the topology of reconfiguration

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collaborators: a. abrams [emory]; v. peterson [uiuc], s. lavalle [uiuc]

re:configuration spaces

there are numerous settings in which coordination is handled via configuration spaces

usually `smooth’

- points in 2-d or 3-d
- mechanical linkages
- vehicle dynamics

however, there are settings for which `discrete’ spaces are more natural…reconfiguration

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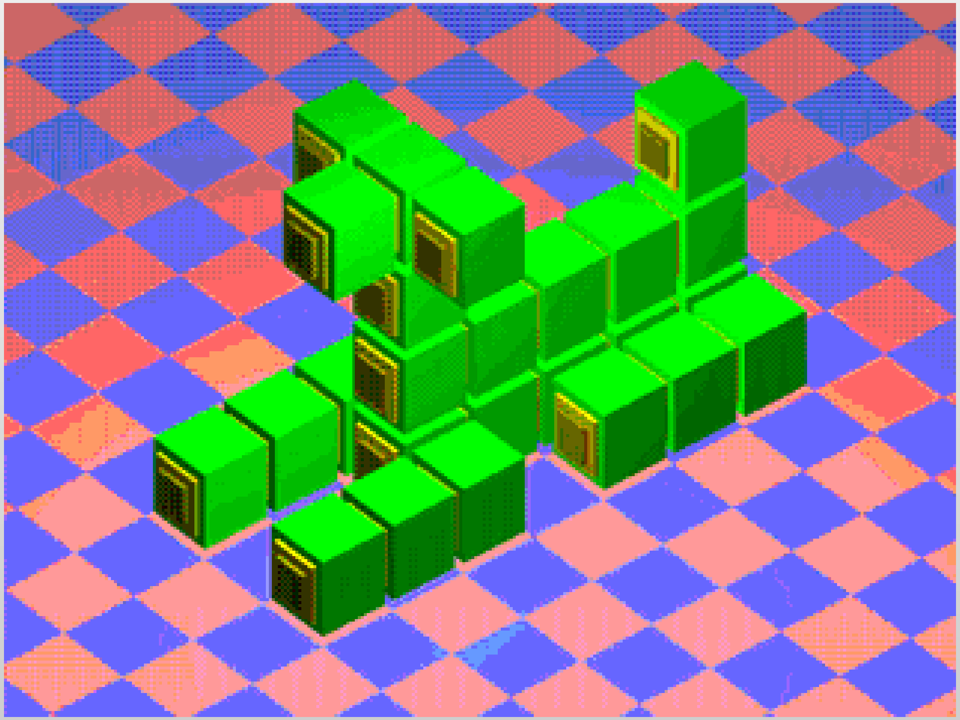
motivation: metamorphic robots

Chirikjian, JHU
Rus & Vona, MIT
Yim, PARC



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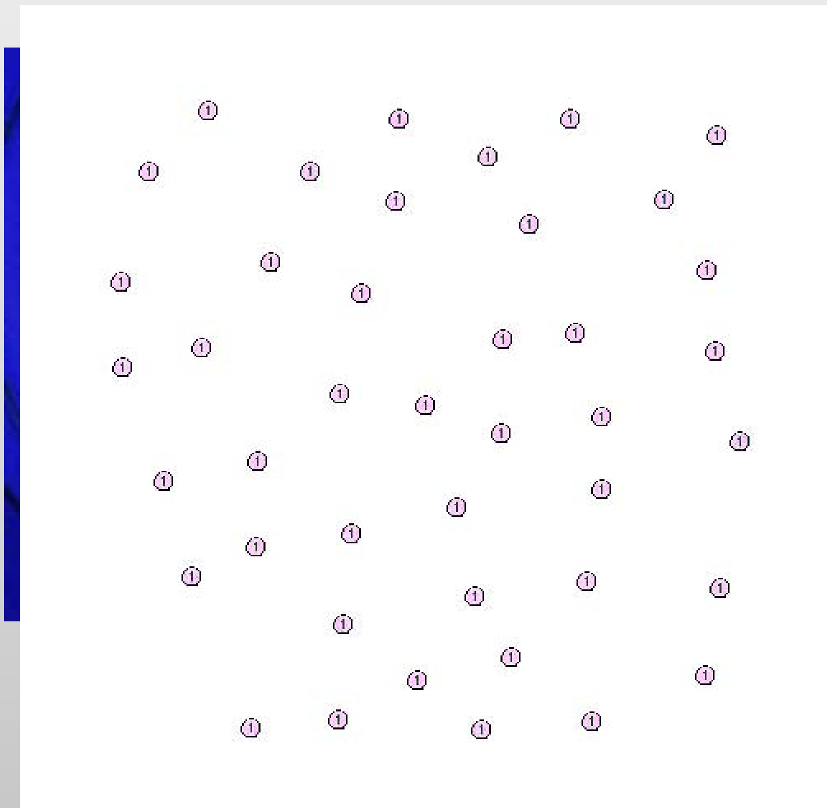
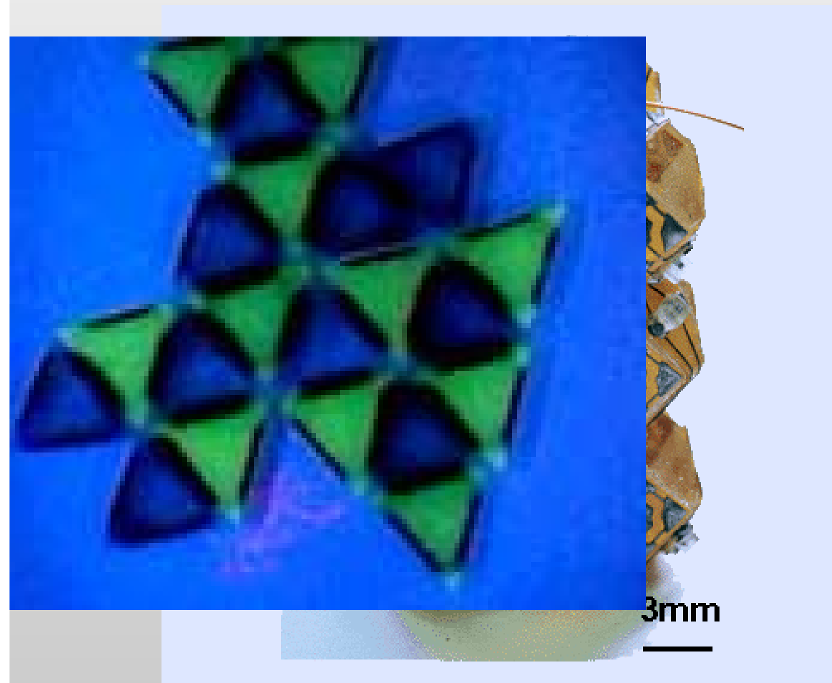
motivation: self-assembly

through passive and active means



E. Klavins, U. Wash.



G. Whitesides, Harvard

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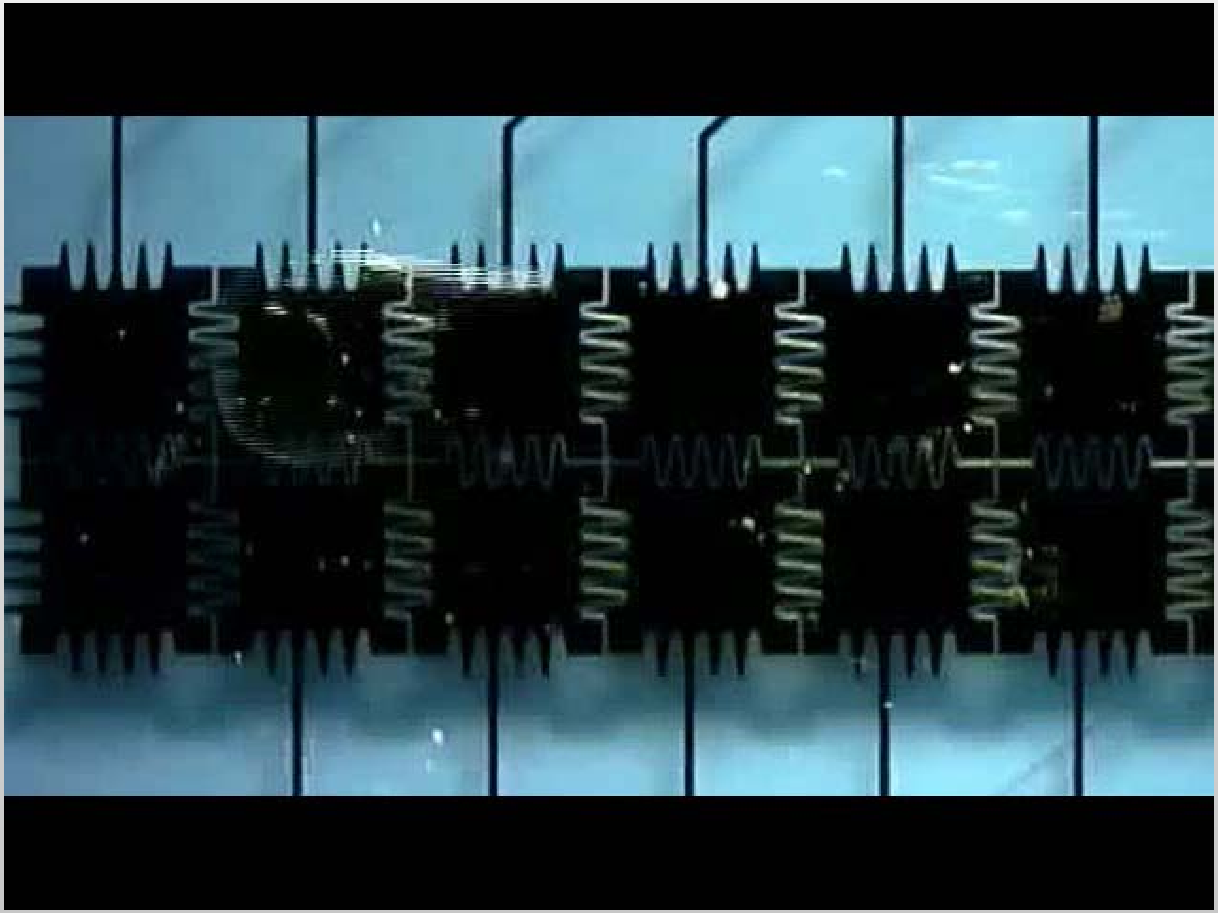
motivation: digital microfluidics

droplet can be manipulated on a grid via electrowetting



duke univ.

(r. fair)

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motivation: group theory

group theory has been using discrete configuration spaces to study groups for well over 100 years…

cayley graphs

↓

more sophisticated geometric objects

this is our perspective…

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Definitions and examples…

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reconfigurable systems

domain = graph G (lattice)

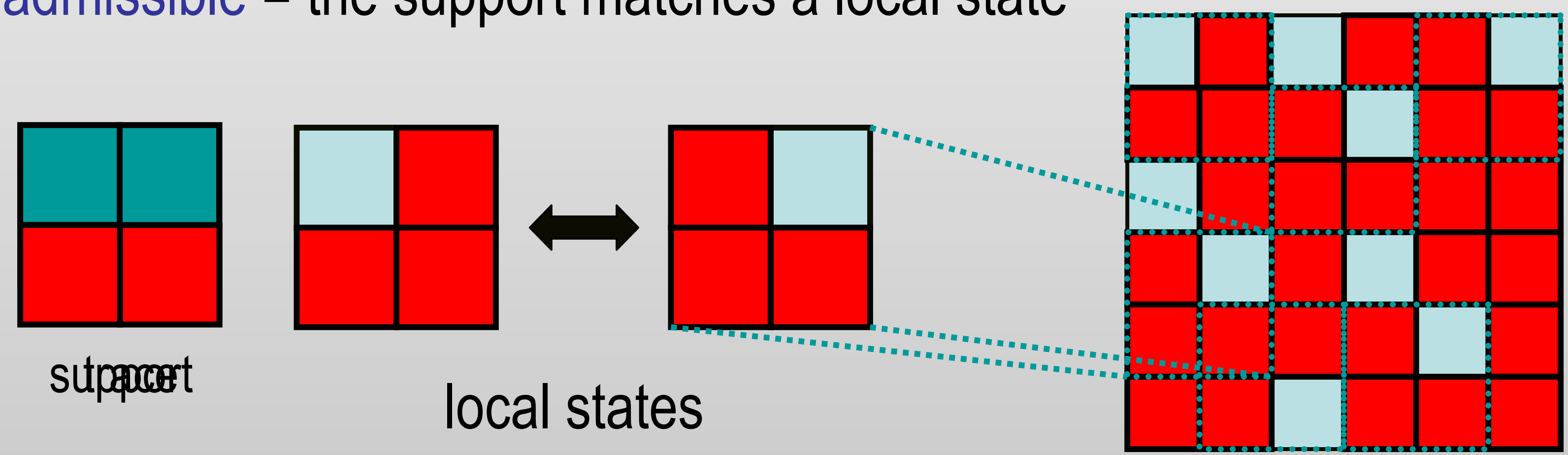
states = labelings of vertices of G by an alphabet $\mathcal{A}$ (Z_2)

generator = (support, trace; two unordered local states on support)

support = subgraph of G;

trace = subgraph of support (where things move)

admissible = the support matches a local state



a reconfigurable system is a set of states closed under generators

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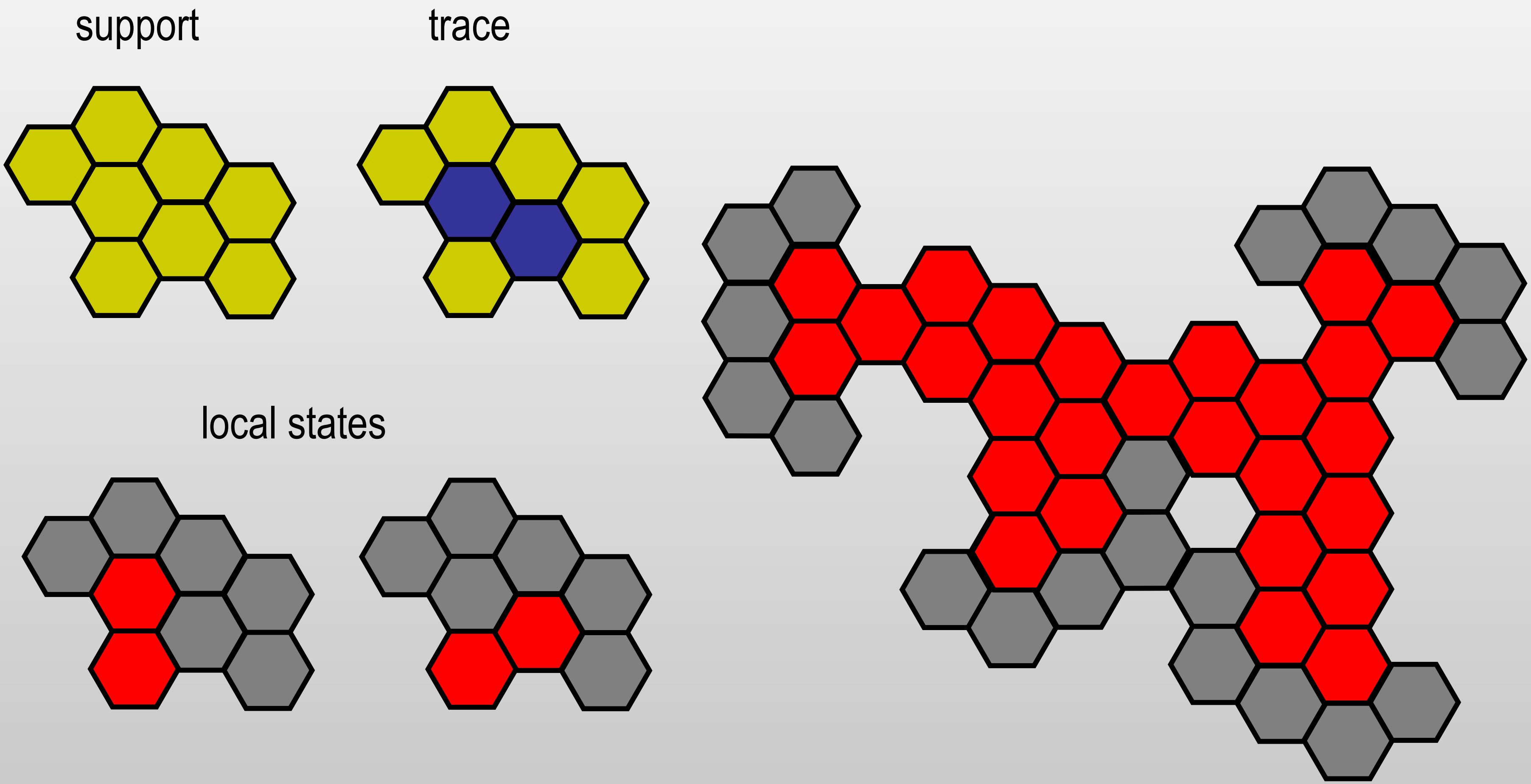
example: 2-d hexagonal

[Chirikjian et al.]



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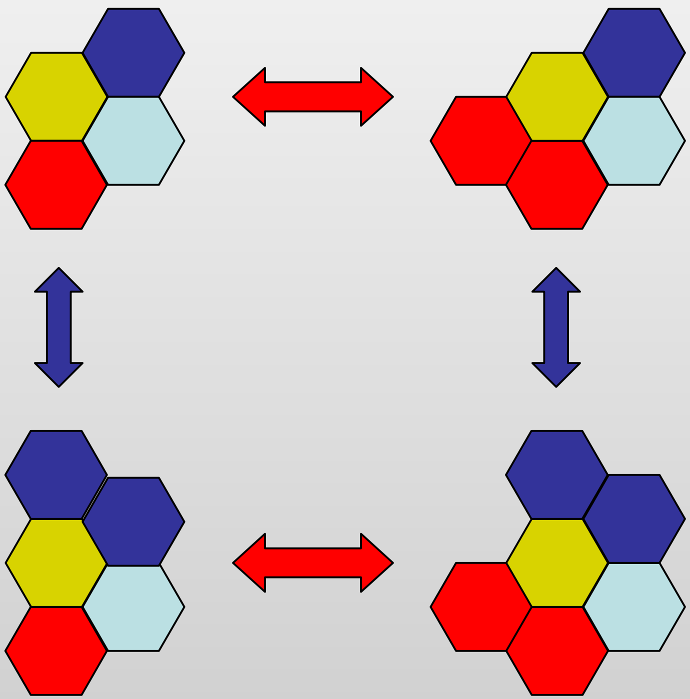
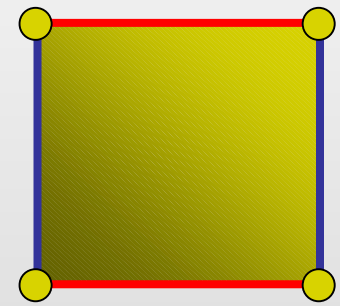
the state complex

idea: transition graph ⇔ 1-skeleton of a cubical complex





generators $\{ \phi_i \}_{i=1..K}$ commute

⇕

$SUP(\phi_m) \cap TR(\phi_n) = 0 : m \neq n$

each cube of dimension K corresponds to K commutative moves…

undirected version of high-dimensional automata [pratt]
geometric concurrency… [goubault, jardine, raussen, et al.]

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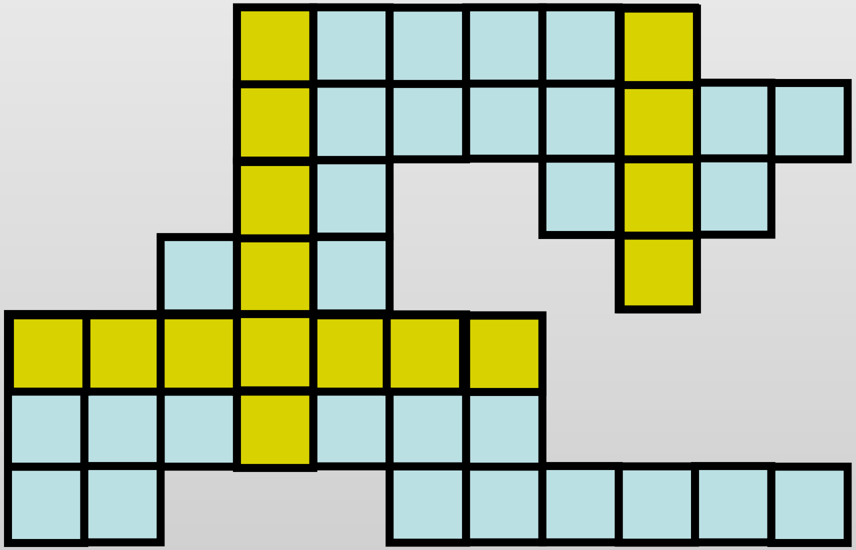
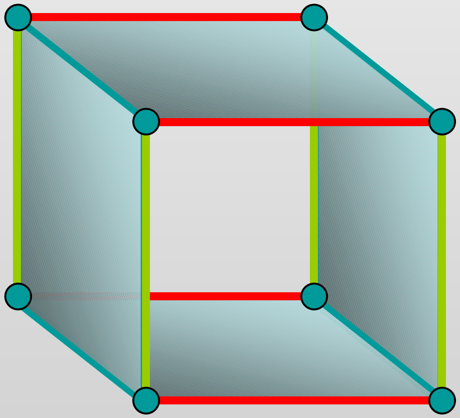
example: planar sliding system

tiles: 2-d, square

local moves: row, column slides





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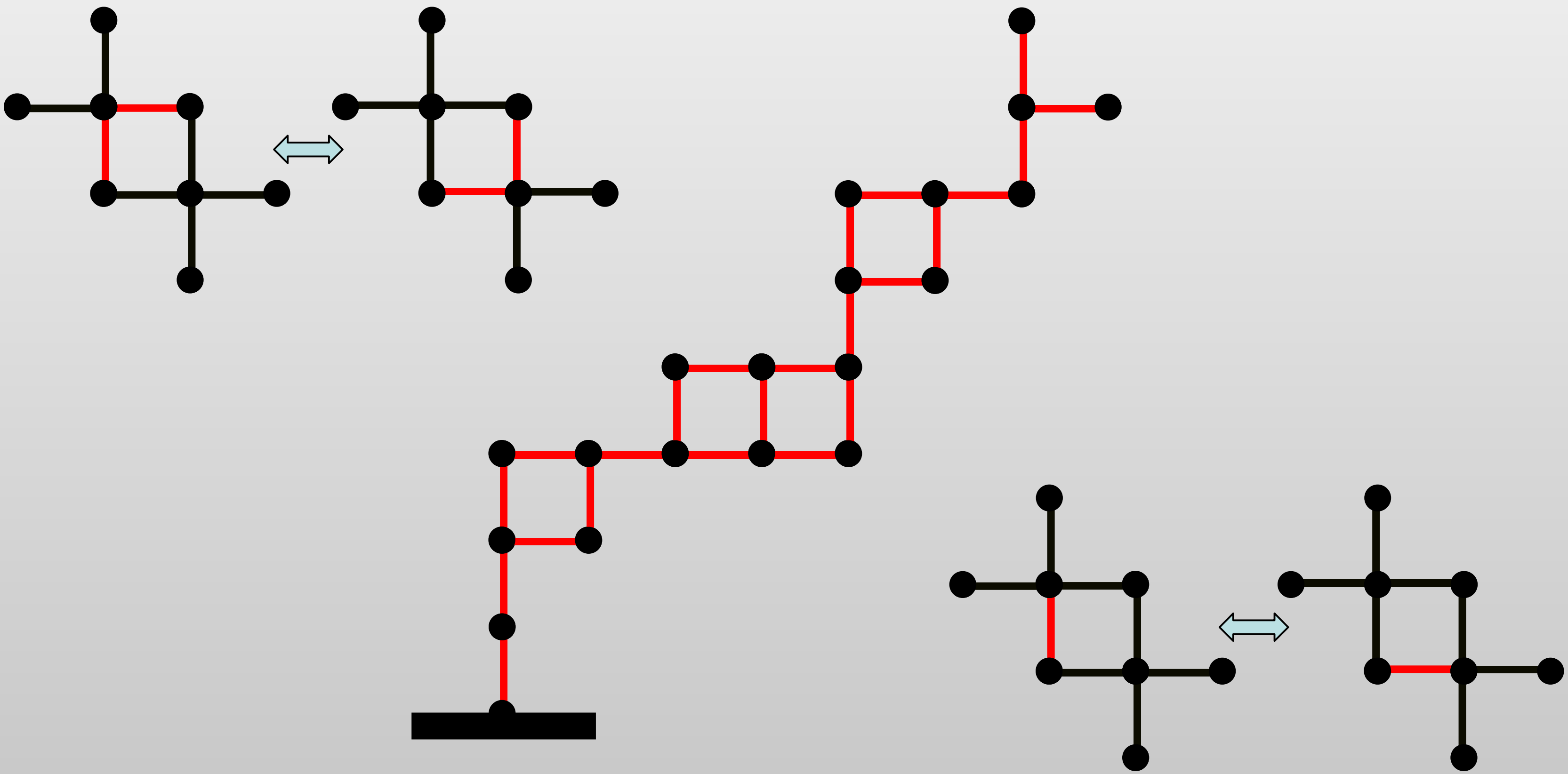
example: articulated robot arm

states = length N chain in planar lattice

moves = rotating end; flipping corners



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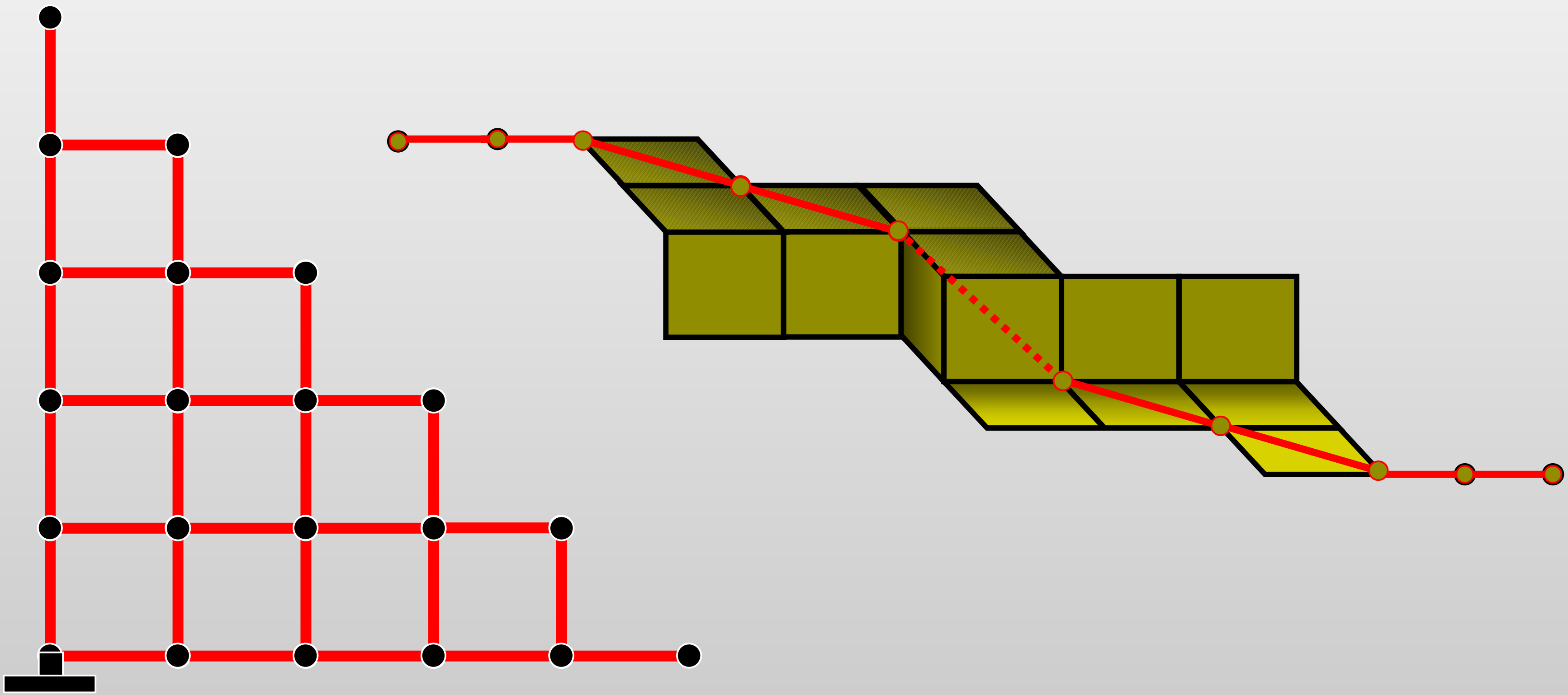
example: articulated robot arm



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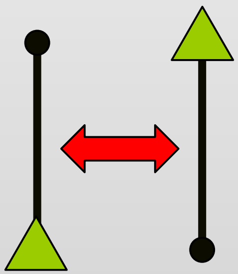
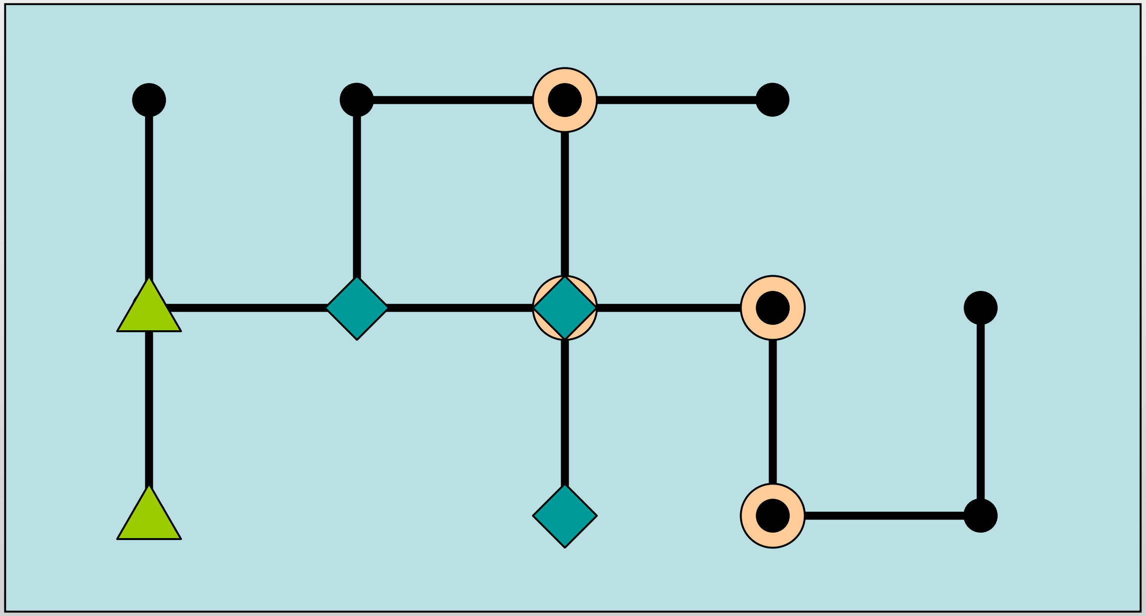
example: points on a graph Γ

states = arrangements of N points on vertices

moves = move a point along a free edge





this models microfluidic arrays, as well as configuration spaces of mobile robots on tracks

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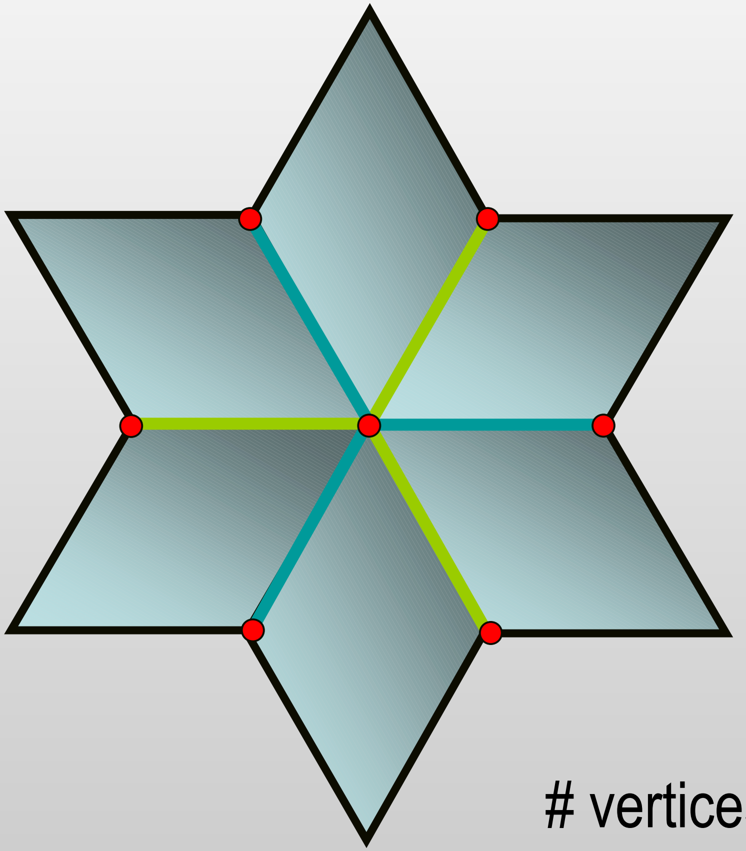
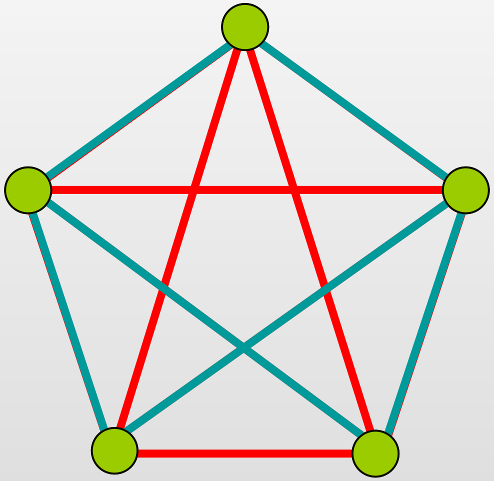
configuration spaces of graphs

[A. Abrams, 2000]





1. determine the local structure
2. euler characteristic computation

vertices = (5)(4) = 20

local euler characteristic = 1 – 3 + 3/2 = –1/2

euler characteristic = –10 ⇔ genus = 6

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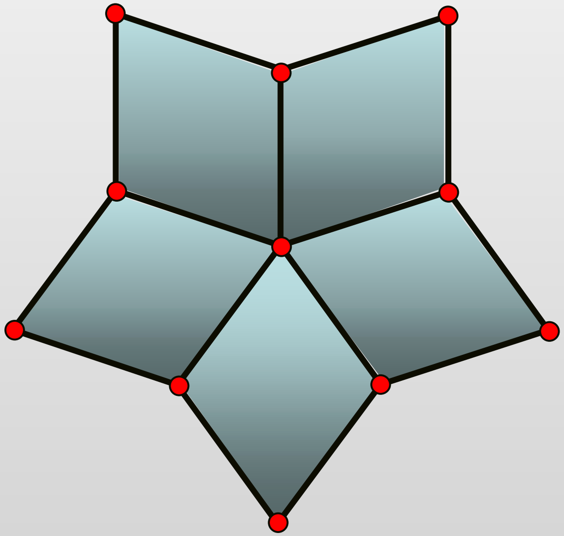
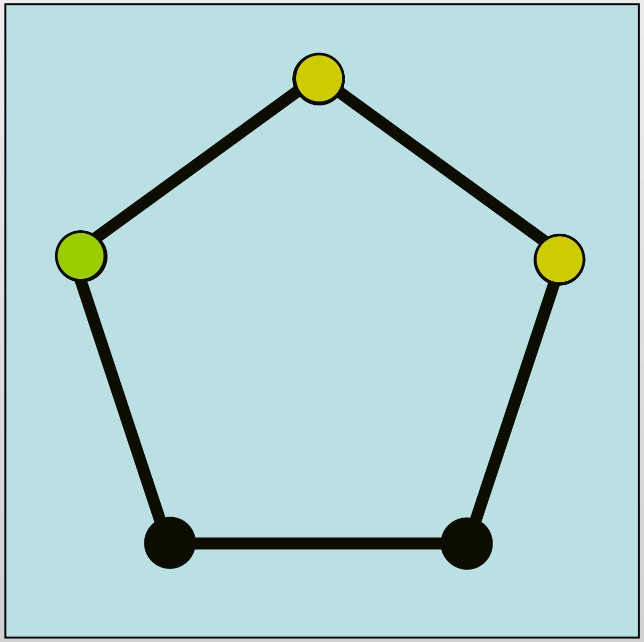
labeled graphs and permutohedra

states = labeled vertices of a graph

generators = exchange distinct labels on neighboring vertices





5-gon with 5 labels: genus 16

filled-in cayley graph of S_5 via transpositions

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notice…

all of these surfaces we construct as state complexes are of genus > 1

can you construct a sphere?

can you construct interesting three-manifolds?

surprisingly, the answer to these topological questions depends almost completely on the geometry of these objects…

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and now for some geometry…

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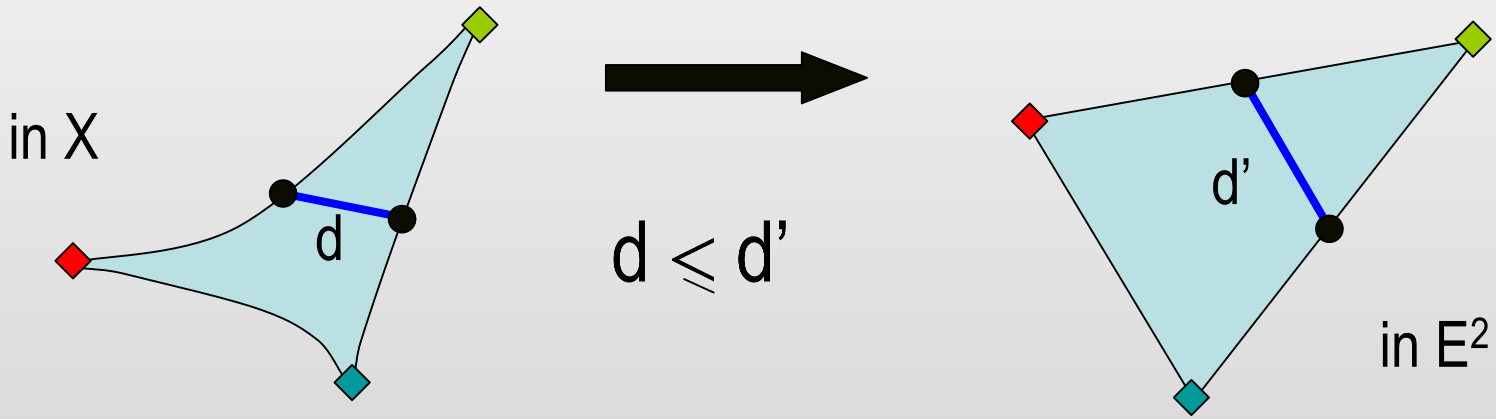
cat(0) geometry:

let X be a metric space with local geodesics…



X is cat(0) ⇔ all triangles are no fatter than in E^2

X is nonpositively curved [npc] iff X is locally cat(0)

(cat(0) = npc + simply connected)

this approach to geometry has a rich history…

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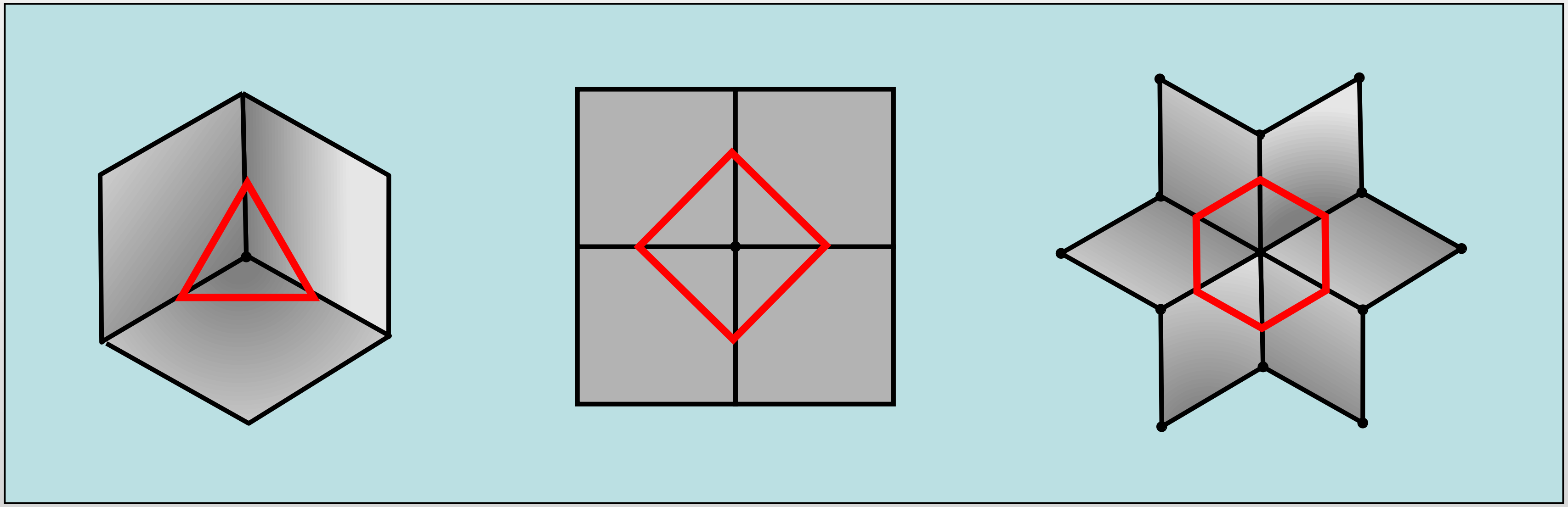
gromov's link condition:

link: simplicial complex of incident cells…



theorem:

cube complex is npc ⇔ link of each vertex is a flag complex

if the edges look like a k-simplex, there really is a k-simplex spanning them…

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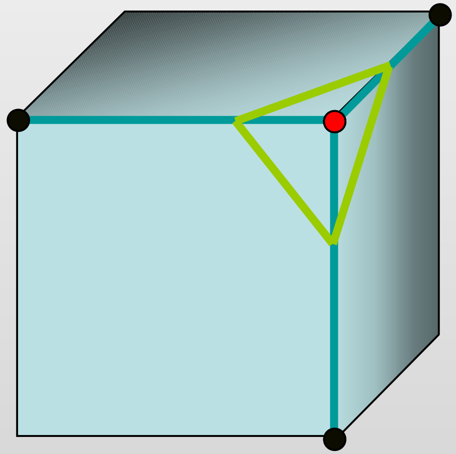
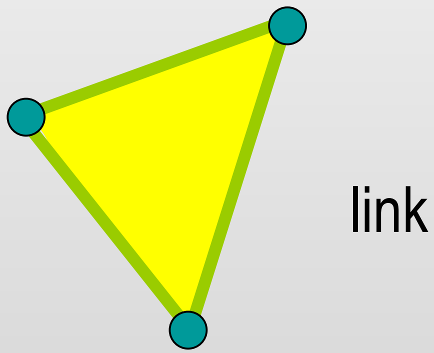
gromov's link condition:

link: simplicial complex of incident cells…





theorem [gromov]:

cube complex is npc ⇔ link of each vertex is a flag complex

if the edges look like a k-simplex, there really is a k-simplex spanning them…

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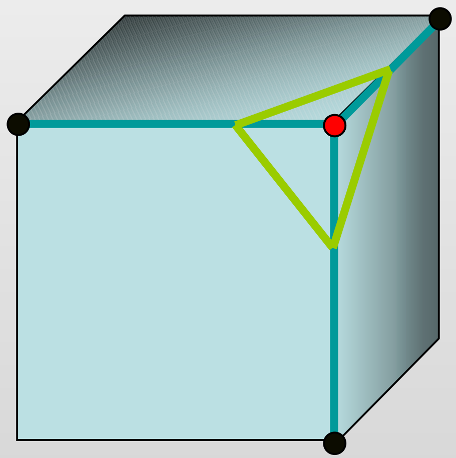
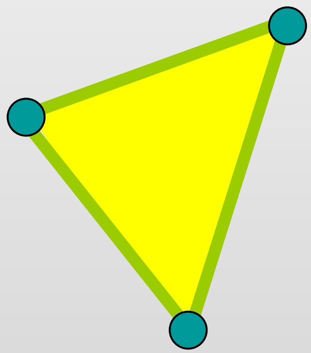
theorem:

[A+G] all state complexes are nonpositively curved.





proof: simple application of the link condition

commutativity as defined is pairwise determined…

corollary: all higher homotopy groups vanish;

π_1 is torsion-free

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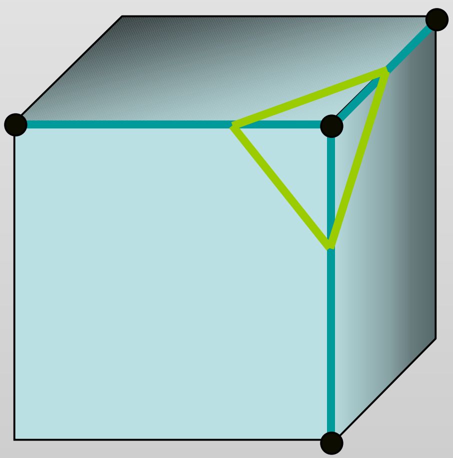
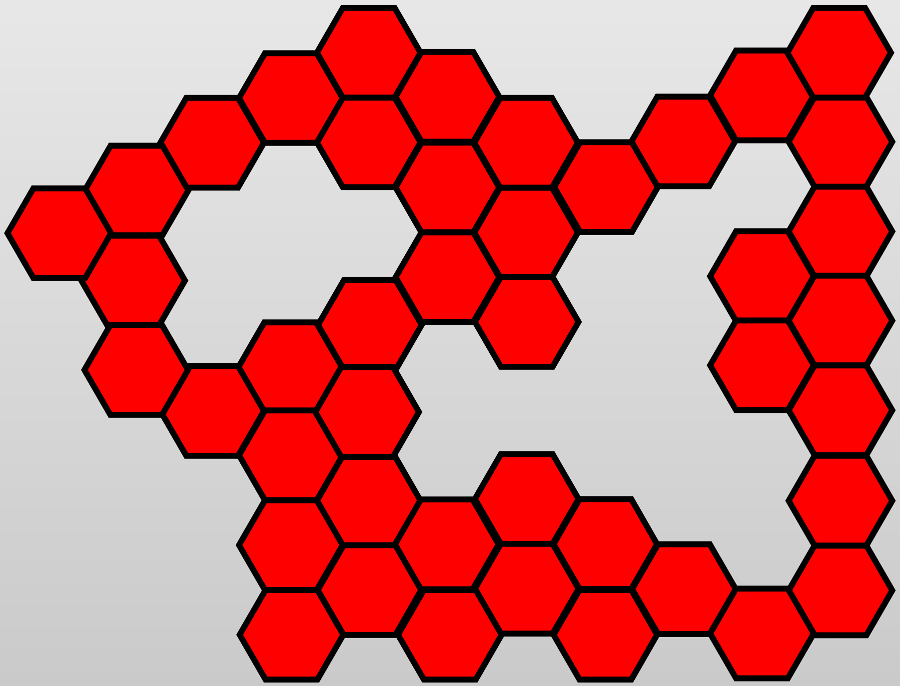
where npc can fail…

note: you really need local rules…





the realization problem:

now we know what to look for: what is possible?

proposition: let L be any simplicial complex which is flag. Then there exists a reconfigurable system whose state complex has L as the link of each and every vertex.

proof: G = L ; one generator for each vertex v in L
support = L-boundary(star(v))
trace = v
change the label on v from 0 ⇔ 1

show that two `vertex generators' commute iff they share an edge

(cf. results of M. Davis in geometric group theory…)

this yields closed `hyperbolic' manifolds of all dimensions

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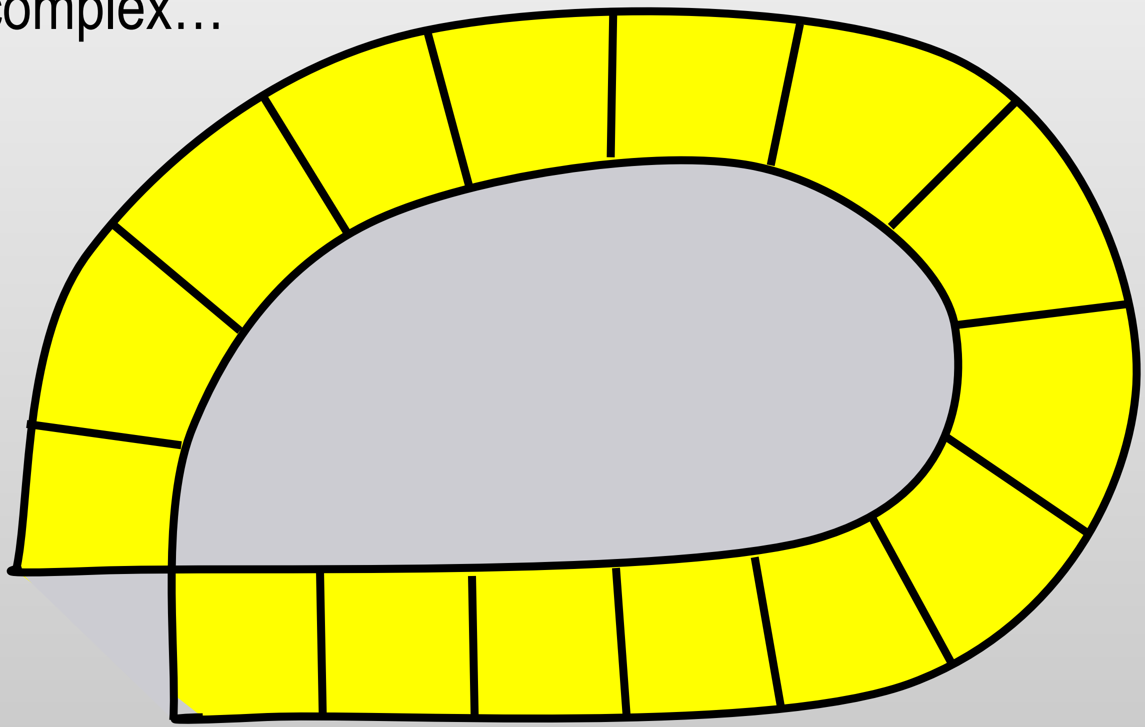
the realization problem:

so…is anything <u>not</u> possible?

yes: the following npc space cannot arise as a subcomplex of a state complex…



conjecture: this is the only obstruction.

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toward a complete characterization…

theorem: [AGP] any npc subcomplex of a product of graphs can be realized as a state complex.

this has a partial converse:

theorem: [AGP] any simply connected state complex is a subcomplex of some cube (of sufficiently high dimension)

the best completion of which would be the following:

conjecture: any state complex can be realized as a subcomplex of a product of graphs…

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well ok then…

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but what is this good for?

theorem: [AGP] the space of all time-minimal paths between two points in a state complex has the homotopy type of a finite set of points.

proof:

fact: geodesics are unique on a cat(0) space

fact: time-minimal paths ⇔ L^∞ geodesics

deform the set of L^∞ geodesics to the unique geodesic

for an npc space, do this in each homotopy class

compactness argument: finite number of classes

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but what is this good for?

this is very useful for optimization problems

stratagem:

given some state trajectory from a distributed, ad hoc, or probabilistic path-planner…

1. perform curve shortening on state complex
2. this must converge to the global minimum obtainable from the initial path…

npc => no local minima at which to get stuck…

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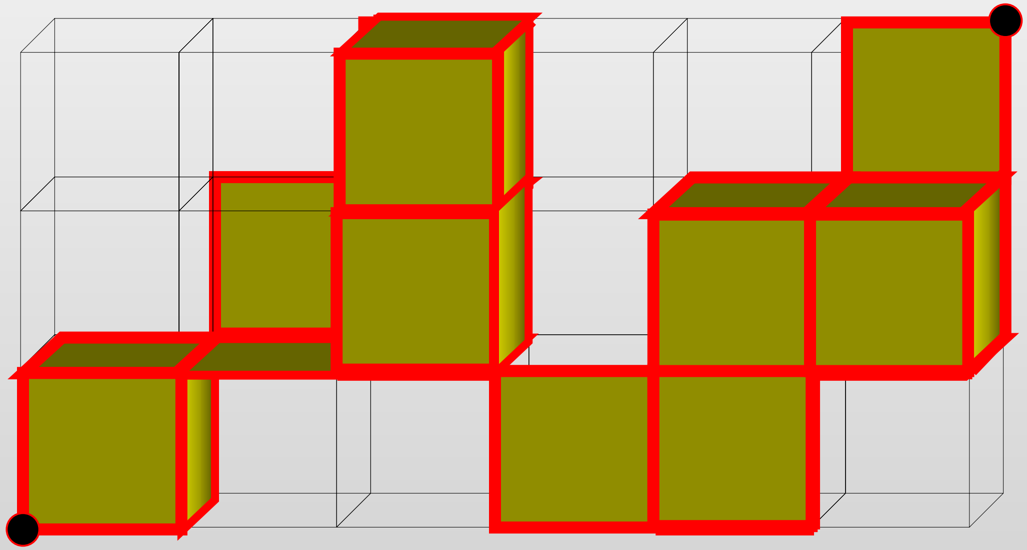
curve-shortening algorithm

given: edge-path in S…
march along checking commutativity…
multiple sweeps through path reduce length…



theorem [AG]: converges to the global minimal-time state trajectory obtainable from initial path.

you do not need to construct the entire state complex

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optimal scheduling

let $\Gamma=(\Gamma_1,\ldots,\Gamma_N)$ denote N graphs, denoting either :
roadmaps for N different robots in a C-space;
N different processes with shared resources

coordination space:

$$C(\Gamma) = \Gamma_1 \times \ldots \times \Gamma_N - O$$

O = obstacle set (open) where, e.g.,
robots collide or processes interfere…

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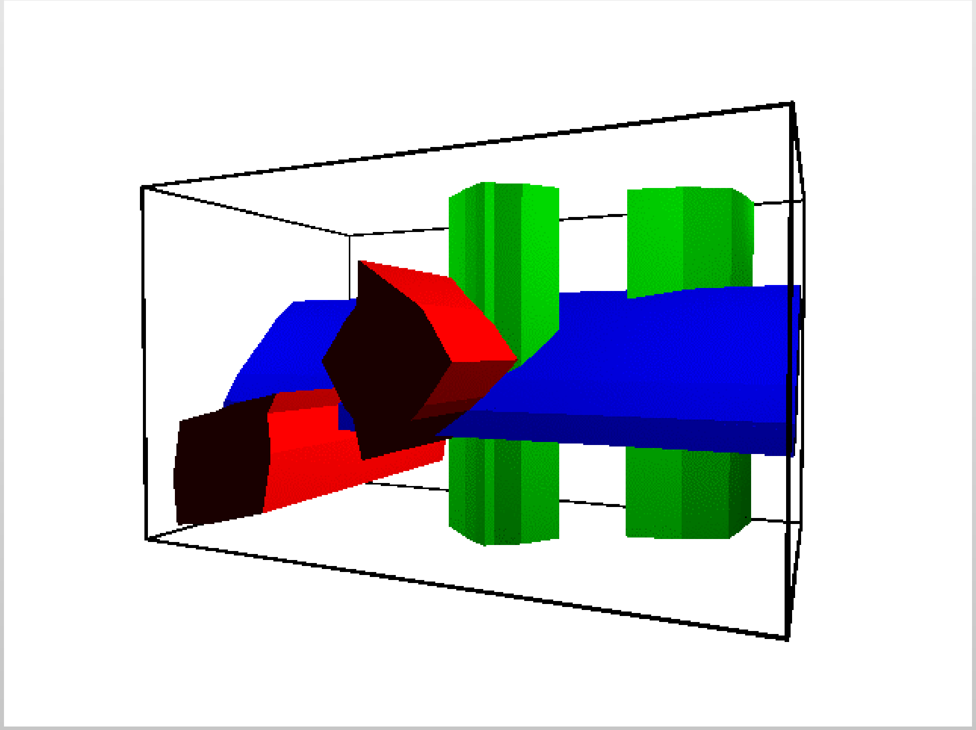
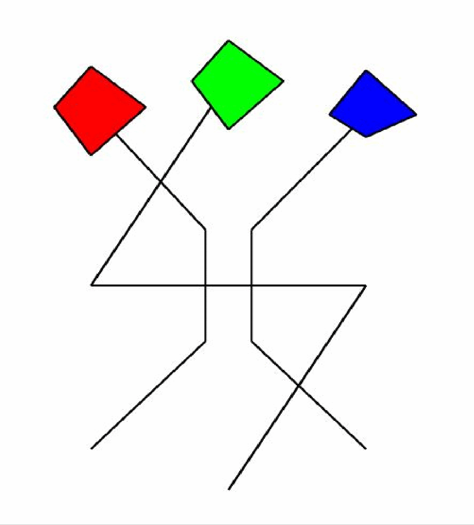
optimal scheduling

for obstacle sets O defined by collisions,
the coordination space is cylindrical…





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pareto optimality

goal: optimal coordination/scheduling

but: each robot/process has its own cost function (e.g., elapsed time)

---

we –could- use

average cost

maximal cost

nonlinear weighted cost?

---

use pareto-optimization

---

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pareto optimality

definition: a path is pareto-optimal iff it is minimal with respect to the partial order on cost vectors.

cost vectors $A=(a_1,\ldots,a_N)$; $B=(b_1,\ldots,b_N)$

$A \leq B \Leftrightarrow a_i \leq b_i$ for all i

incomparable if

$a_i < b_i$ and $a_j > b_j$ for some i, j

equivalent if

$a_i = b_i$ for all i

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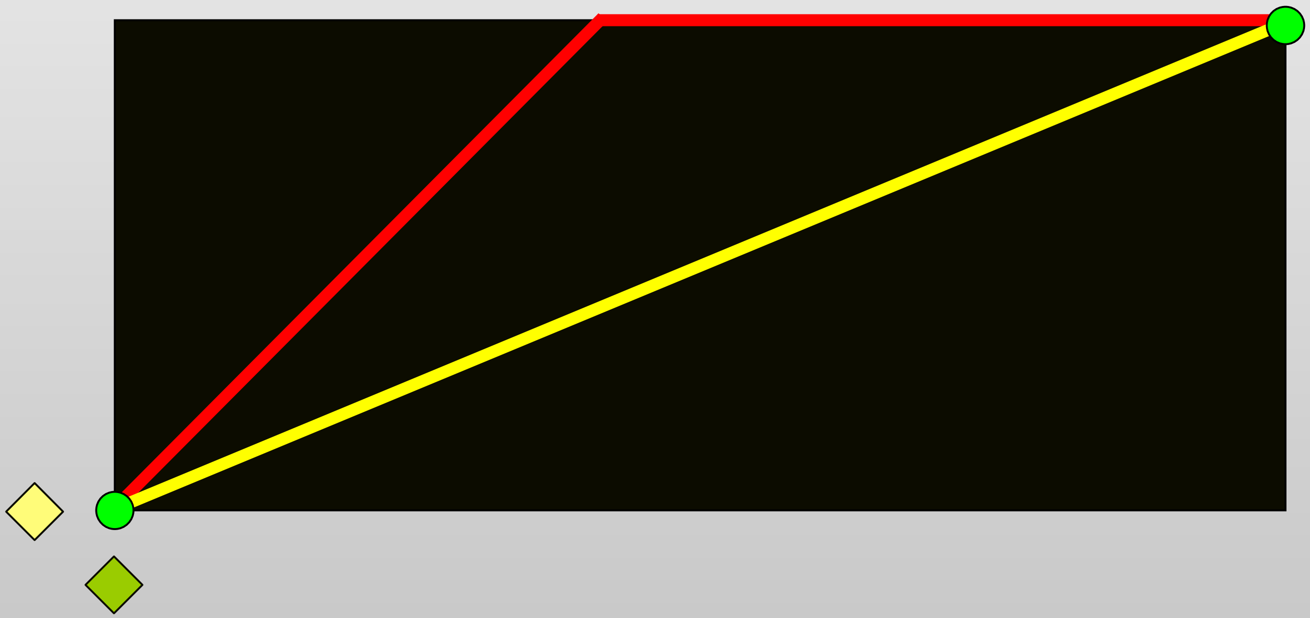
pareto optimality

problem: classify & compute pareto optimal path classes



pareto optimality

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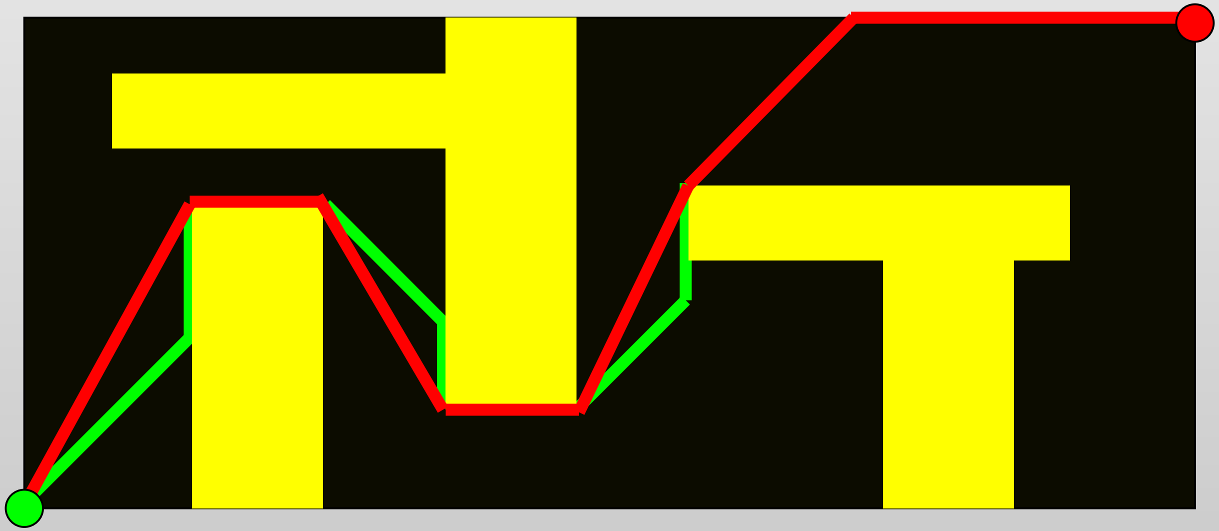

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why pareto?

lemma: any optimum for any monotone scalarization of the cost functions is in fact a pareto optimum.

precomputing the pareto optima is good:

set of all possibilities

changing priorities / costs

hopefully, this is a small set of optima

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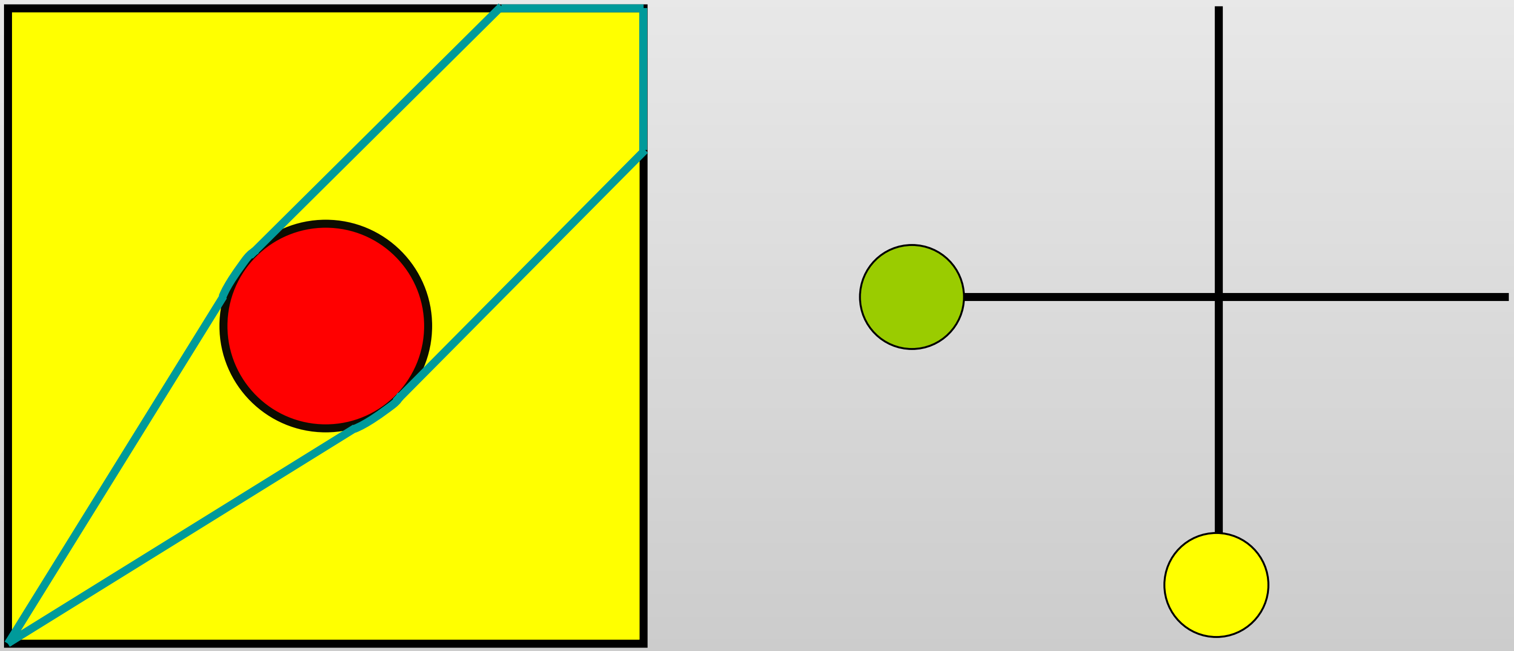
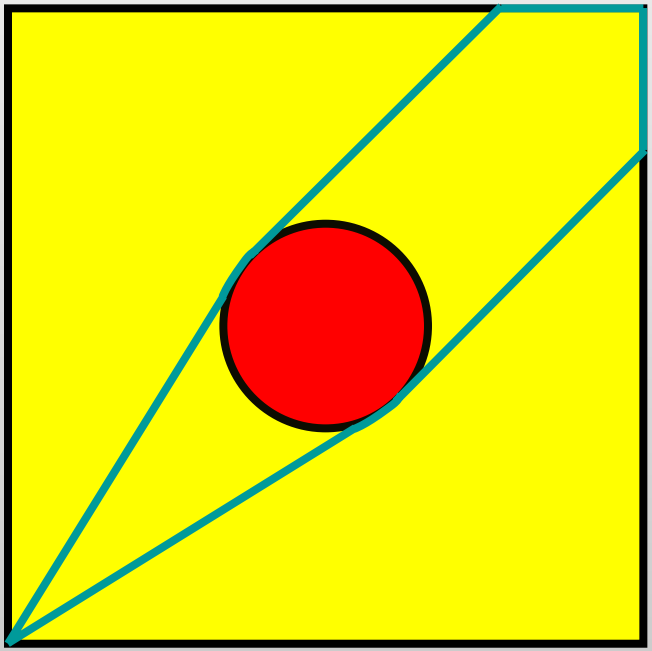
example

cylindrical

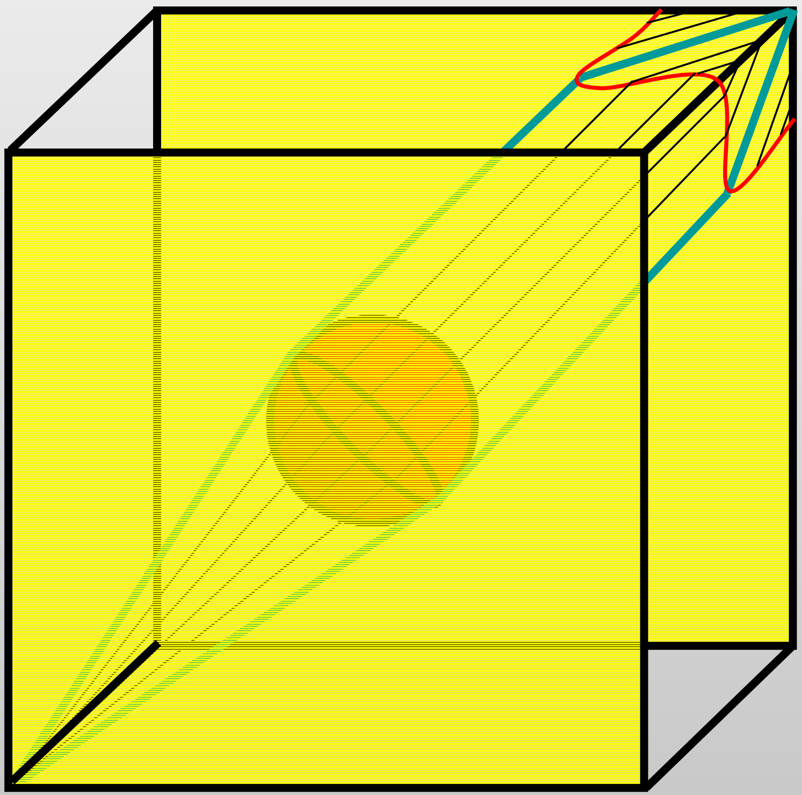


example

cylindrical



noncylindrical



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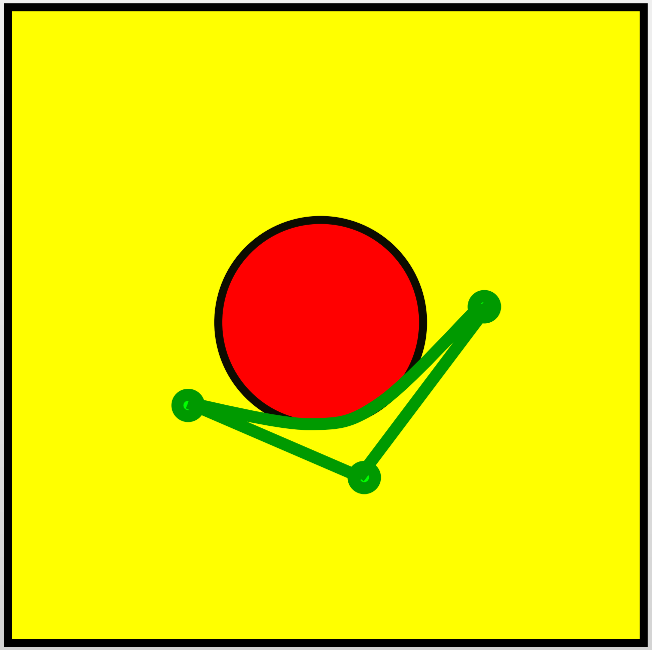
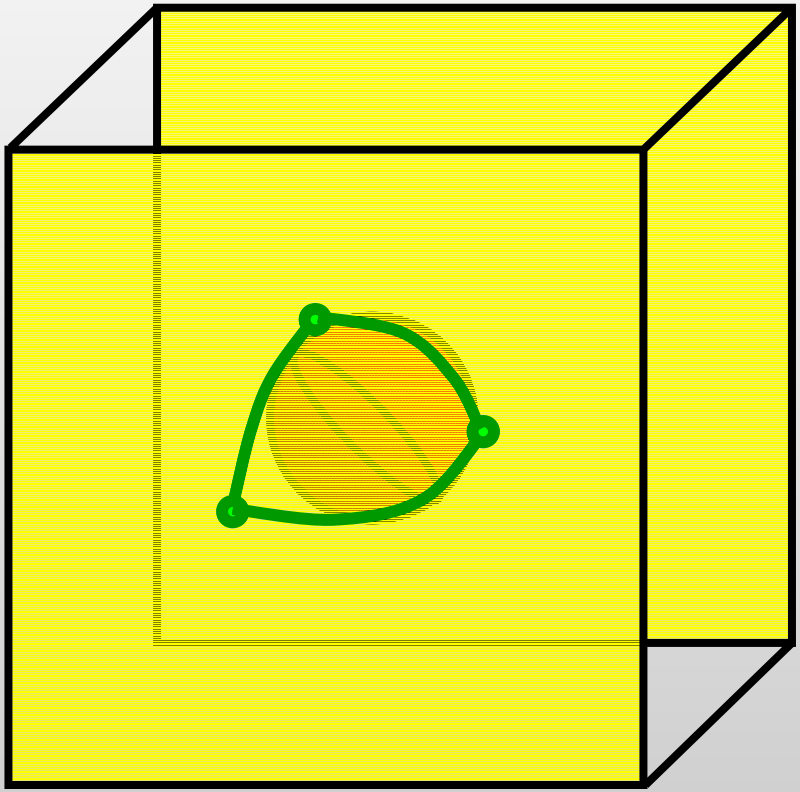
what is the difference?





(non)positive curvature

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pareto optimality

theorem: [GL] for C cylindrical (obstacles defined by pairwise collisions), there is a finite number of pareto optimal classes

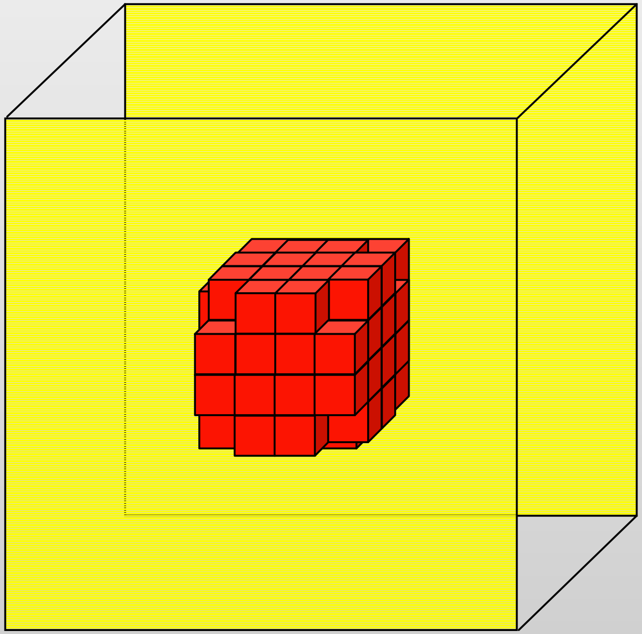
proof:

* given a pareto-optimal path in C
* approximate it by a cube path in $C^{(n)}$
* deform it to a left-greedy cube path (unique on npc space)
* verify that cost functions have not changed
* use compactness to go from discrete to finite

pareto optimality

why doesn't this work in the case
of positive curvature?

...all the steps work
except one



* can approximate by cube paths

* left-greedy cube paths are
pareto optima

* there is no longer a unique left-greedy path

exponential blow-up in #paths

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open questions…

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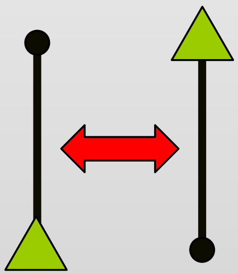
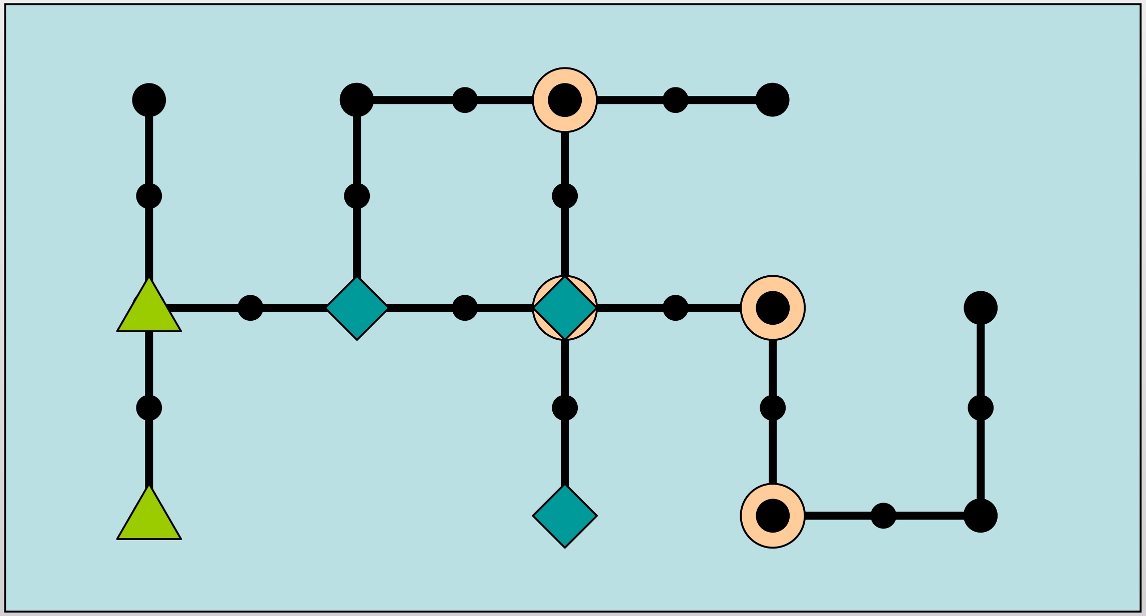
open question 1: convergence

what is the `limit' of a reconfigurable system under refinement?





theorem: [a. abrams] refinements converge in homotopy type to the (`smooth') configuration space of the graph

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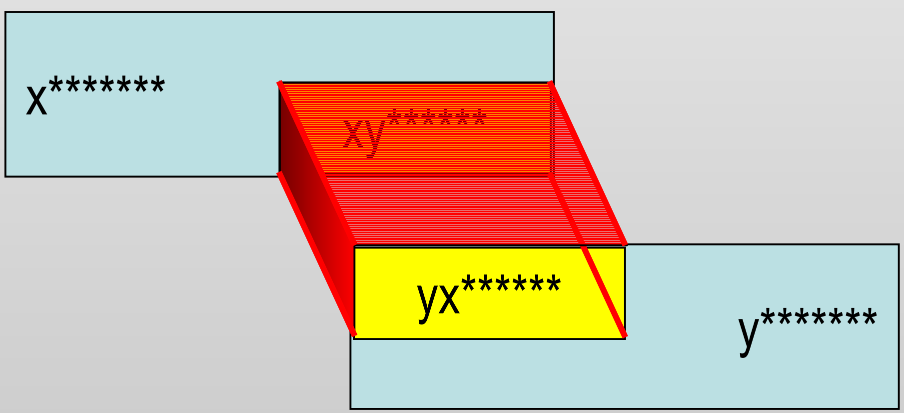
open question 1: convergence

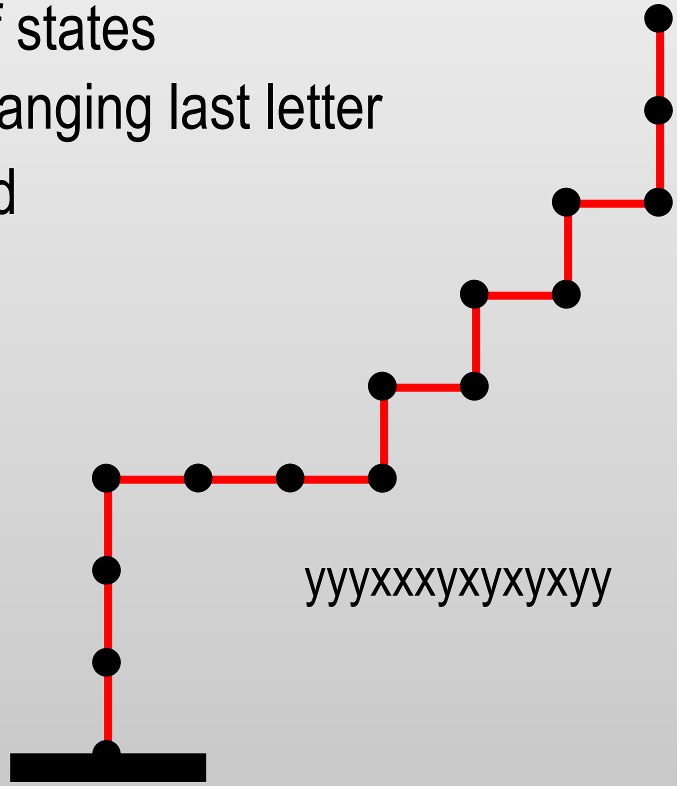
proposition: refinements of the articulated arm always give contractible state complexes

proof: use symbolic representation of states

actions = transposing x⇔y; changing last letter

induct on the length of the word





state complex “converges” to a smooth configuration space

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open question 1: convergence

when do state complexes converge to a smooth configuration space under refinement?

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open question 2: complexity

meta-theorem: algorithmic problems are

intractable		positive	
quadratic	for groups of	nonpositive	curvature
linear		negative	

cf: shortest path problem in 3-d

np hard [canny-reif]; but this relies on positive curvature…

recall: on a cat(0) space, there is a unique geodesic

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open question 2: complexity

is it true that NP + CAT(0) yields P ?

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