.

Recall that if M is a graded E-module and $a \in E^1$, then M may be regarded as a chain complex with differential given by multiplication by a. Following [1], an element $a \in E^1$ is said to be M-singular if the cohomology of (M, a) is nonzero. Let $V(M)$ be the variety of all M-singular elements. Say $a \in E^1$ is M-regular if $a \notin V(M)$.

We shall also want a weaker property: for an integer d, say $a \in E^1$ is M-regular *in degree* d exactly when $H^d(M, a) = 0$; then a is M-regular if and only if it is M-regular for all d.

If $a = \sum_{i=1}^n \alpha_i e_i$, let $\mathrm{supp}(a) = \{i\colon \alpha_i \neq 0\}$, and consider the cohomology of $\mathbf{k}\langle K\rangle = E/J$, regarded as a chain complex with differential given by multiplication by the element a. We recall the following result of Aramova, Avramov and Herzog [1] (with indexing corrected). (See also [22, Theorem 5.5].)

Proposition 4.3. *The cohomology of* $(E/J, a)$ *depends only on* $\mathrm{supp}(a)$. *Let* $\mathcal{I} = \mathrm{supp}(a)$. *Then*

$$H^q(E/J, a) \cong \bigoplus_{\sigma\in K\colon \sigma\cap\mathcal{I}=\emptyset} \widetilde{H}^{q-|\sigma|-1}(\mathrm{link}_{K_\mathcal{I}}\sigma, \mathbf{k}), \quad \textit{and} \tag{14}$$

$$H^q(J_{K^*}, a) \cong \bigoplus_{\sigma\in K\colon \sigma\cap\mathcal{I}=\emptyset} \widetilde{H}_{n-q-|\sigma|-1}(\mathrm{link}_{K_\mathcal{I}}\sigma, \mathbf{k}) \tag{15}$$

where by definition $\mathrm{link}_{K_\mathcal{I}}\sigma = \{\tau \in K_\mathcal{I}\colon \tau \cup \sigma \in K\}$.

Proof. Formula (14) appears as Proposition 4.3 in [1]. Their proof expresses $(E/J, a)$ as a direct sum of simplicial cochain complexes. Since $(E/J)^* = J_{K^\star}$, the second formula (15) can by obtained from the same argument by considering the $\mathbf{k}$-dual chain complexes. □

In particular, this characterizes the E/J-singular elements. A sequence of elements $a_1, \ldots, a_r \in E^1$ is called M-regular if a_1 is M-regular and a_i is $M/(a_1, \ldots, a_{i-1})$-regular for each i, $2 \leq i \leq r$. Continuing our weakening, say the sequence is M-regular *in degree* d if a_1 is M-regular in degree d, and a_i is $M/(a_1, \ldots, a_{i-1})$-regular in degree d.

Proposition 4.4. *For any coordinate map* $f\colon [n] \to [m]$*, the vector space*

$$H_q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m])$$

is finite-dimensional (over $\mathbf{k}$*) if and only if the elements* $\{h_1, \ldots, h_m\}$ *defined in Lemma 3.2(2) form a* $J_{K^\star}$*-regular sequence in degree* $n - q$*.*

Proof. If $m = 1$, $h_1 = a = \sum_{i=1}^n e_i$. Then the ideal (a) has an injective resolution

$$0 \longrightarrow (a) \longrightarrow E \xrightarrow{a} E \xrightarrow{a} \cdots$$

and since $\mathrm{Hom}_E(\mathbf{k}\langle K\rangle, E) = J_{K^\star}$, we find $\mathrm{Ext}_E(\mathbf{k}\langle K\rangle, (a))$ is the cohomology of the infinite complex

$$J_{K^\star} \xrightarrow{a} J_{K^\star} \xrightarrow{a} \cdots$$

It follows $\mathrm{Ext}^0_E(\mathbf{k}\langle K\rangle, (a)) = \mathrm{ann}(a) \cap J_{K^\star}$, and

$$\begin{aligned} \mathrm{Ext}^p_E(\mathbf{k}\langle K\rangle, (a))_{n-q} &\cong H^{n-q}(J_{K^\star}, a) \\ &\cong \widetilde{H}_{q-1}(K, \mathbf{k}) \end{aligned}$$

for all $p \geq 1$. Then $\mathrm{Ext}_E(\mathbf{k}\langle K\rangle, (a))$ is finite-dimensional iff these vanish for $p \geq 1$, which is to say a is $J_{K^\star}$-regular in degree $n - q$. The claim follows for $H_q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}])$ by Theorem 3.

The general result follows in the same way, using the Cartan complex (see [1, Remarks 3.4], where this argument is implicit.) □

Corollary 4.5. *If the generators* $\{h_1, \ldots, h_m\}$ *of the ideal* $\mathfrak{a}_f$ *form a* $\mathbf{k}\langle K\rangle$*-regular sequence in degree* $n - q$*, then there are isomorphisms of (trivial)* S*-modules*

$$H_q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) \cong (J_{K^\star} \cap \mathfrak{a}_f)^{n-q}, \tag{16}$$

$$H^q_c(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) \cong E/(J_K + \mathfrak{a}_f)^{n-q}. \tag{17}$$

Proof. First, since $J_{K^\star}$ and $k\langle K\rangle$ are E-dual, one can check that a sequence is $\mathbf{k}\langle K\rangle$-regular if and only if it is $J_{K^\star}$-regular. Given a regular sequence in degree $n - q$, by Theorem 3,

$$\begin{aligned} H_q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) &\cong \mathrm{Ext}_E(\mathbf{k}\langle K\rangle, (a_f))_{n-q} \\ &= \mathrm{Hom}_E(\mathbf{k}\langle K\rangle, (a_f))_{n-q} \end{aligned}$$

since finite-dimensionality here means higher Ext's vanish. This, and the analogous statement for cohomology with compact support, follow from the isomorphism $\mathrm{Hom}_E(E/\mathfrak{a}_f, E) \cong (a_f)$ (see §3.2.) □

We end this section by observing an easy necessary condition for the existence of a regular sequence. For $i \geq 0$, let $f_i = f_i(K)$ be the number of simplices of K with i vertices, so that $F(K,t) = \sum_{i\geq 0} f_i t^i$ is the Hilbert series of $\mathbf{k}\langle K\rangle$.

Proposition 4.6. *If there exists a regular sequence of degree 1-elements $\{h_1, \dots, h_m\}$ for $\mathbf{k}\langle K\rangle$, then $F(K,t)$ is divisible by $(1+t)^m$.*

Proof. For $i \geq 0$, let $A_i = \mathbf{k}\langle K\rangle/(h_1, \dots, h_i)$. By definition of regularity, multiplication by h_i gives an exact sequence of graded A_i-modules

$$\begin{tikzcd} 0 \arrow[r] & A_i(-1) \arrow[r, "h_i"] & A_{i-1} \arrow[r] & A_i \arrow[r] & 0 \end{tikzcd}$$

By induction on i, the Hilbert series of A_i equals $F(K,t)/(1+t)^i$, for all $i \leq m$. ☐

In particular, if the ideal $\mathfrak{a}_f$ is generated by a regular sequence, then

$$\sum_{q\geq 0} \dim_{\mathbf{k}} (E/(J_K + \mathfrak{a}_f))^q t^q = F(K,t)/(1+t)^m. \tag{18}$$

Note that one may reformulate this condition in terms of vanishing of the h-vector of the simplicial complex K.

4.3. **Cohomology ring.**

Theorem 4. *If f is a coordinate map for which the generators $\{h_1, \dots, h_m\}$ of the ideal $\mathfrak{a}_f$ form a $\mathbf{k}\langle K\rangle$-regular sequence, then there is a natural isomorphism of graded algebras*

$$H^\bullet(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) \cong E/(J_K + \mathfrak{a}_f). \tag{19}$$

In particular, the cohomology of the coordinate subgroup of the right-angled Artin group is a quotient of the exterior Stanley-Reisner ring by linear equations.

Proof. By hypothesis, the isomorphisms (16) hold for each q, so (dually) there is an additive isomorphism (19).

Now $H^\bullet(\mathcal{Z}_K(S^1), \mathbf{k}) \cong E/J_K$ by Theorem 1, so to show (19) is an algebra isomorphism, it is enough to check the natural map

$$\mathrm{res}\colon H^\bullet(\mathcal{Z}_K(S^1), \mathbf{k}) \to H^\bullet(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m])$$

is surjective. For this, consider the spectral sequence

$$E_2^{pq} = H^p(\mathbb{Z}^m, H^q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m])) \Rightarrow H^{p+q}(\mathcal{Z}_K(S^1), \mathbf{k}).$$

Since the action of $\mathbb{Z}^m$ on $H^q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m])$ is trivial (from (9)),

$$\begin{aligned} E_2^{pq} &\cong H^p(\mathbb{Z}^m, \mathbf{k}) \otimes_{\mathbf{k}} H^q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) \\ &\cong \mathbf{k}\langle f_1, \dots, f_m\rangle^p \otimes_{\mathbf{k}} H^q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]). \end{aligned}$$

where the former is an exterior algebra. Counting dimensions with (18) shows that

$$\sum_{p,q} \dim_{\mathbf{k}} E_2^{pq} t^{p+q} = F(K,t).$$

But this is the Hilbert series of E_∞ by Theorem 1, so the spectral sequence collapses at E_2. It follows that the restriction map agrees with the (surjective) edge homomorphism

$$H^q(\mathcal{Z}_K(S^1),\mathbf{k}) \to E_\infty^{0q} \cong H^q(\mathcal{Z}_K(S^1),\mathbf{k}[\mathbb{Z}^m]),$$

for all $q \geq 0$. □

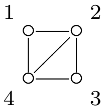
Example 2 (continued)**.** For the Bestvina-Brady group (where $f\colon [n] \to [1]$), the only summand in Proposition 4.3 is indexed by the empty simplex, since $a = h_1$ has support $[n]$. Then, for all $q \geq 0$, $H^q(J_{K^\star}, a) = \widetilde{H}_{n-q-1}(K,\mathbf{k})$. By the argument above, $H_q(\mathcal{Z}_K(S^1),\mathbf{k}[\mathbb{Z}]) = H_q(H_\Gamma,\mathbf{k})$ is finite-dimensional if and only if $\widetilde{H}_{q-1}(K,\mathbf{k}) = 0$, in which case

$$\begin{aligned} H_q(H_\Gamma,\mathbf{k}) &\cong (\operatorname{ann}(a)\cap J_{K^\star})^{n-q} \\ &\cong ((a)\cap J_{K^\star})^{n-q} \quad \text{and} \\ H^q(H_\Gamma,\mathbf{k}) &\cong (E/(J+a))^q, \end{aligned}$$

taking duals. If K is acyclic, the latter gives an isomorphism of $\mathbf{k}$-algebras. This agrees with a result of Leary and Saadetoğlu, [16, Theorem 13], whose result is more general in not requiring field coefficients.

Example 3. Consider the right-angled Artin group of the graph:

$\Gamma =$



Let $K = K_\Gamma$ be the flag complex of the graph Γ, given by adding 2-cells $1,2,4$ and $2,3,4$ to Γ. By Corollary 4.1, $H_q(G_\Gamma^{ab},\mathbf{k}) = 0$ except for $q = 0,1$. The dimension of H_1 over $\mathbf{k}[\mathbb{Z}^n]$ is 2, since this is the size of the largest disconnected subgraph of Γ.

Here, $J_K = (e_1e_3)$. K is acyclic, so H_Γ is FP. The ith Betti number is the coefficient of t^i in the Hilbert series of $J_{K^\star}\cap(a)$, which is $1+3t+2t^2$.

Now choose coordinate maps $f_i\colon [4] \to [2]$ for $i = 1,2$ by letting $f_1(1) = f_1(2) = 1$, $f_1(3) = f_1(4) = 2$, and $f_2(1) = f_2(3) = 1$, $f_2(2) = f_2(4) = 2$. Let N_1, N_2 denote the corresponding subgroups (§2.3). By direct calculation, one can verify that $\{e_1+e_2, e_3+e_4\}$ is a regular sequence for $\mathbf{k}\langle K\rangle$, for any field $\mathbf{k}$, so $H_\bullet(N_1,\mathbf{k})$ is finite-dimensional, with Betti numbers given by $1+2t$.

On the other hand, e_1+e_3 is not a regular element, using (14), so $\{e_1+e_3, e_2+e_4\}$ is not a regular sequence, and the subgroup N_2 is not FP_2. By (18), there can be no regular sequence of length 3.

Remark 2. Theorem 4 is formally similar to the Theorem of Danilov-Jurkiewicz (see [8]) on the cohomology of projective toric varieties. It would be interesting to know if the moment-angle complex point of view could suggest any useful analogies.

5. DUALITY

We single out the case of Cohen-Macaulay complexes K for their particularly nice properties. First, recall the Theorem of Eagon and Reiner in [10]. For this, let I_K denote the defining ideal of the Stanley-Reisner ring of a complex K, and write $\mathbf{k}[K] = S/I_K$.

Theorem 5 ([10])**.** *The ideal $I_{K^\star}$ has a linear, free resolution over S if and only if the complex K is Cohen-Macaulay.*

If K is a Cohen-Macaulay simplicial complex of dimension d, consider the Cartan complex: this is the complex $(\mathbf{k}\langle K\rangle \otimes S, \omega)$, whose differential is given by multiplication by the element $\omega = \sum_{i=1}^{n} e_i \otimes u_i$. Let $F_K = H^{d+1}(\mathbf{k}\langle K\rangle^\bullet \otimes S, \omega)$.

The next Lemma is a basic consequence of Bernstein-Gelfand-Gelfand duality for the module F_K. We refer to [11, 12, 13] for background.

Lemma 5.1. *The following are equivalent.*

1. *K is Cohen-Macaulay of dimension d.*
2. *$H^p(\mathbf{k}\langle K\rangle \otimes S, \omega) = 0$ for $p \neq d+1$.*
3. *$\operatorname{Tor}^S(F_K, \mathbf{k}) \cong \mathbf{k}\langle K\rangle^*(n-d+1) \cong J_{K^\star}(n-d+1)$ as graded E-modules.*

An interesting special case is that of K a homology sphere (Gorenstein* complex). From [10], it follows that:

Lemma 5.2. *$F_K \cong I_{K^\star}$ as S-modules.*

Then BGG duality gives the following reformulation of Theorem 3.

Theorem 6. *For any Cohen-Macaulay complex K of dimension d and coordinate map $f\colon [n] \to [m]$, there is an isomorphism of S-modules for $q \geq 0$:*

$$H_q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) \cong \operatorname{Ext}_S^{d+1-q}(F_K, R).$$

Dually,

$$H_c^q(\mathcal{Z}_K(S^1), \mathbf{k}[\mathbb{Z}^m]) \cong \operatorname{Tor}^S_{d+1-q}(F_K, R). \tag{20}$$

Proof. By Lemma 5.1(2), the Cartan complex is a free resolution of F_K over S. By applying $\operatorname{Hom}_S(-, R)$, we obtain the cellular chain complex for $\mathcal{Z}_K(S^1) \times_{\mathbb{Z}^n} \mathbb{Z}^m$ of Proposition 3.1(7), and the result follows.

The dual statement is proven in the same way. □

Corollary 5.3. *Suppose K is the clique complex of a graph Γ and K is Cohen-Macaulay of dimension d. For any coordinate map $f\colon [n] \to [m]$, we have*

$$H^q(G_\Gamma, \mathbf{k}[\mathbb{Z}^m]) = 0 \quad \text{for } q \leq m-n+d \text{ and } q > d+1.$$

Proof. Since $\operatorname{pd}_S R = n-m$, $\operatorname{Tor}_i^S(F_K, R) = 0$ for $i > n-m$. Now apply (20). □

Example 1 (continued)**.** Suppose K is the clique complex of Γ, and K is Cohen-Macaulay of dimension d. Then the compactly-supported cohomology of the universal abelian cover is concentrated in dimension $d+1$; more precisely, as modules over $\mathbf{k}[G_\Gamma^{ab}]$,

$$H^q(G_\Gamma, \mathbf{k}[G_\Gamma^{ab}]) \cong \begin{cases} \mathfrak{m}^{-1}F_K & \text{For } q = d+1; \\ 0 & \text{otherwise.} \end{cases}$$

It should be noted that the vanishing of cohomology here and in the Corollary are closely related to results of Brady and Meier [3]: they show that G_Γ is a duality group if and only if K is a Cohen-Macaulay complex. However, there does not seem to be an easy derivation of these results from theirs.

Example 2 (continued)**.** Under the same hypotheses, the homology of the Bestvina-Brady group is given for $q \geq 0$ by

$$H_q(H_\Gamma, \mathbf{k}) \cong \operatorname{Ext}_S^{d+1-q}(F_K, \mathbf{k}[s]).$$

Acknowledgement. The author would like to thank MSRI for its hospitality and support during this project in Fall 2006.

References

1. Annetta Aramova, Luchezar L. Avramov, and Jürgen Herzog, *Resolutions of monomial ideals and cohomology over exterior algebras*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 579–594. MR MR1603874 (2000c:13021)
2. Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470. MR MR1465330 (98i:20039)
3. Noel Brady and John Meier, *Connectivity at infinity for right angled Artin groups*, Trans. Amer. Math. Soc. **353** (2001), no. 1, 117–132. MR MR1675166 (2001b:20068)
4. Victor M. Buchstaber and Taras E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, vol. 24, American Mathematical Society, Providence, RI, 2002. MR MR1897064 (2003e:57039)
5. Kai-Uwe Bux and Carlos Gonzalez, *The Bestvina-Brady construction revisited: geometric computation of Σ-invariants for right-angled Artin groups*, J. London Math. Soc. (2) **60** (1999), no. 3, 793–801. MR MR1753814 (2001f:20083)
6. Ruth Charney, *An introduction to right-angled Artin groups*, Geom. Dedicata, to appear.
7. Ruth Charney and Michael W. Davis, *Finite $K(\pi, 1)$s for Artin groups*, Prospects in topology (Princeton, NJ, 1994), Ann. of Math. Stud., vol. 138, Princeton Univ. Press, Princeton, NJ, 1995, pp. 110–124. MR MR1368655 (97a:57001)
8. V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk **33** (1978), no. 2(200), 85–134, 247. MR MR495499 (80g:14001)
9. Graham Denham and Alexandru Suciu, *Moment-angle complexes, monomial ideals, and Massey products*, Pure and Applied Math. Quarterly, (Special volume in honor of Robert MacPherson), **3** (2007), no. 1, pp. 25–60.
10. John A. Eagon and Victor Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, J. Pure Appl. Algebra **130** (1998), no. 3, 265–275. MR MR1633767 (99h:13017)
11. David Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, New York, 2005, A second course in commutative algebra and algebraic geometry. MR MR2103875 (2005h:13021)
12. David Eisenbud, Gunnar Fløystad, and Frank-Olaf Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4397–4426 (electronic). MR MR1990756 (2004f:14031)
13. Jürgen Herzog, *Koszul algebras and modules*, Advances in algebra and geometry (Hyderabad, 2001), Hindustan Book Agency, New Delhi, 2003, pp. 25–37. MR MR1986140 (2004f:13017)
14. C. Jensen and J. Meier, *The cohomology of right-angled Artin groups with group ring coefficients*, Bull. London Math. Soc. **37** (2005), no. 5, 711–718. MR MR2164833 (2006m:20054)
15. K. H. Kim and F. W. Roush, *Homology of certain algebras defined by graphs*, J. Pure Appl. Algebra **17** (1980), no. 2, 179–186. MR MR567067 (82e:05114b)
16. Ian Leary and Müge Saadetoğlu, *The cohomology of Bestvina-Brady groups*, preprint, 2006.
17. John Meier, Holger Meinert, and Leonard VanWyk, *Higher generation subgroup sets and the Σ-invariants of graph groups*, Comment. Math. Helv. **73** (1998), no. 1, 22–44. MR MR1610579 (99f:57002)
18. ———, *On the Σ-invariants of Artin groups*, Topology Appl. **110** (2001), no. 1, 71–81, Geometric topology and geometric group theory (Milwaukee, WI, 1997). MR MR1804699 (2001j:20058)

19. Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR MR2110098 (2006d:13001)
20. Stefan Papadima, personal communication.
21. Stefan Papadima and Alexander I. Suciu, *Algebraic invariants for Bestvina-Brady groups*, Journal of the London Math. Society, to appear, math.GR/0603240.
22. ———, *Algebraic invariants for right-angled Artin groups*, Math. Ann. **334** (2006), no. 3, 533–555. MR MR2207874 (2006k:20078)
23. Tim Römer, *Generalized Alexander duality and applications*, Osaka J. Math. **38** (2001), no. 2, 469–485. MR MR1833633 (2002c:13029)
24. Gunnar Sjödin, *A set of generators for* $\mathrm{Ext}_R(k,k)$, Math. Scand. **38** (1976), no. 2, 199–210. MR MR0422248 (54 #10239)
25. John Stallings, *A finitely presented group whose 3-dimensional integral homology is not finitely generated*, Amer. J. Math. **85** (1963), 541–543. MR MR0158917 (28 #2139)
26. Neil Strickland, *Notes on toric spaces*, available at `http://www.shef.ac.uk/personal/n/nps/papers/`, 1999.

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7
URL: `http://www.math.uwo.ca/~gdenham`