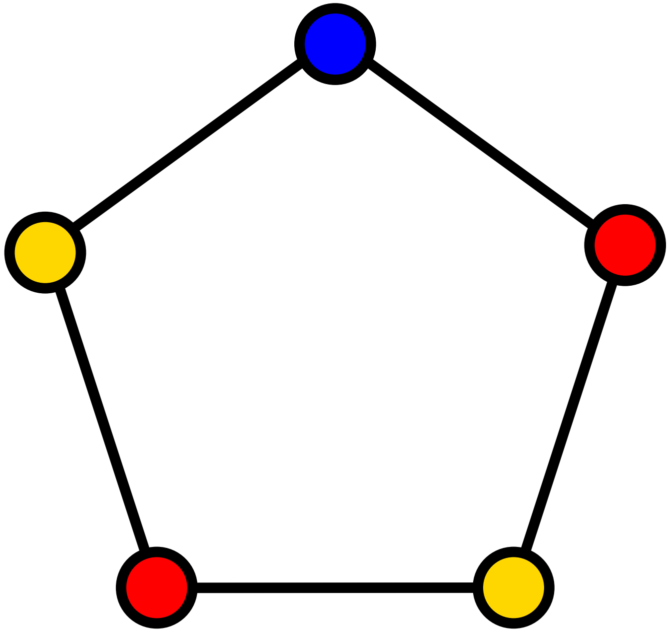
# Graph Coloring Manifolds

Frank H. Lutz

(TU Berlin/ZIB)



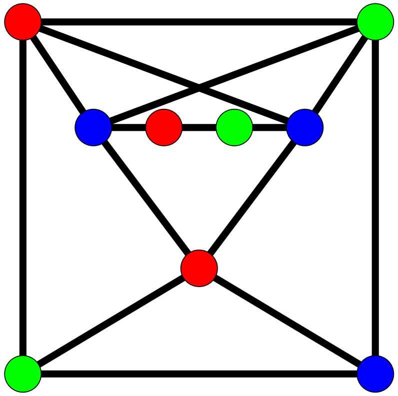
1. Graph coloring
2. The neighborhood complex and
   Lovász' proof of the Kneser conjecture
3. Hom complexes
4. Sphere bundles
5. Graph coloring manifolds
6. Flag 1-spheres
7. Flag 2-spheres

(Joint work with Péter Csorba.)

# 1. Graph coloring

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$.

- $\chi(G) :=$ chromatic number of $G$



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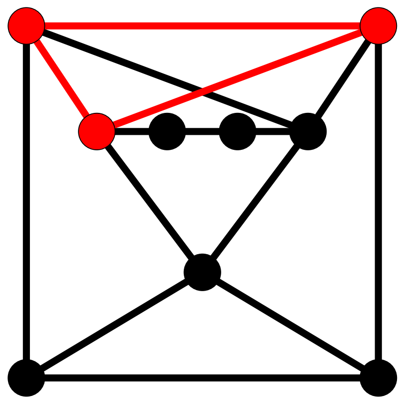
- $\omega(G) :=$ clique number of $G$

$$\omega(G) \leq \chi(G)$$



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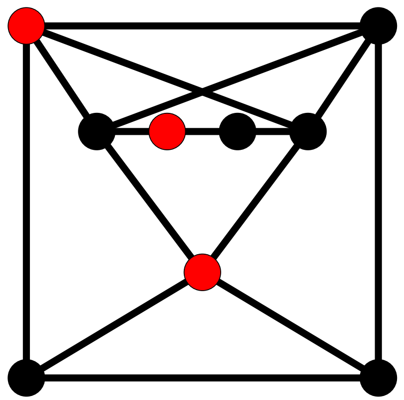

- $\chi(G) :=$ chromatic number of $G$
- $\omega(G) :=$ clique number of $G$

$$\omega(G) \leq \chi(G)$$

- $\alpha(G) :=$ stable set number of $G$

$$\frac{|V|}{\alpha(G)} \leq \chi(G)$$

**THEOREM** KARP (1972)

It is NP-complete to determine $\chi(G)$.

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**THEOREM** COOK (1971)

It is NP-complete to determine $\alpha(G)$
(and also $\omega(G) = \alpha(\overline{G})$).

Difficulties with lower bounds:

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- often rather weak
- hard to compute
- yield good results only in specific cases

Topological approach:

$G \mapsto K(G)$ simplicial complex/cell complex

"Compute" topological invariants of $K(G)$,
e.g., connectivity.

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"If $K(G)$ is highly connected, then $G$ has high chromatic number."

- $K(G) = \mathcal{N}(G)$, neighborhood complex (simplicial), Lovász
- $K(G) = \mathrm{B}(G)$, box complex (simplicial), Matoušek and Ziegler
- $K(G) = \mathrm{Hom}(H, G)$, Hom complex (cellular), Lovász

# 2. The neighborhood complex and Lovász’ proof of the Kneser conjecture

*Neighborhood complex* $\mathcal{N}(G)$ of a graph $G$:

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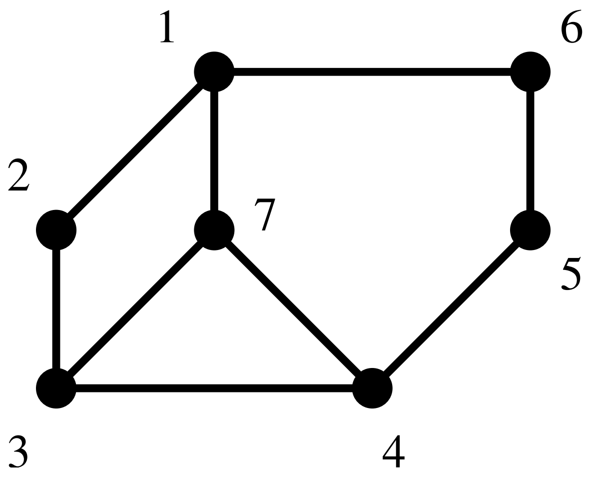
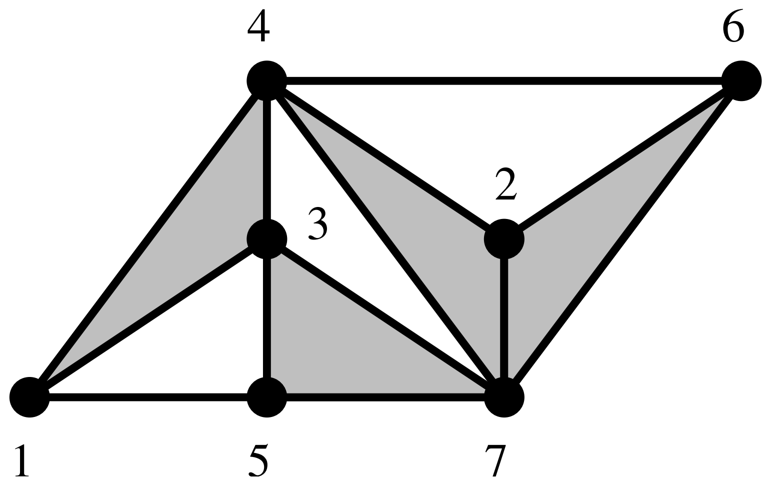
- simplicial complex:

  vertices := nodes of $G$
  simplices := sets of nodes that have common neighbors

$G$




$\mathcal{N}(G)$



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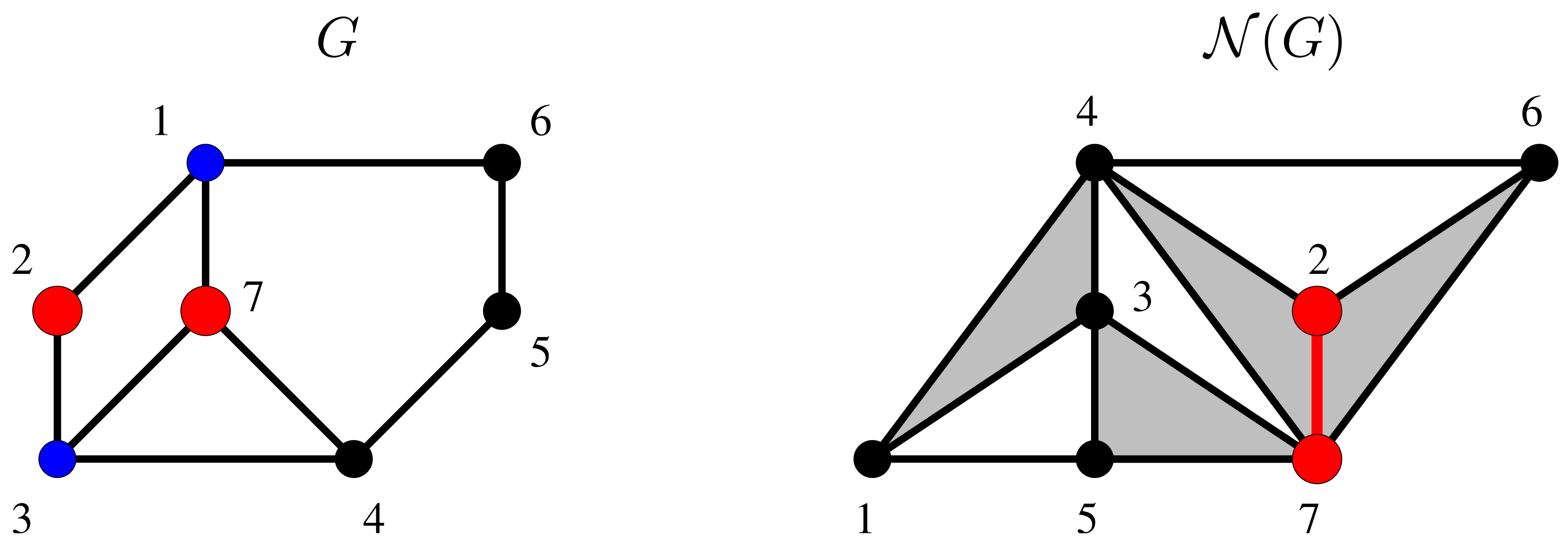

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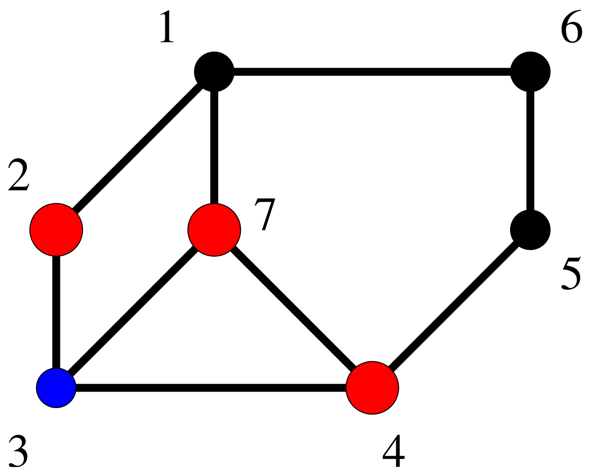


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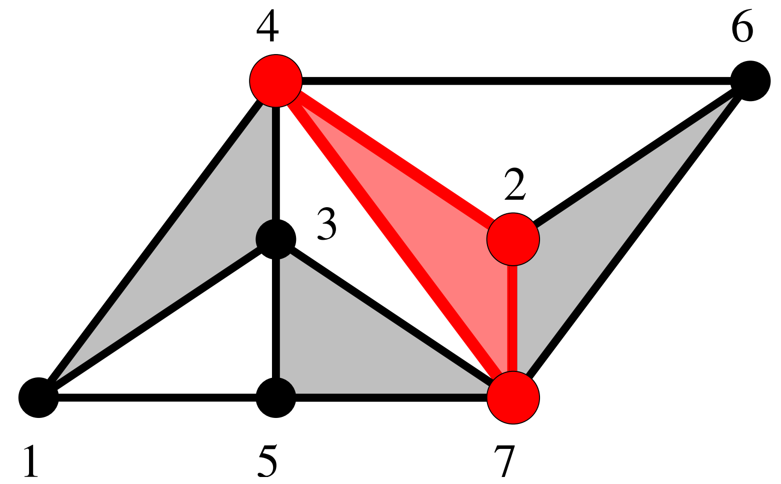

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Kneser graph $KG_{n,k}$:

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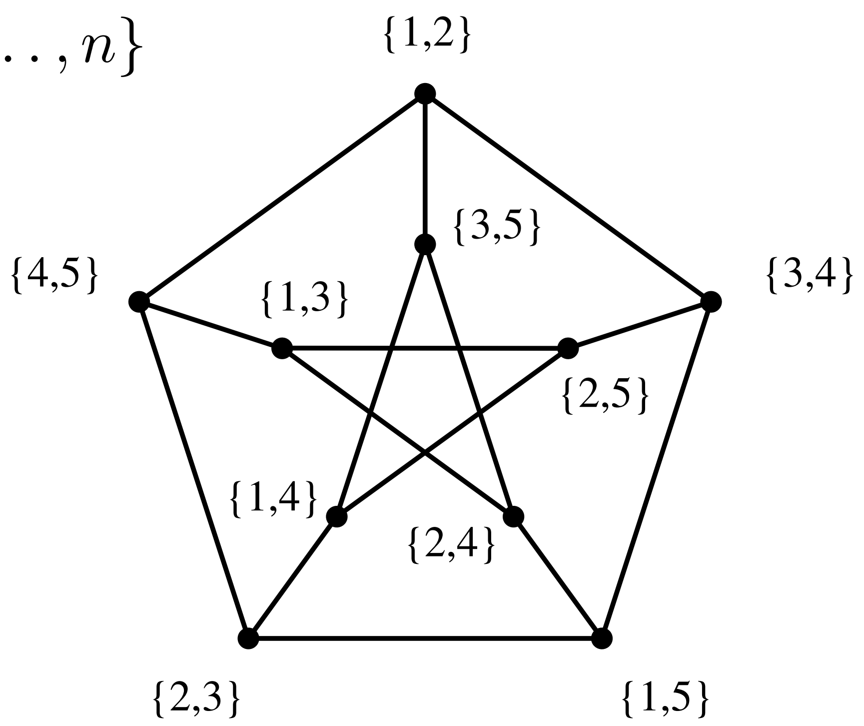
$KG_{5,2}$ is the Petersen graph:



# Kneser's conjecture (1955)

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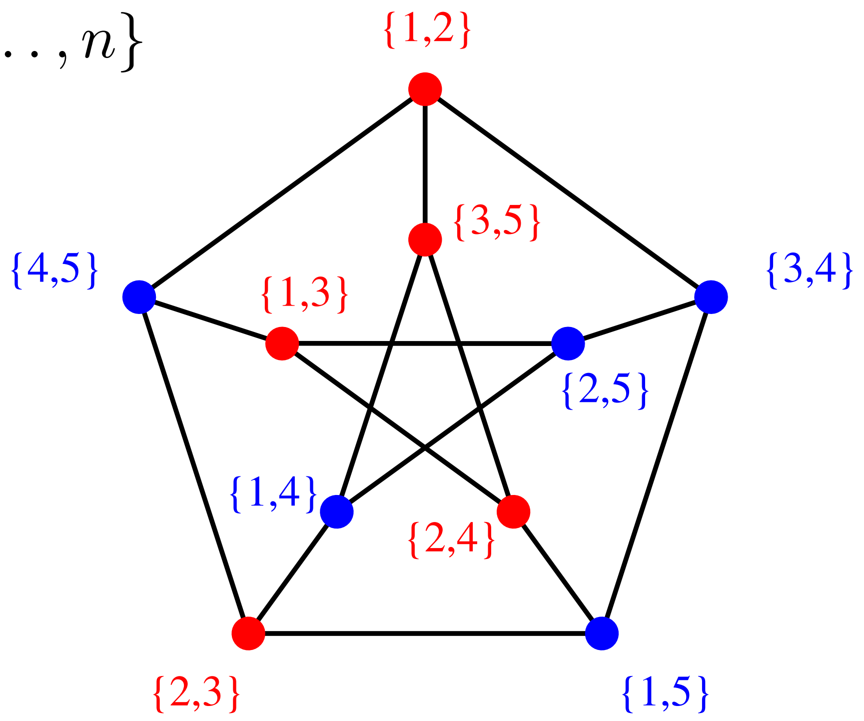

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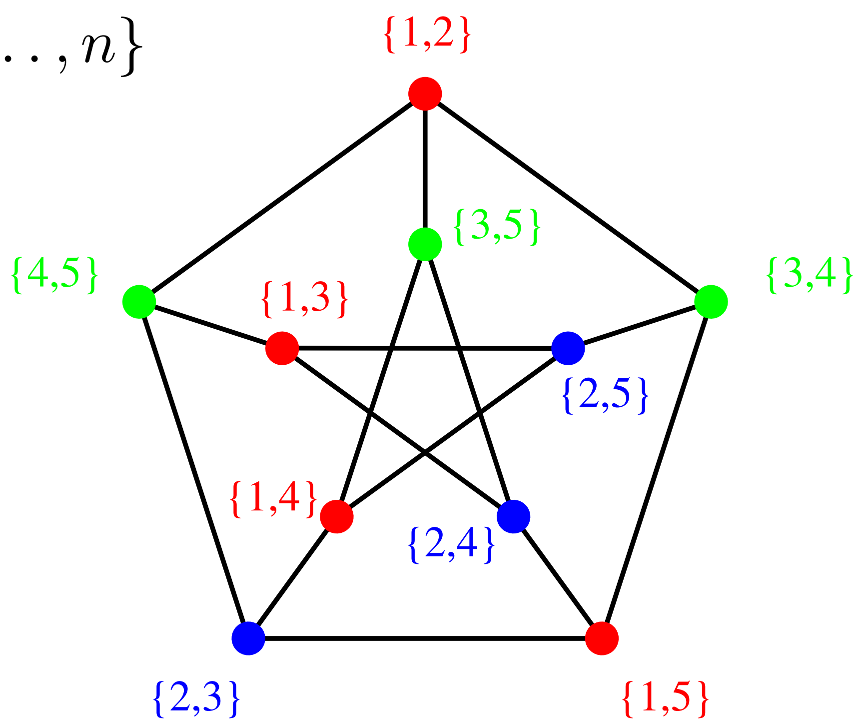

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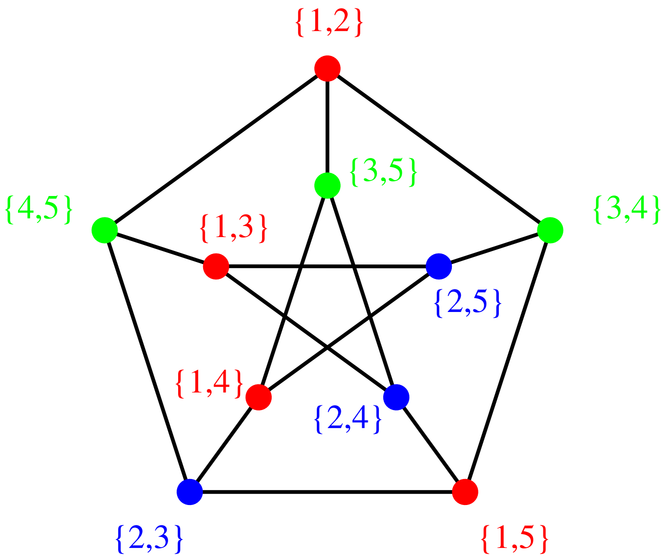


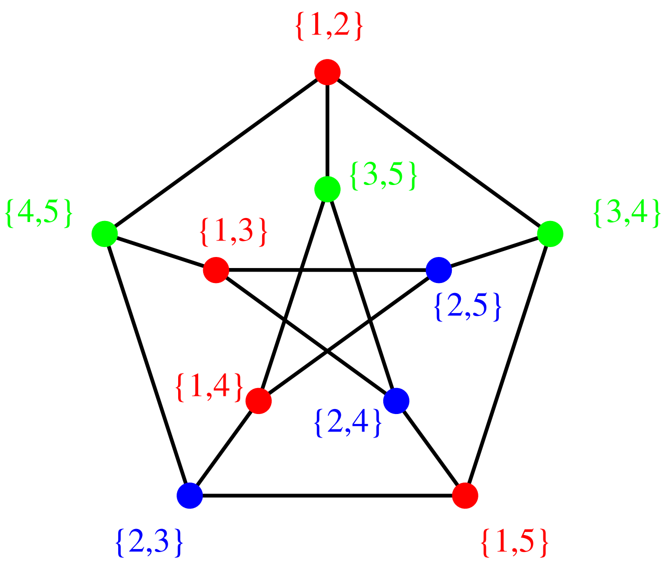

$K_i :=$ collection of $k$-sets with first element $i$.

Any two sets from the $n-2k+2$ classs

$$K_1, K_2, \ldots, K_{n-2k+1}, K_{n-2k+2} \cup \cdots \cup K_{n-k+1},$$

have a common element [since $n-(n-2k+1) = 2k-1$].

- For $n = 3k - 1$: $\omega(KG_{3k-1,k}) = 2$.



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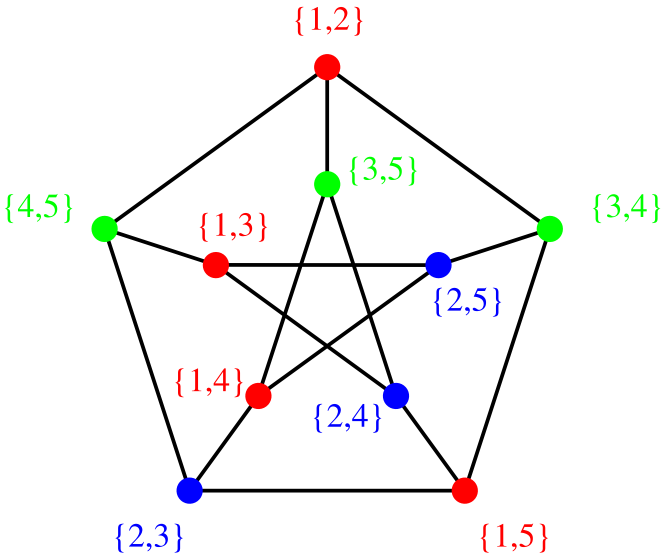


$$\frac{V(KG_{n,k})}{\alpha(KG_{n,k})} = \binom{n}{k} \Big/ \binom{n-1}{k-1} = \frac{n}{k}.$$

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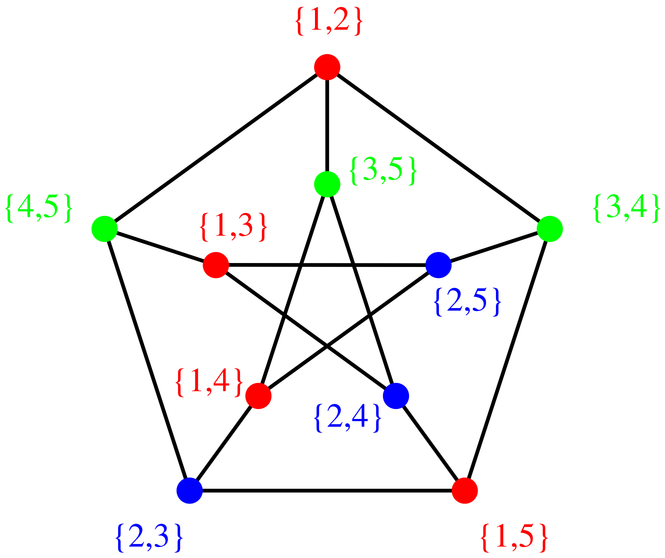


$$\frac{V(KG_{n,k})}{\alpha(KG_{n,k})} = \binom{n}{k} / \binom{n-1}{k-1} = \frac{n}{k}.$$

- For $n = 3k - 1$: $\dfrac{V(KG_{3k-1,k})}{\alpha(KG_{3k-1,k})} = \dfrac{3k-1}{k} < 3.$

**THEOREM** LOVÁSZ (1978)

The chromatic number of the Kneser graph $KG_{n,k}$ is $\chi(KG_{n,k}) = n - 2k + 2$.

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- Lovász uses the theorem of Borsuk-Ulam.

## Lovász' proof of the Kneser conjecture

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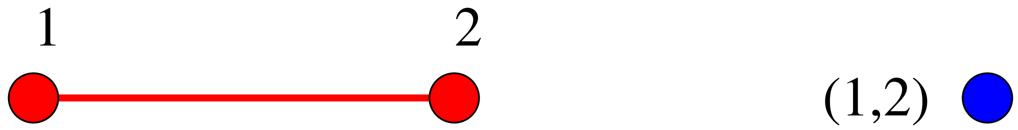
**THEOREM** — LOVÁSZ (1978)

$\mathcal{N}(KG_{n,k})$ is $(n-2k-1)$-connected.

# 3. Hom complexes

The Hom complex $\mathrm{Hom}(K_2, K_3)$:



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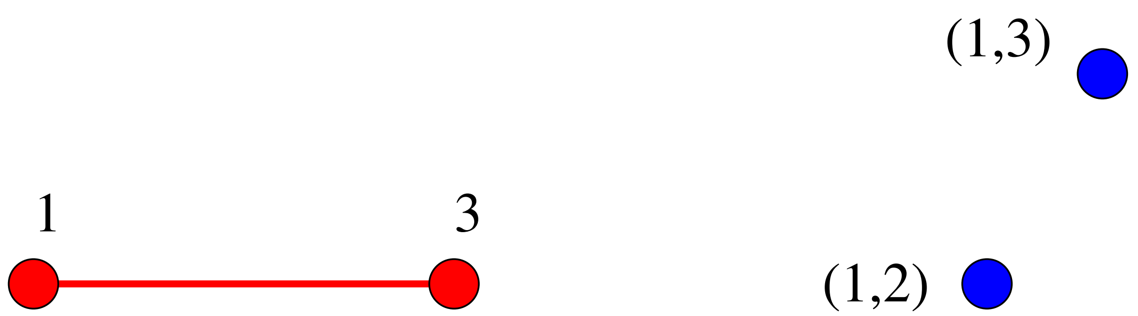

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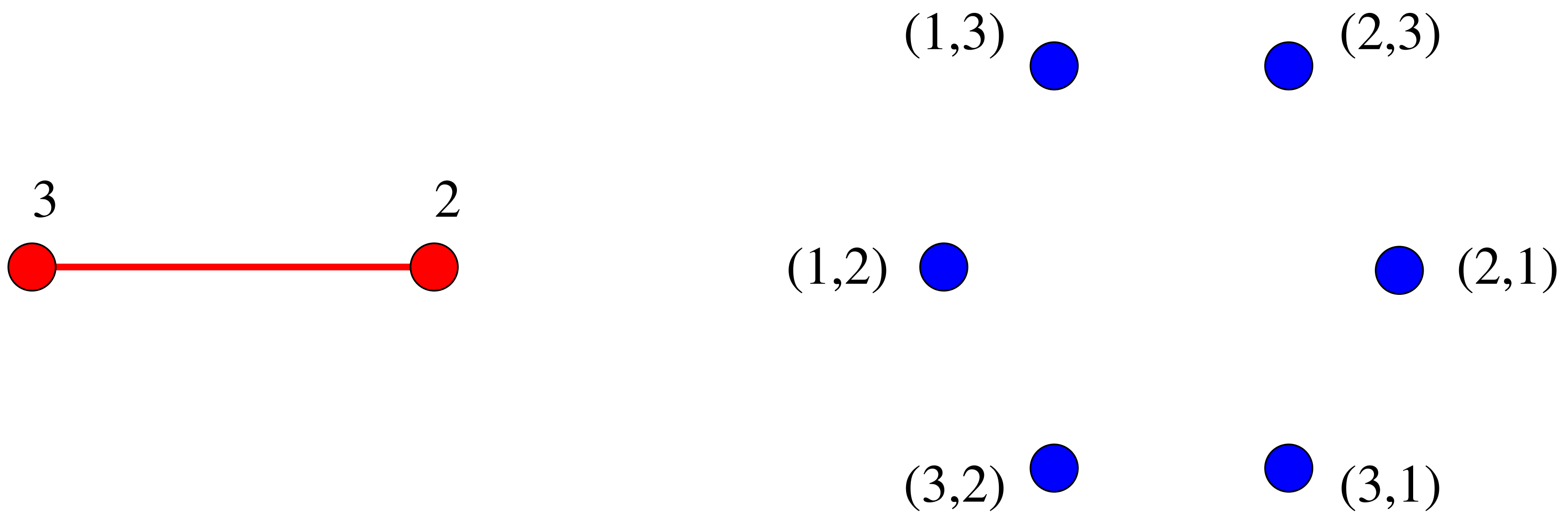

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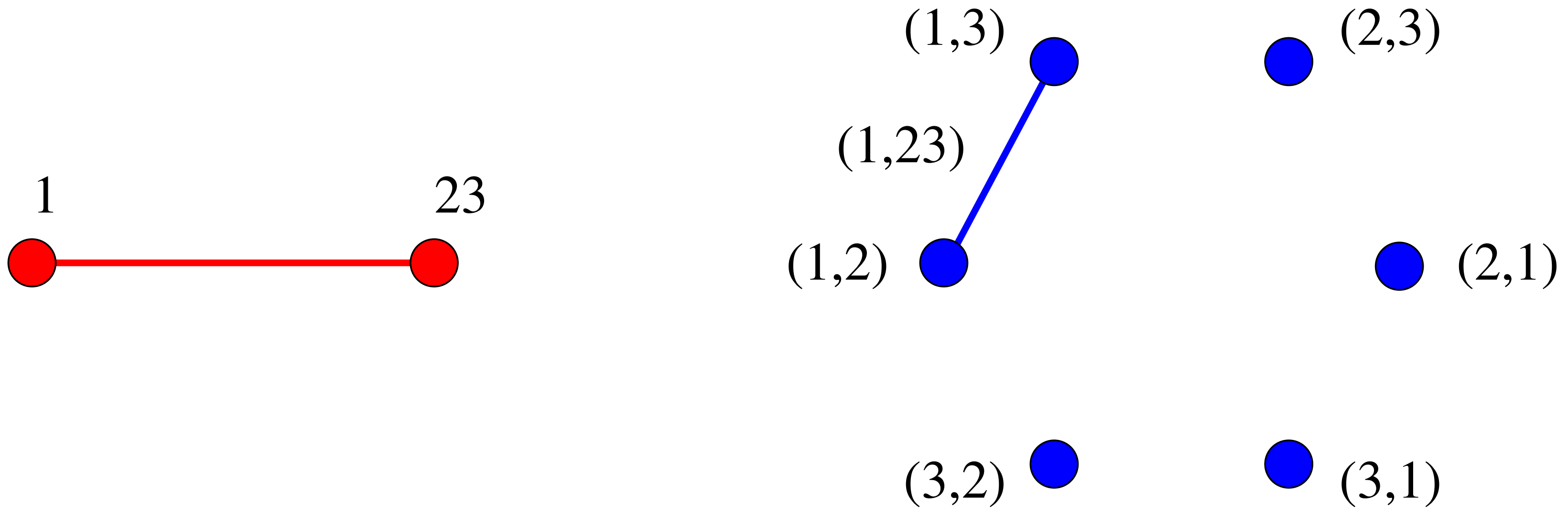
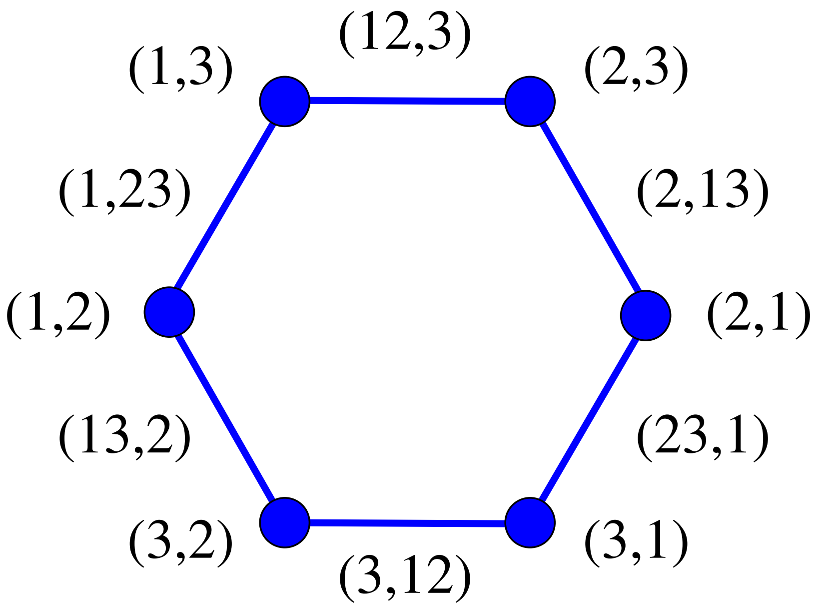


The Hom complex $\mathrm{Hom}(\mathrm{K}_2, \mathrm{K}_3)$:

The *Hom complex* $\mathrm{Hom}(H, G)$ of two graphs $H$ and $G$:

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- polyhedral complex with:

vertices := graph homomorphisms $\phi : V(H) \to V(G)$,

i.e., for $e = (i, j) \in E(H)$
we have $(\phi \times \phi)(e) = (\phi(i), \phi(j)) \in E(G)$

cells := indexed by all funktions $\eta : V(H) \to 2^{V(G)} \backslash \{\emptyset\}$,

i.e., for $(i, j) \in E(H)$
we have for all $\tilde{i} \in \eta(i)$ and $\tilde{j} \in \eta(j)$
that $(\tilde{i}, \tilde{j}) \in E(G)$

LOVÁSZ/BABSON & KOZLOV (2003):

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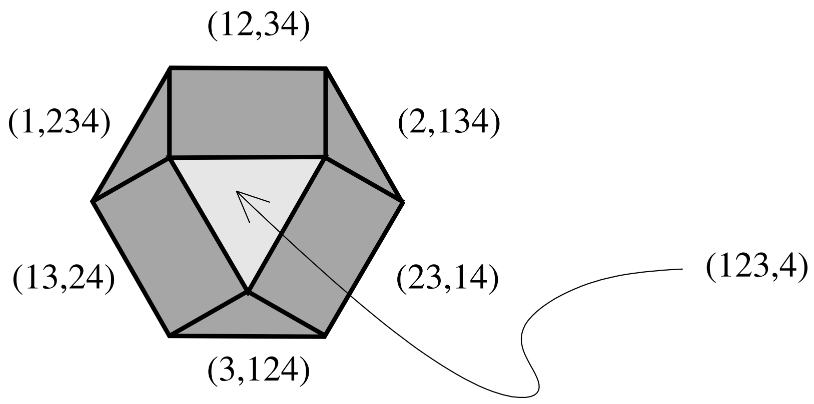
- $\mathrm{Hom}(G \dot{\cup} H, K) = \mathrm{Hom}(G, K) \times \mathrm{Hom}(H, K)$.
- $\mathrm{Hom}(\mathrm{K}_2, \mathrm{K}_{\mathrm{n}+2}) \cong \mathrm{S}^\mathrm{n}$,

$\mathrm{Hom}(\mathrm{K}_2, \mathrm{K}_3) \subseteq \mathrm{Hom}(\mathrm{K}_2, \mathrm{K}_4)$:



**THEOREM** BABSON & KOZLOV (2003)

If $\mathrm{Hom}(\mathrm{C}_{2\mathrm{r}+1}, \mathrm{G})$ is $k$-connected, then $\chi(G) \geq k + 4$.

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If $\mathrm{Hom}(\mathrm{C_{2r+1}}, \mathrm{G})$ is $k$-connected, then $\chi(G) \geq k + 4$.

$H_*(\mathrm{Hom}(\mathrm{C_5}, \mathrm{K_n})) = (\mathbb{Z}, 0, \ldots, 0, \mathbb{Z}, \mathbb{Z}, *, \cdots, *)$ if $n$ is odd,
$H_*(\mathrm{Hom}(\mathrm{C_5}, \mathrm{K_n})) = (\mathbb{Z}, 0, \ldots, 0, \mathbb{Z}_2, 0, *, \cdots, *)$ if $n$ is even.

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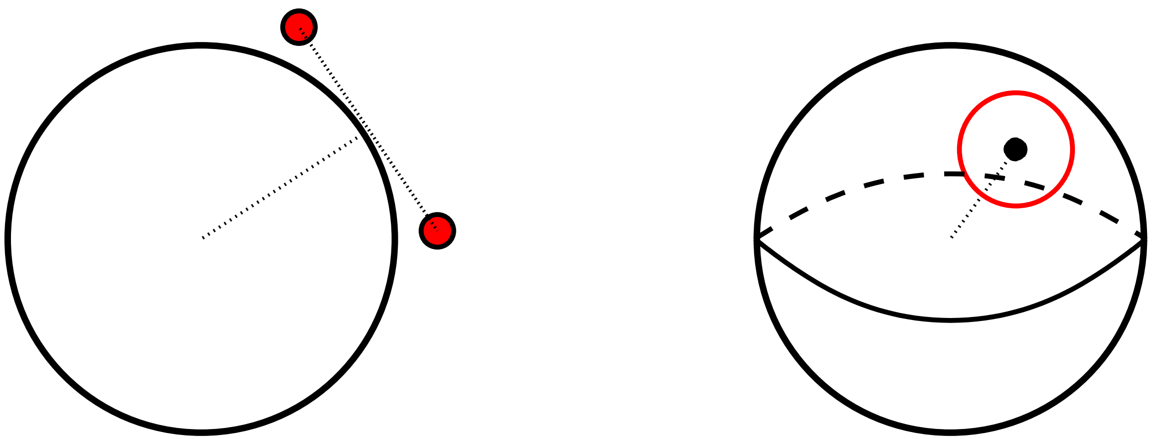
**CONJECTURE** CSORBA (2003)

The Hom complex $\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_{\mathrm{n}+2})$ is homeomorphic to the Stiefel manifold $V_{n+1,2}$.



$$\begin{aligned} V_{n+1,2} &= \{\, (v_1, v_2) \mid v_1, v_2 \in \mathbb{R}^{n+1},\ \|v_1\| = \|v_2\| = 1,\ < v_1, v_2 >= 0 \,\} \\ &= \text{unit tangent bundle to } S^n. \end{aligned}$$

$$\begin{aligned} V_{2,2} &= S^0 \times S^1. \\ V_{3,2} &= \mathbb{R}\mathbf{P}^3. \end{aligned}$$

```
Date: Sun, 22 Jun 2003 10:41:38 +0200 (MET DST)
From: Peter Csorba <pcsorba@inf.ethz.ch>

Dear Frank,

I have a question about the running time of your BISTELLAR Software.
I ran it on the enclosed complex (2160 vertex, 3 dimensional)
for few days (400 Mhz)...

best, Peter
```

# $\mathrm{Hom}(C_5, K_4)$, a 3D puzzle

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....
```

12960 tetrahedra!

- compute *homology* with

  `polymake` (Gawrilow, Joswig et al.),
  `Simplicial Homology` (Dumas, Heckenbach, Saunders, Welker):

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  $\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_4)$ is a manifold.

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- check *vertex links*:

  $\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_4)$ is a manifold.

- compute *fundamental group* with `GAP` (Groups, Algorithms, and Programming):

  $\pi_1(\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_4)) = \mathbb{Z}_2$.

Thurston's geometrization conjecture implies:

$\mathbb{R}\mathbf{P}^3$ is the only $3$-manifold with fundamental group $\mathbb{Z}_2$.

[Proof(?) by Perelman, 2003.]

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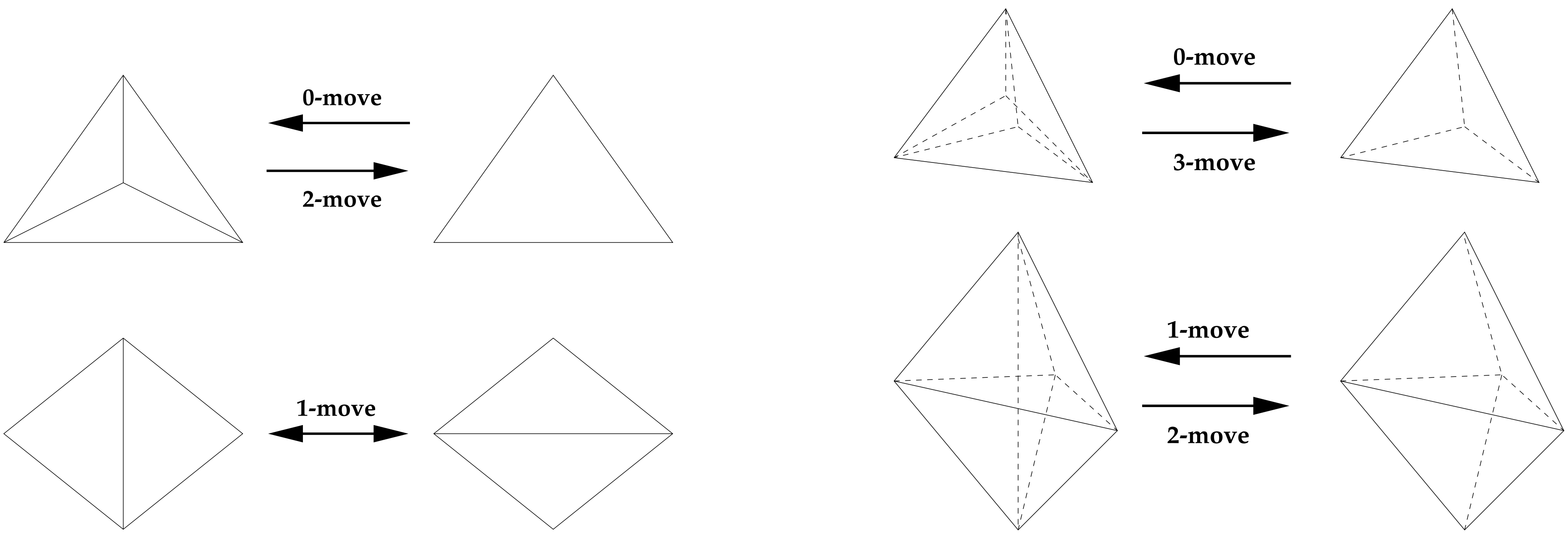
Try alternative proof for $\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_4) = \mathbb{R}\mathbf{P}^3$:

recognition with bistellar flips.

# bistellar flips



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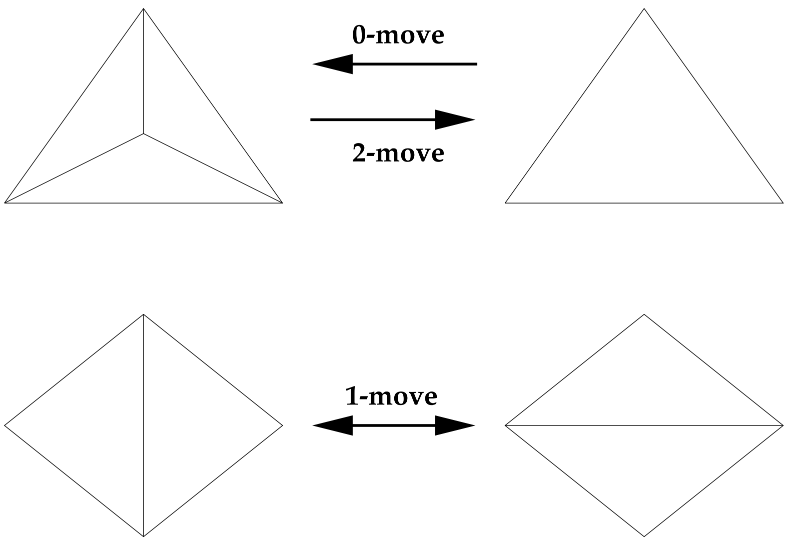
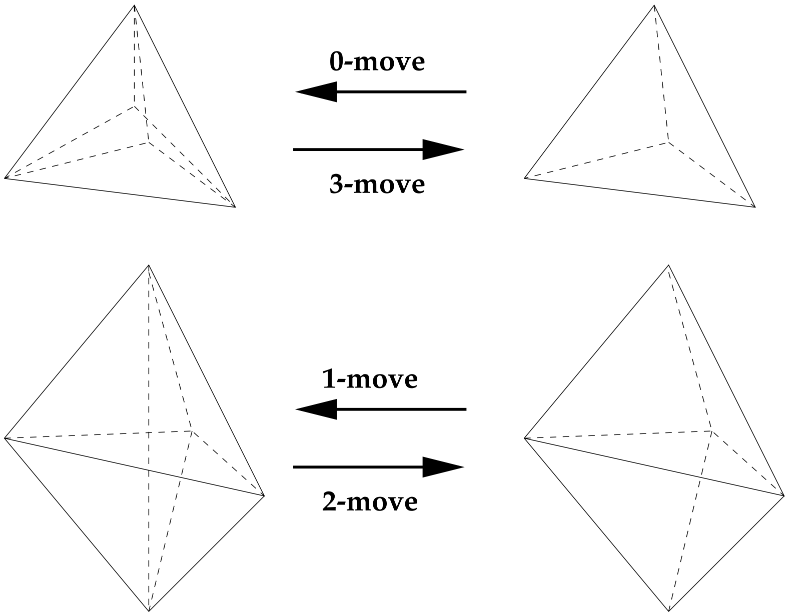
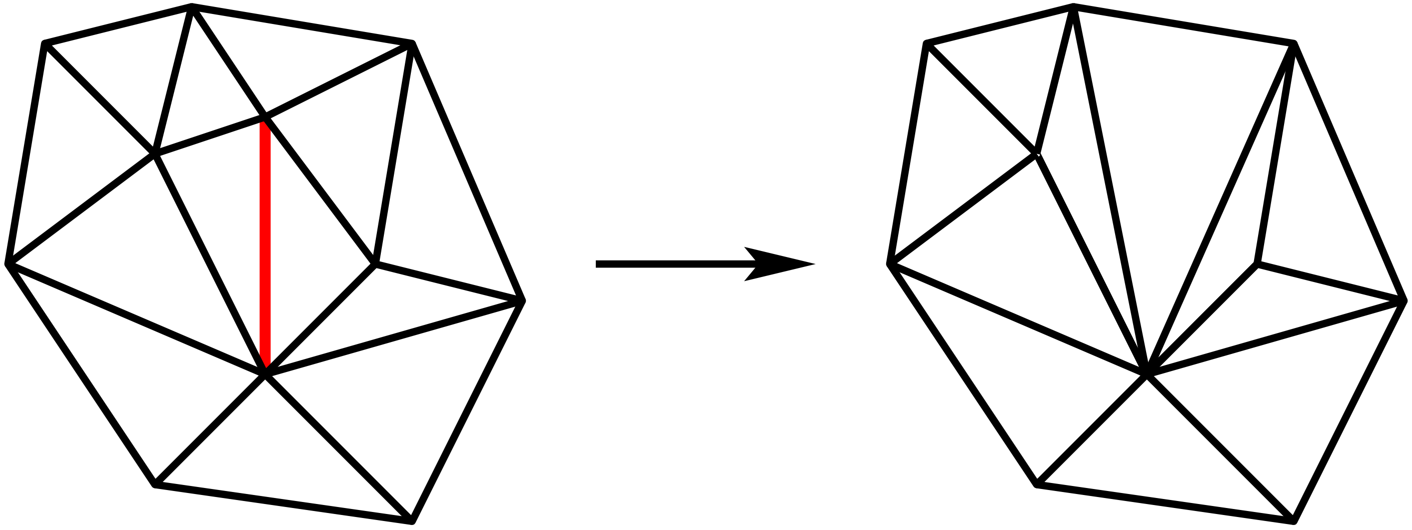
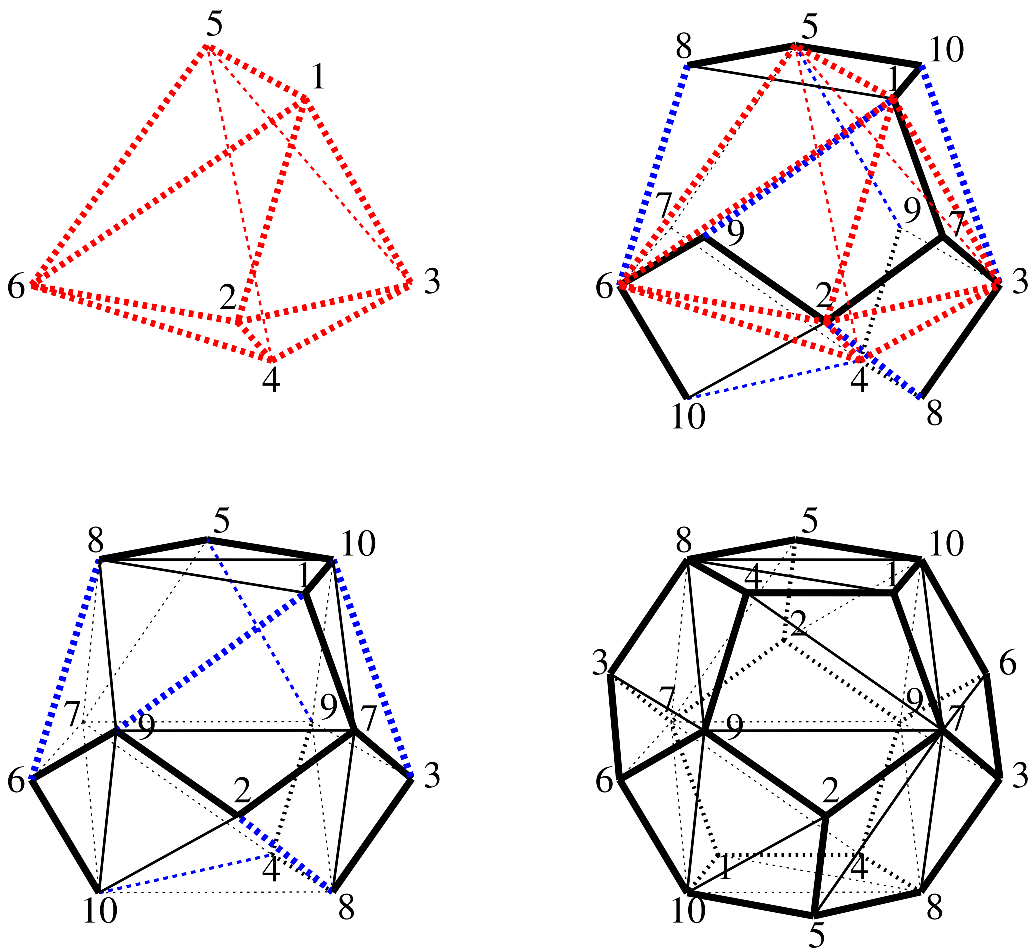



Recognition works well for triangulations of $S^3$ and $\mathbb{R}\mathbf{P}^3$ with up to $200$ vertices.

For triangulations of $S^3$ and $\mathbb{R}\mathbf{P}^3$
with up to $3000$ vertices: need edge contractions.






minimal $11$-vertex triangulation of $\mathbb{R}\mathbf{P}^3$ [Walkup, 1970]

# 4. Sphere bundles

**THEOREM** CSORBA & L. (2003)

$\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_{\mathrm{n}+2})$ is an $S^{n-1}$-bundle over $S^n$.

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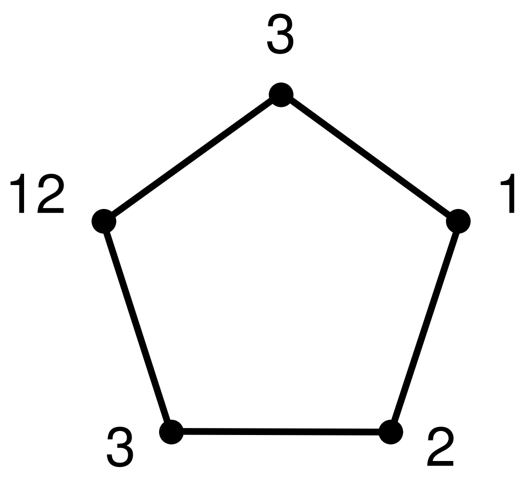
**Proof:**

- $\mathrm{Hom}(C_5, K_{n+2})$ is a pure cell complex of dimension $d = 2(n+2) - 5$.

*Example:* (12,3,1,2,3) is a facet of $\mathrm{Hom}(C_5, K_3)$.



**THEOREM** CSORBA & L. (2003)

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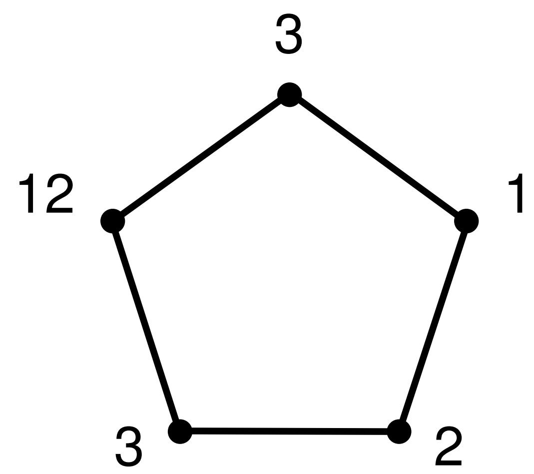


- $\mathrm{Hom}(C_5, K_{n+2})$ is a pure cell complex of dimension $d = 2(n+2) - 5$.

*Example:* (12,3,1,2,3) is a facet of $\mathrm{Hom}(C_5, K_3)$.

- $\mathrm{Hom}(C_5, K_{n+2})$ is a pseudomanifold, i.e., every $(d-1)$-face is contained in exactly two facets.

*Example*: (1~~2~~,3,1,2,3).

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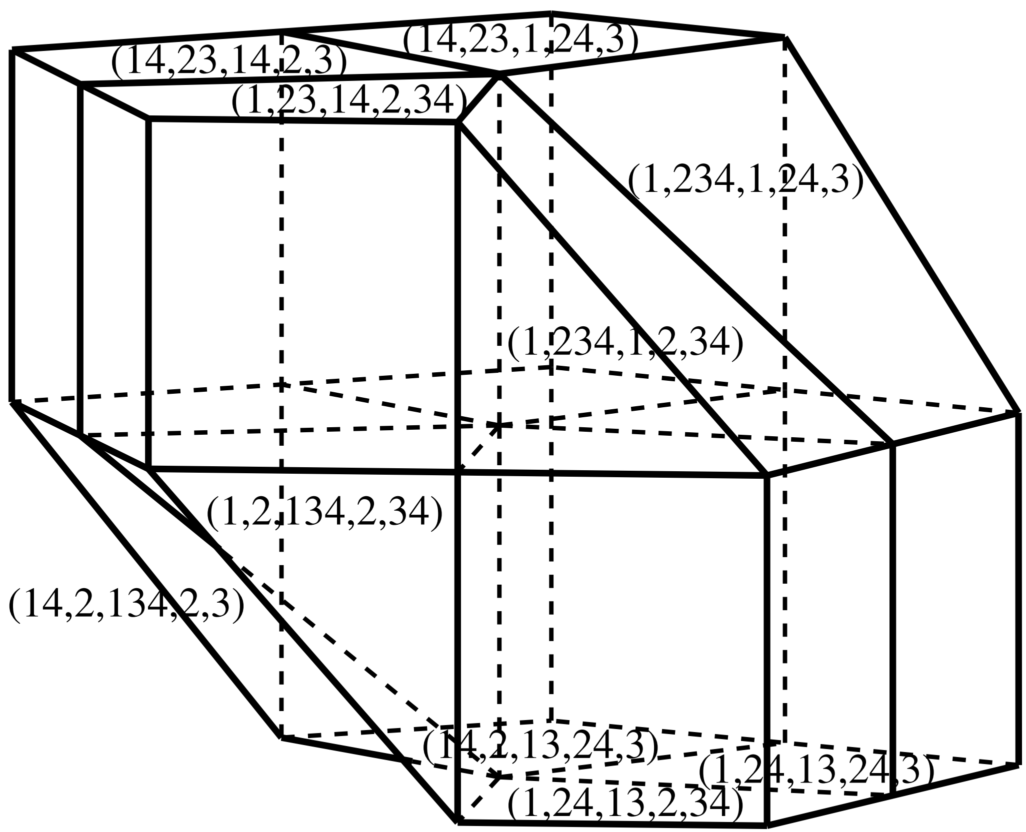
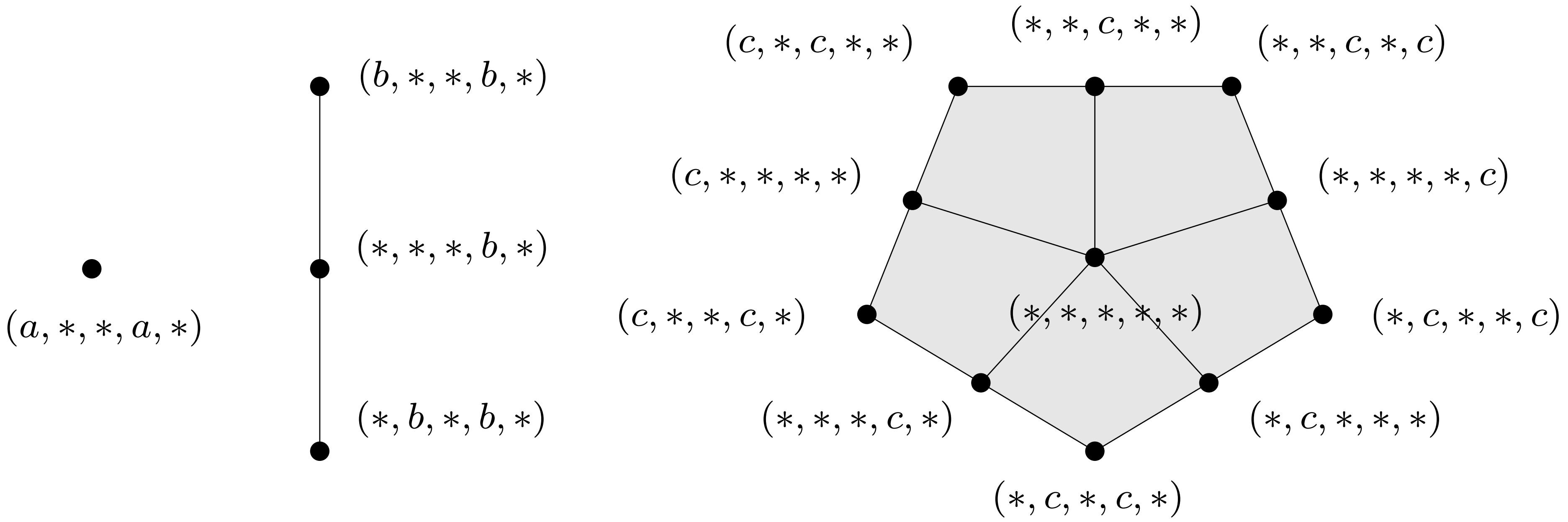
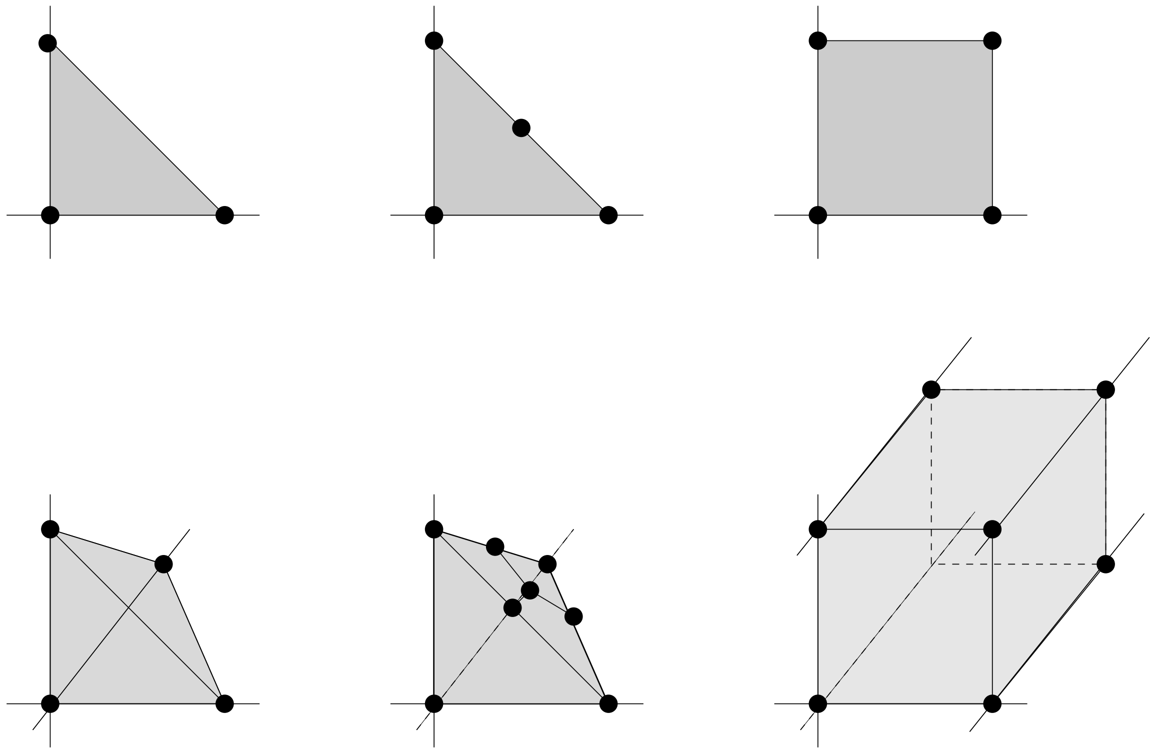
*Example*: Vertex-star of the vertex $(1,2,1,2,3)$ in $\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_4)$:



Cubical complex $X := X_1 \times \cdots \times X_n$:



Subdivision of the vertex-stars in $\mathrm{Hom}(C_5, K_{n+2})$:



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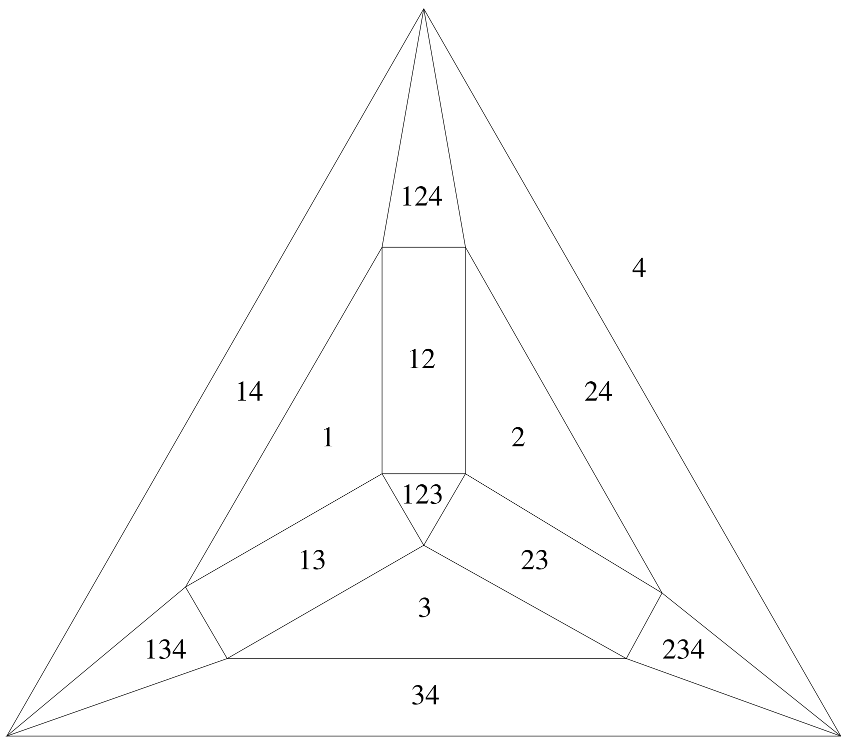
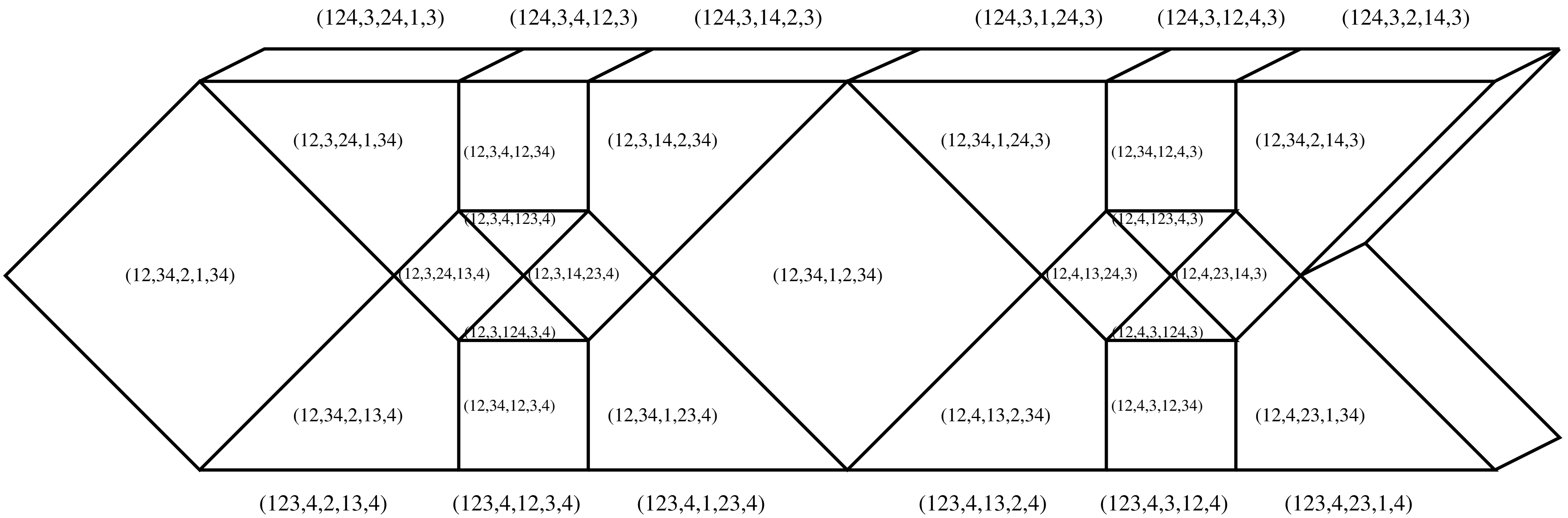



Cell-decomposition of the base sphere $S^2$ in $\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_4)$.

Collection of facets of the form $(12, *, *, *, *)$ in $\mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_4)$:



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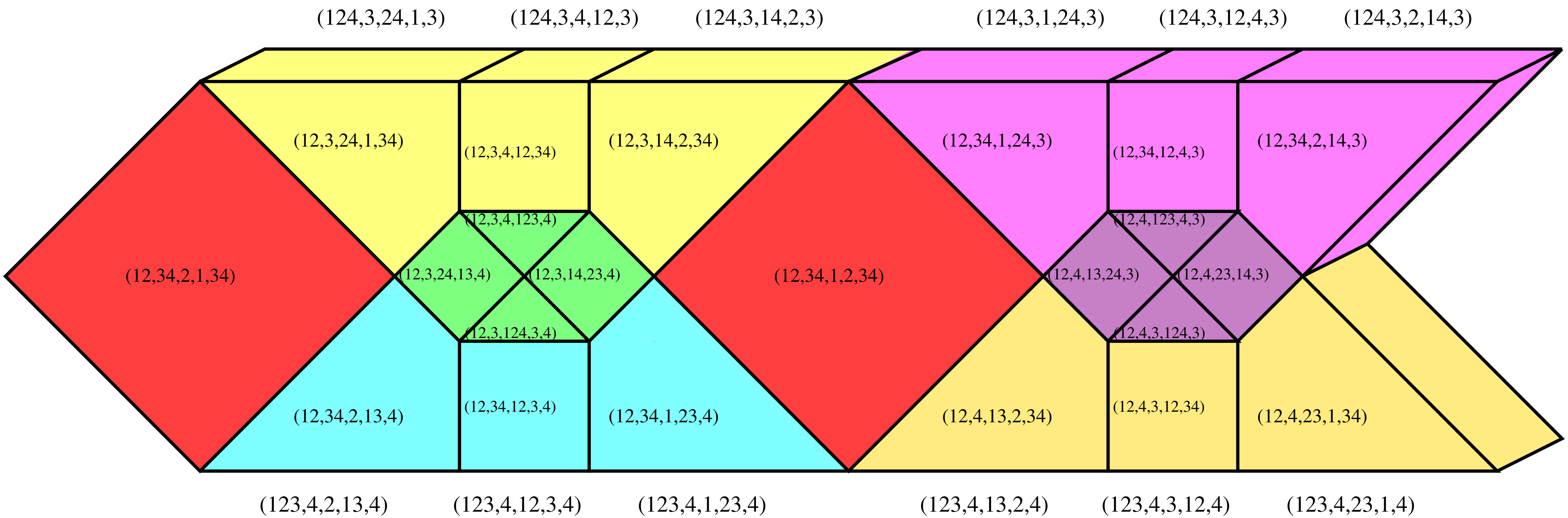
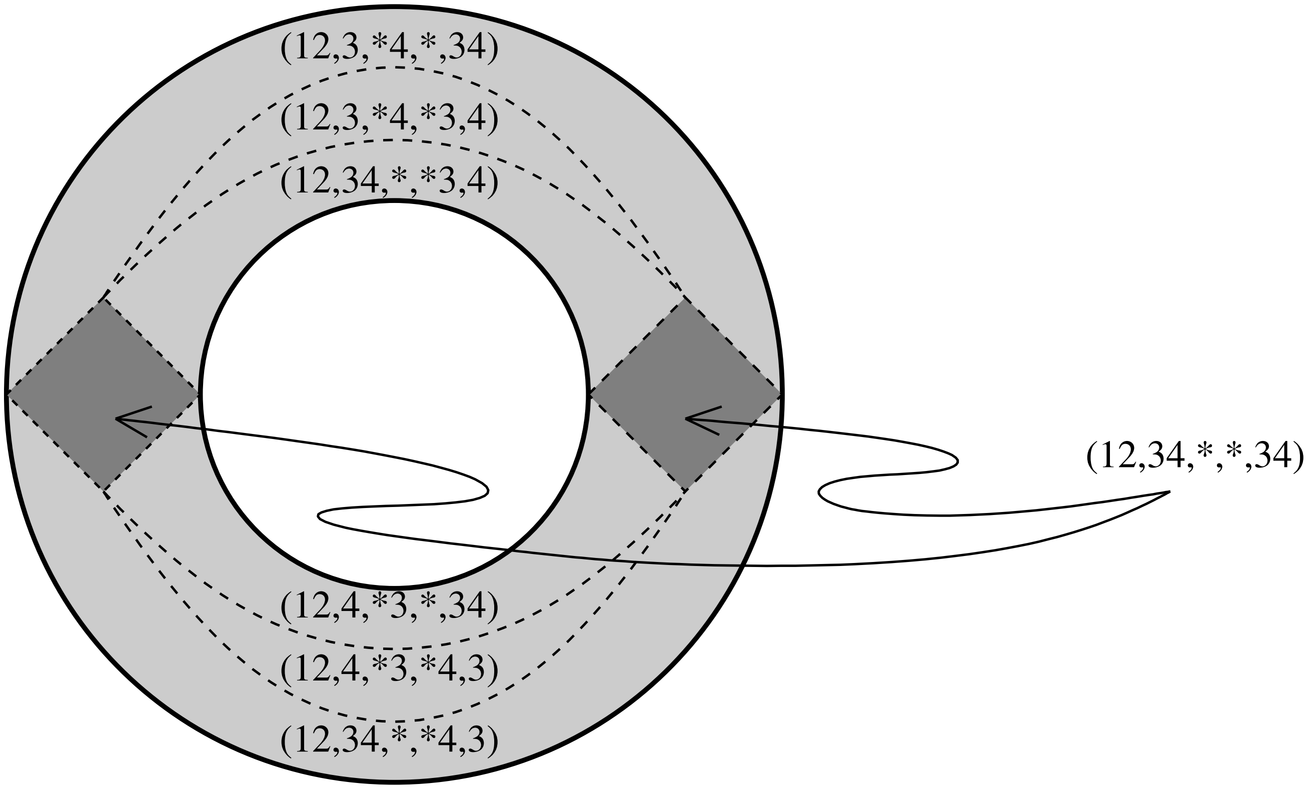
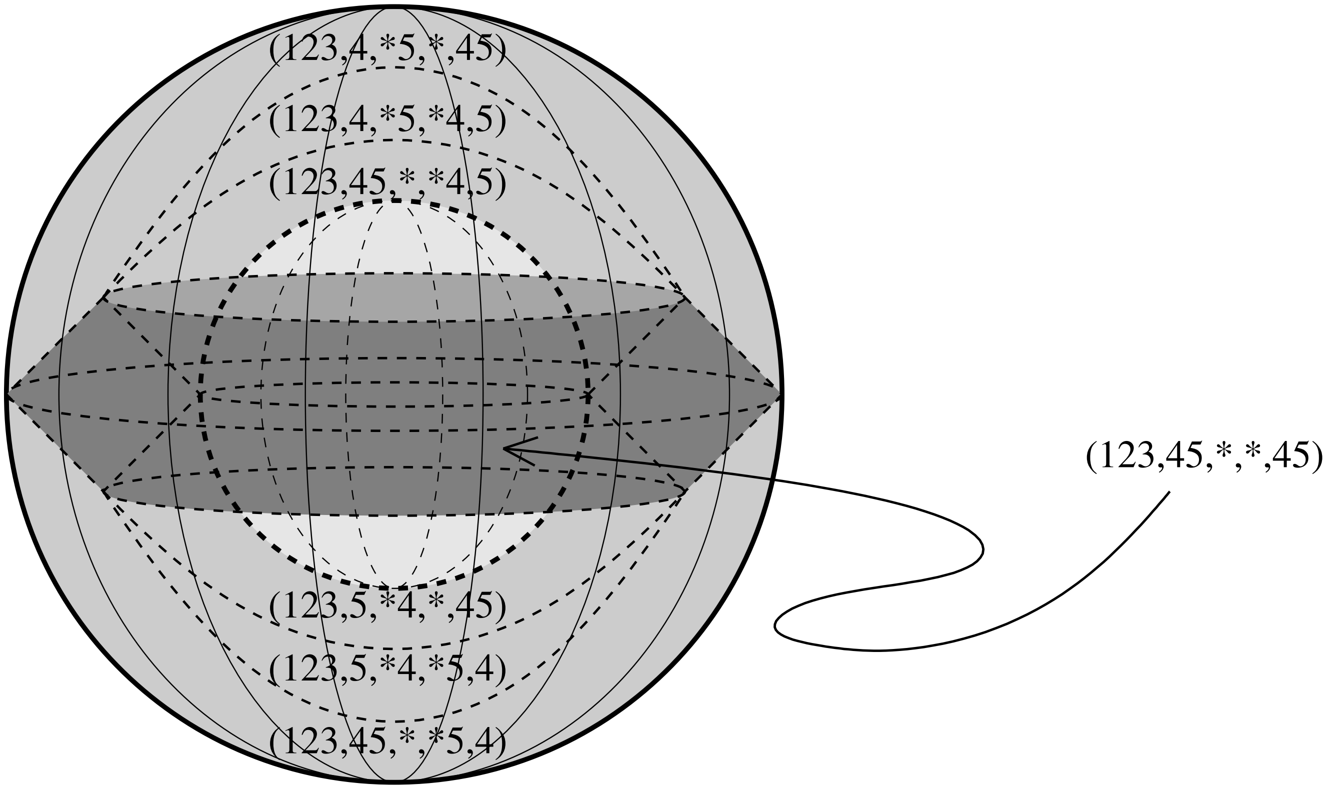


Decomposition of $(12, *, *, *, *) \subseteq \mathrm{Hom}(C_5, K_4)$ into sub-collections of cells:



Decomposition of $(123, *, *, *, *) \subseteq \mathrm{Hom}(\mathrm{C}_5, \mathrm{K}_5)$ into sub-collections of cells:




□

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- Find coordinates for vertices on $V_{n+1,2}$.
- Use topological classification theorems:

$$\mathrm{Hom}(C_5, K_3) = V_{2,2} = S^0 \times S^1,$$
$$\mathrm{Hom}(C_5, K_4) = V_{3,2} = \mathbb{R}\mathbf{P}^3,$$
$$\mathrm{Hom}(C_5, K_5) = V_{4,2} = S^2 \times S^3 \qquad [\pi_2(SO(3)) = 0],$$
$$\mathrm{Hom}(C_5, K_6) = V_{5,2} \qquad \text{[Escher \& Crowley (2003)]},$$
$$\mathrm{Hom}(C_5, K_9) = V_{8,2} = S^6 \times S^7 \qquad [\pi_6(SO(7)) = 0].$$

# 5. Graph coloring manifolds

For which graphs is $\mathrm{Hom}(\mathrm{H},\mathrm{K_n})$ a manifold?

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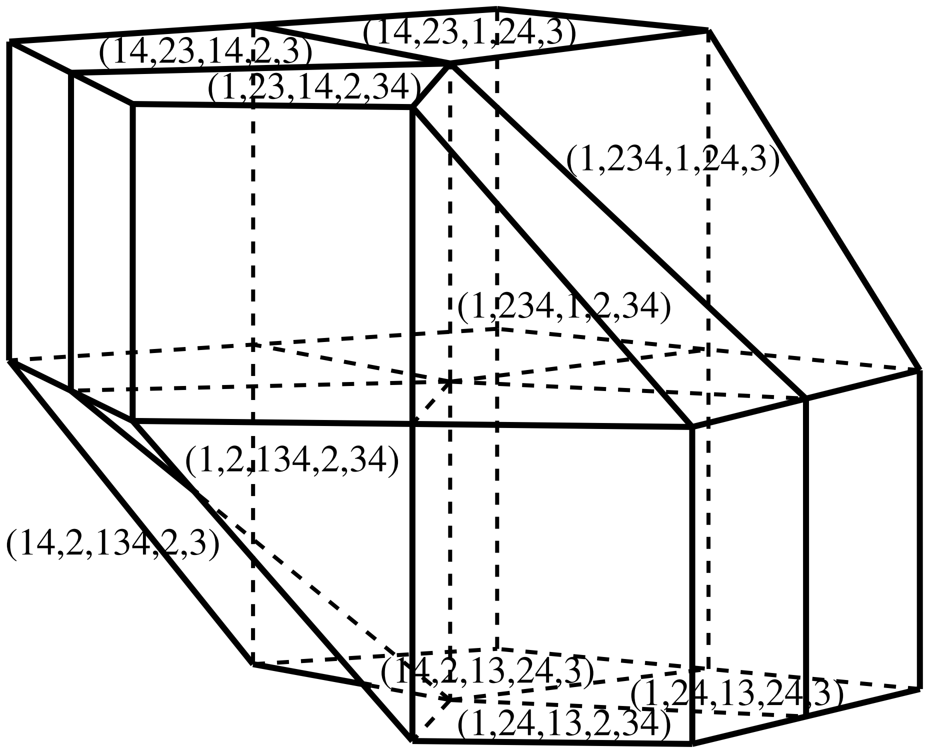

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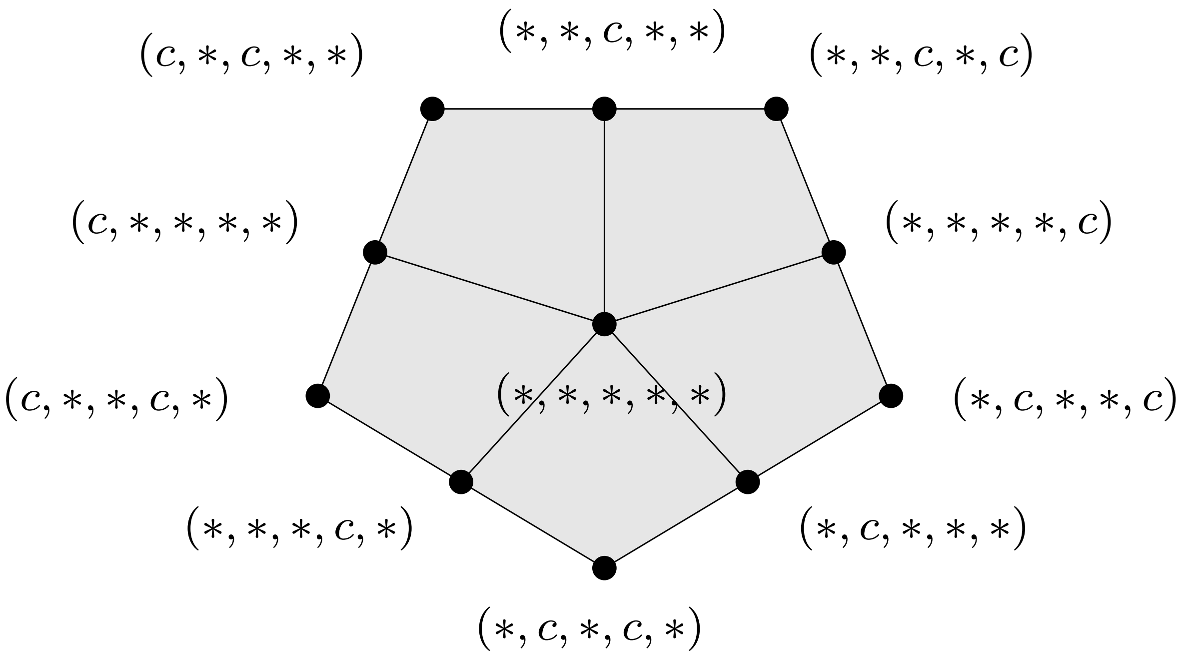

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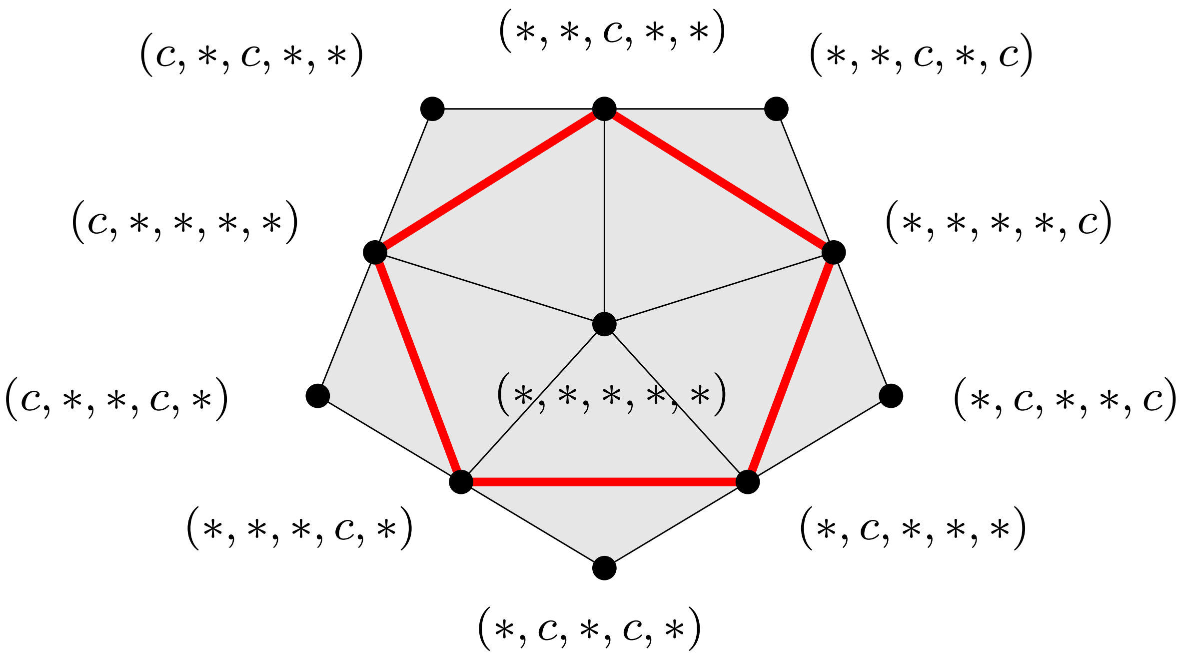

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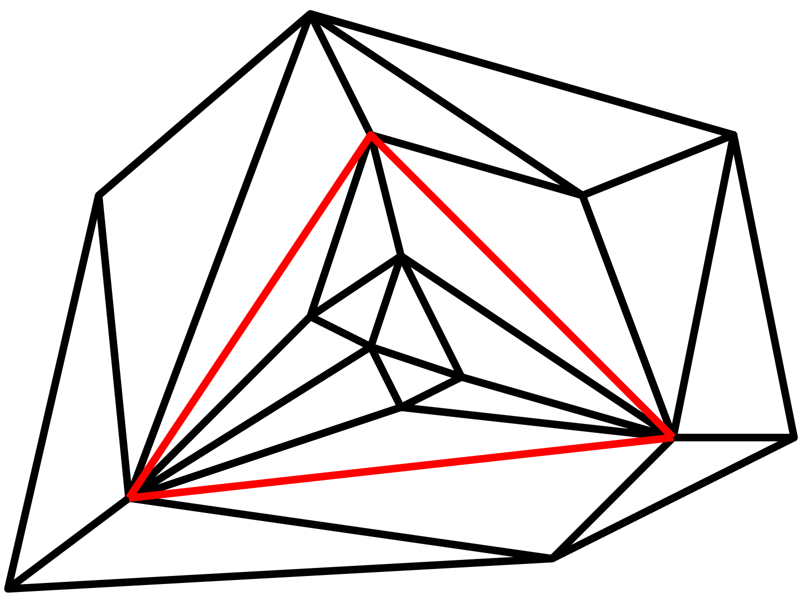
We have:

- $\mathrm{SK}_1(\mathrm{Cliq}(\overline{H})) = \overline{H}$.
- Every clique in $\mathrm{SK}_1(\mathrm{Cliq}(\overline{H}))$ forms a face of $\mathrm{Cliq}(\overline{H})$.

A simplicial complex $K$ is *flag*,
if every clique in its $1$-skeleton $\mathrm{SK}_1(K)$ spans a simplex,
i.e., no “empty simplices”.



**THEOREM** CSORBA & L. (2003)

$\mathrm{Hom(H, K_n)}$ is a PL manifold for all $n \geq \chi(H)$
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iff $H$ is the complement of the $1$-skeleton
of a flag simplicial PL sphere.

For such graphs $H$, we call $\mathrm{Hom}(\mathrm{H}, \mathrm{K_n})$
a *graph coloring manifold*.

# 6. One-dimensional flag spheres

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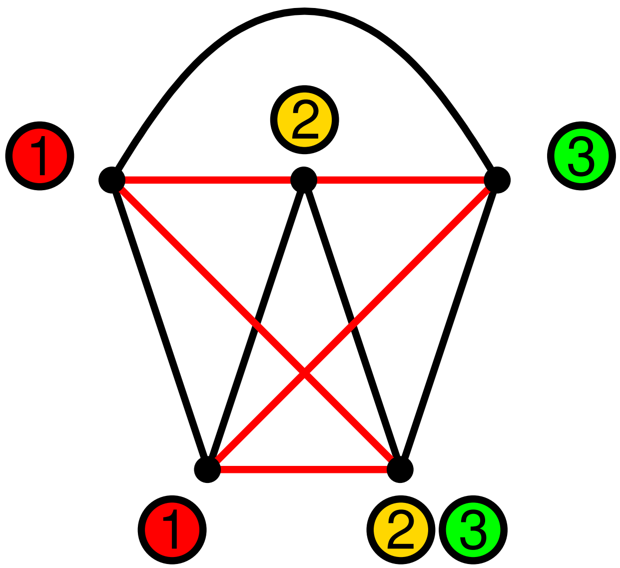


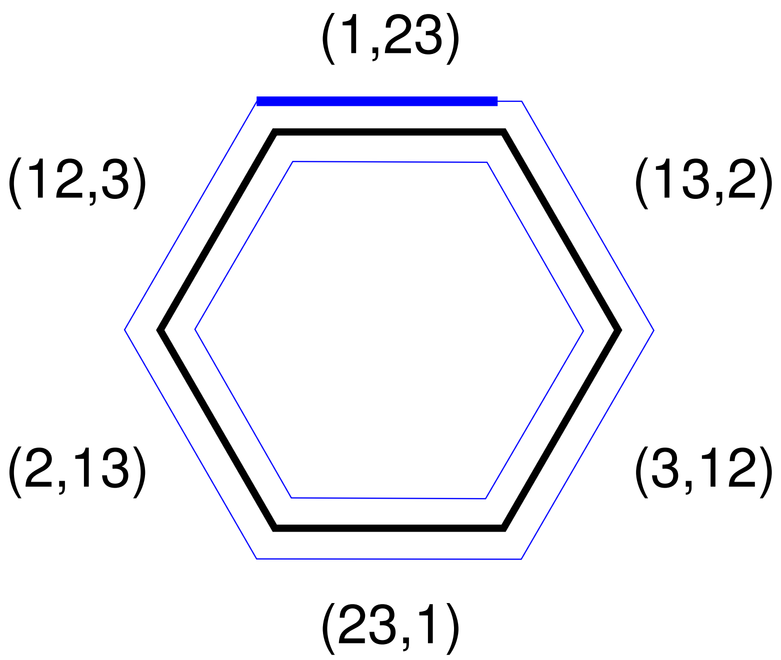

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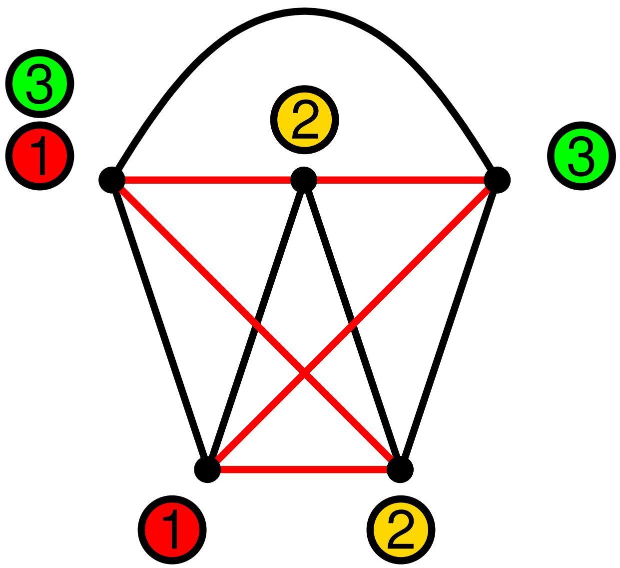


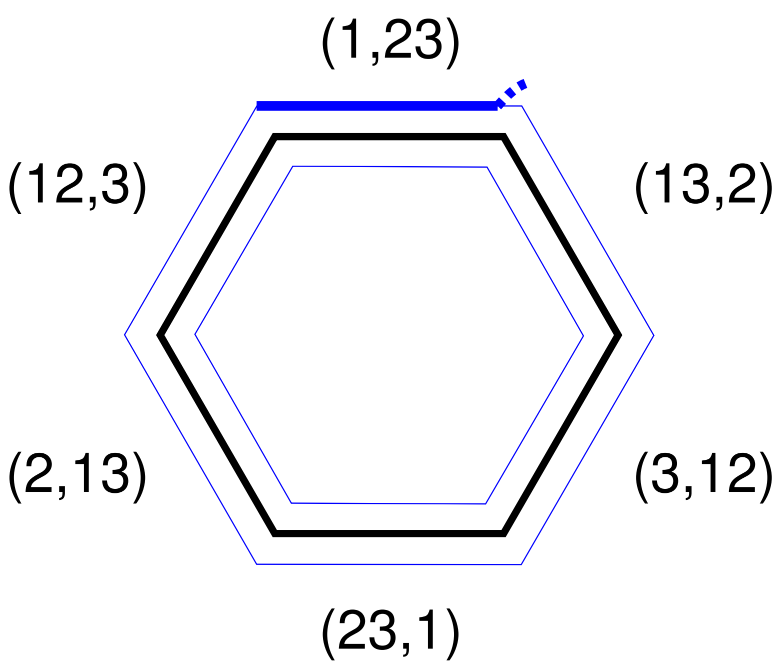

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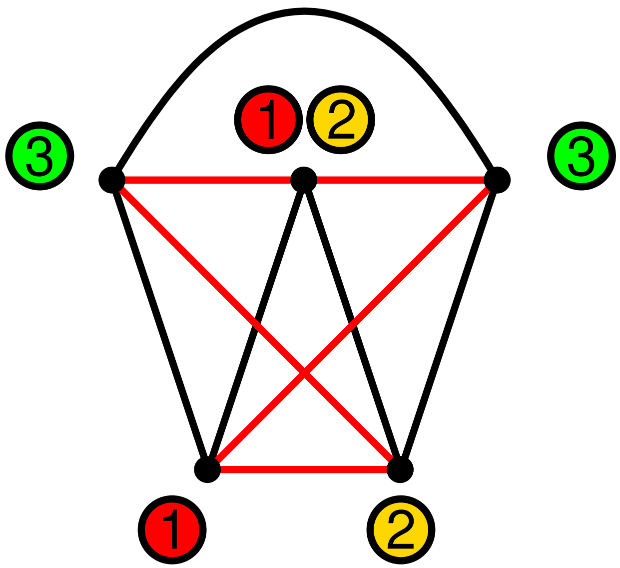


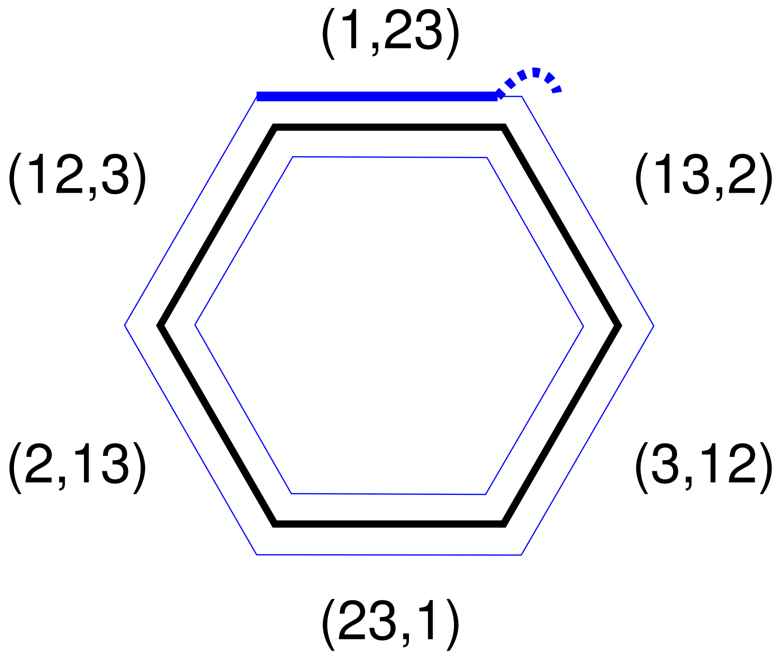

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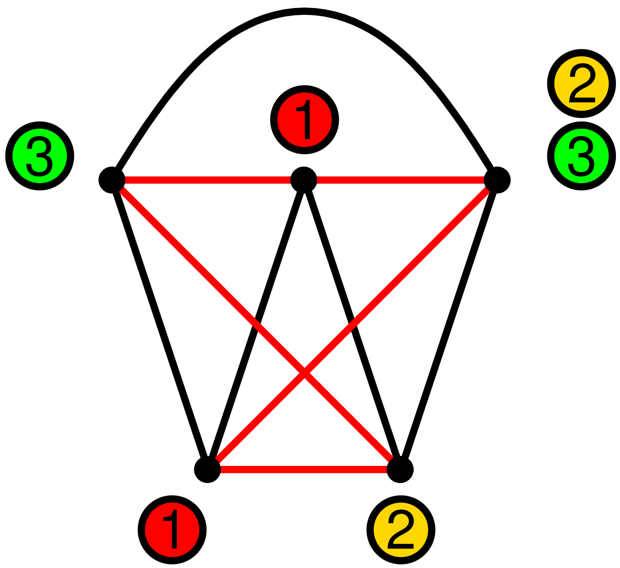


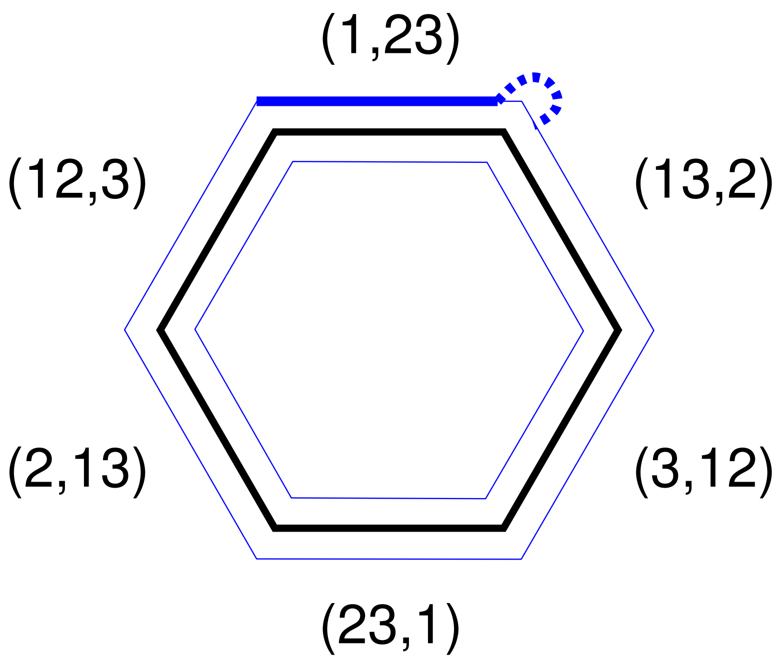

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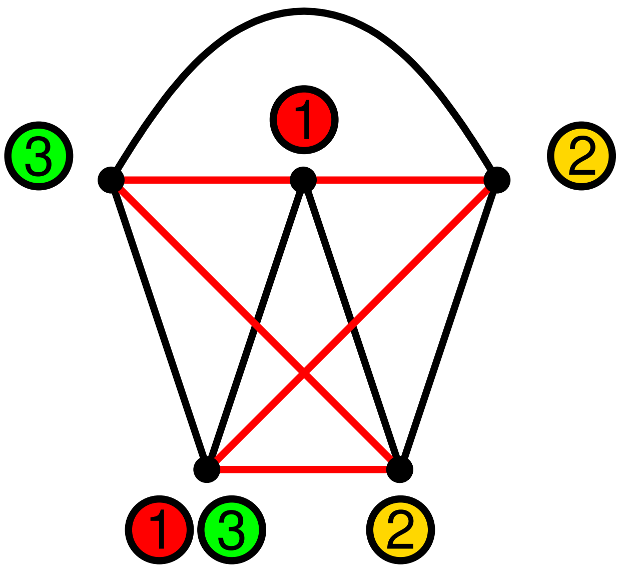


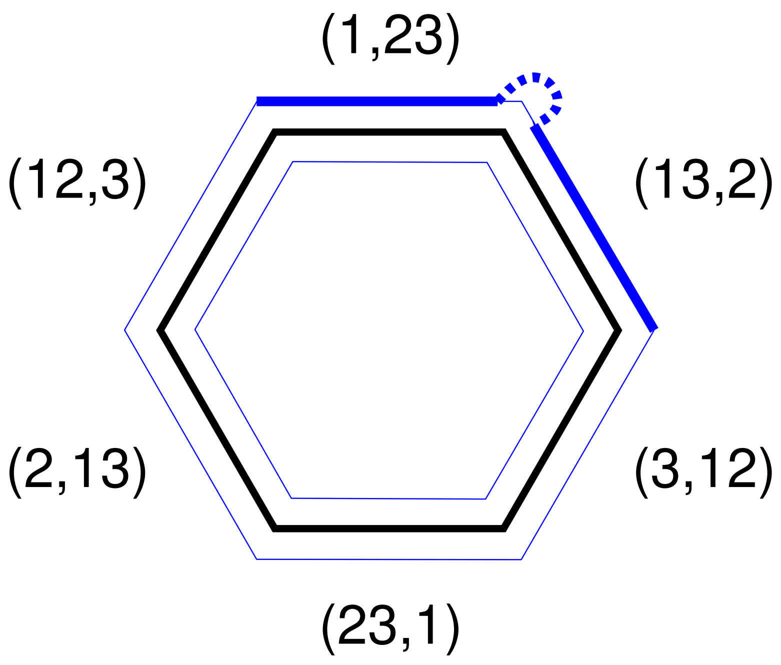

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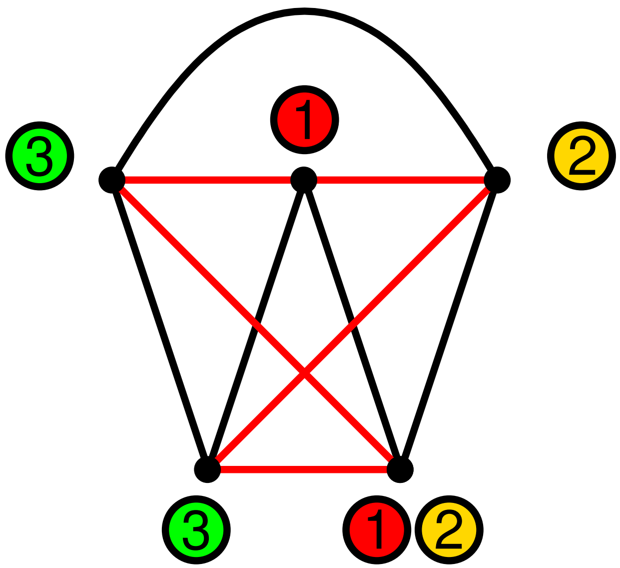


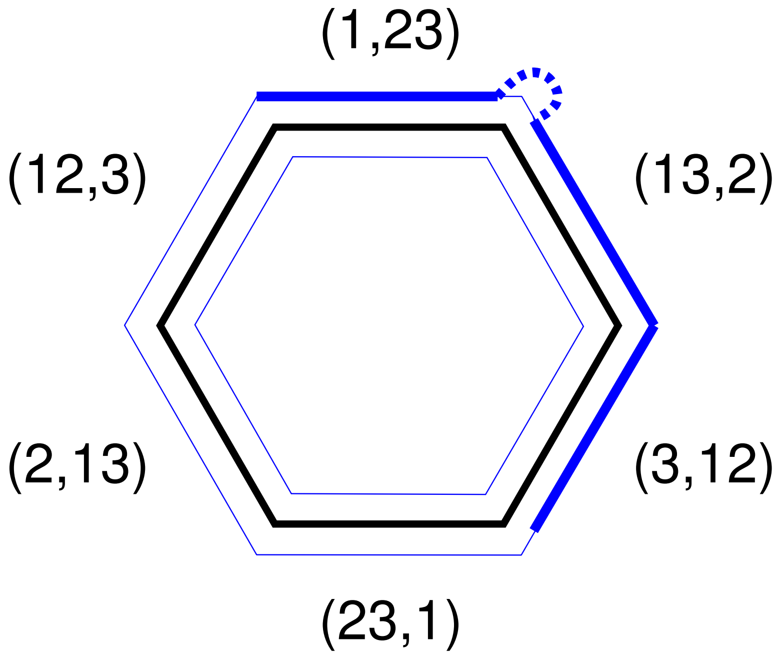

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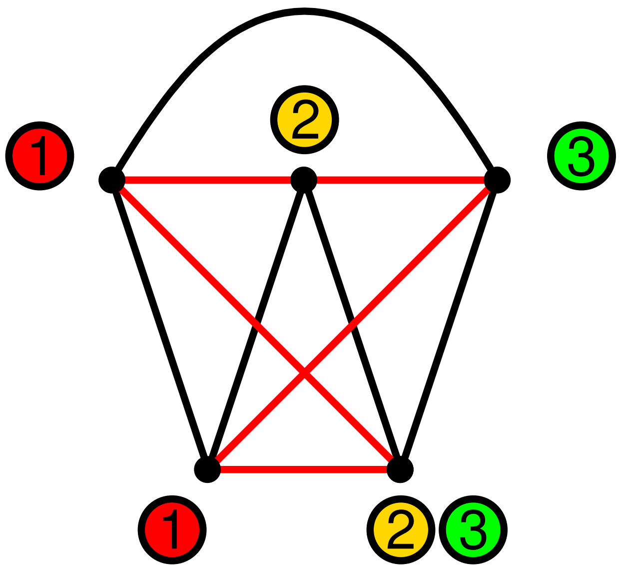


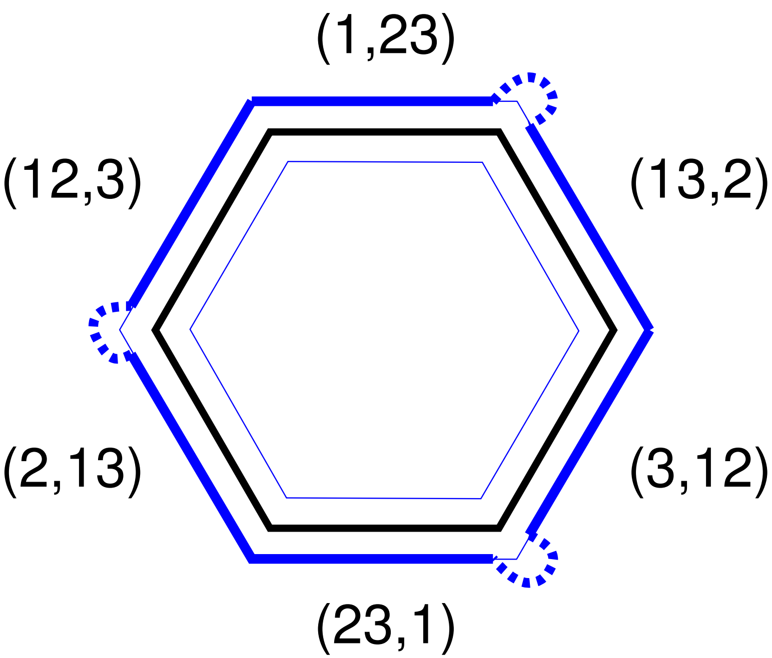

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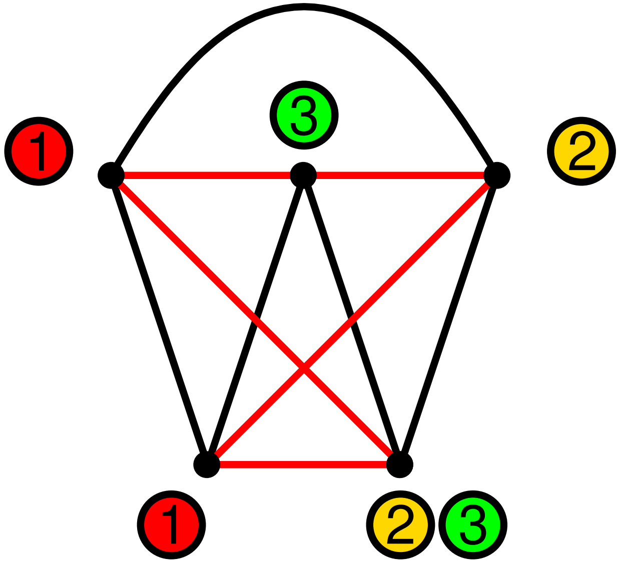


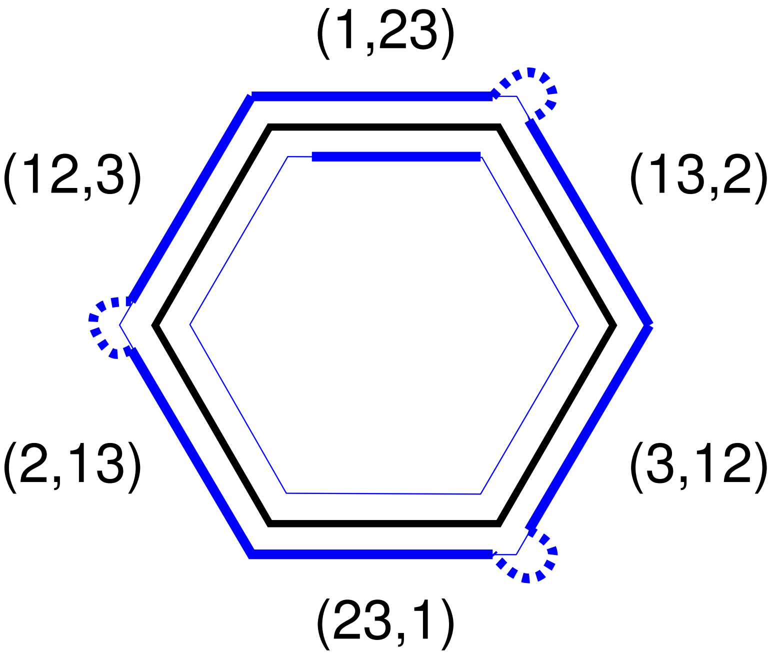

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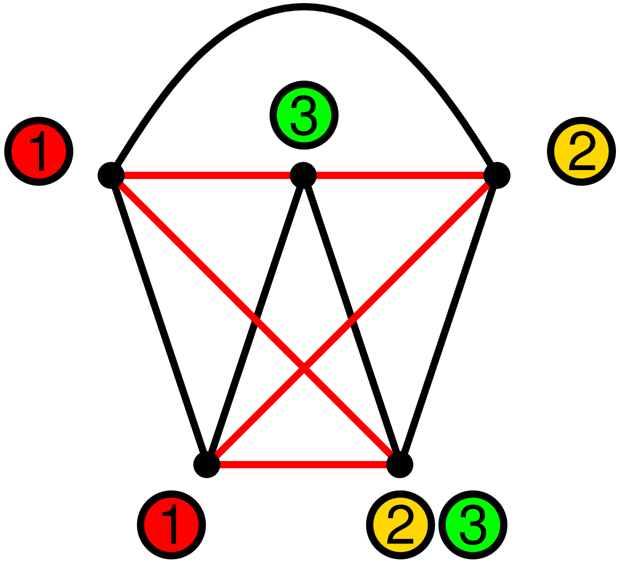
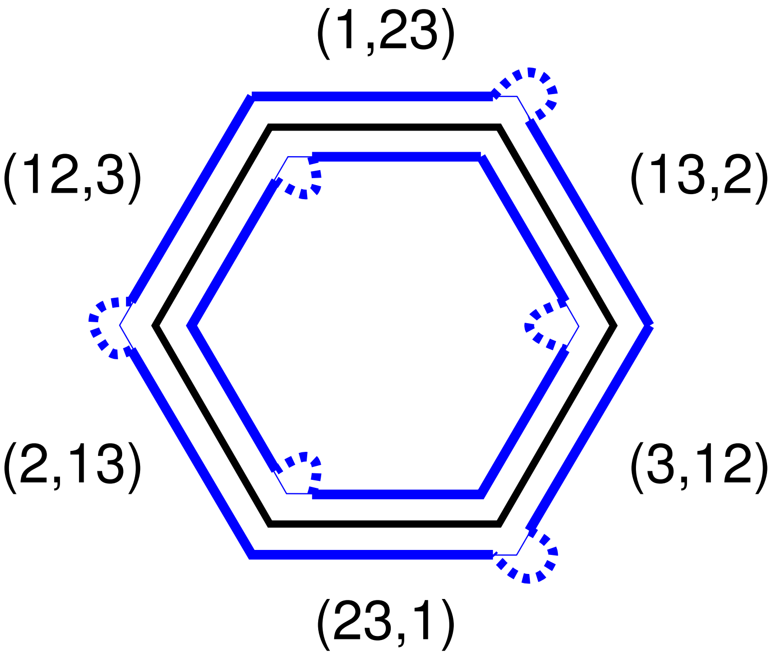
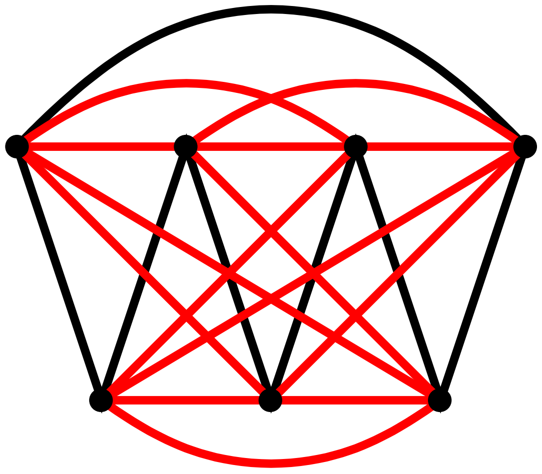


$C_7$ and $\overline{C}_7$:



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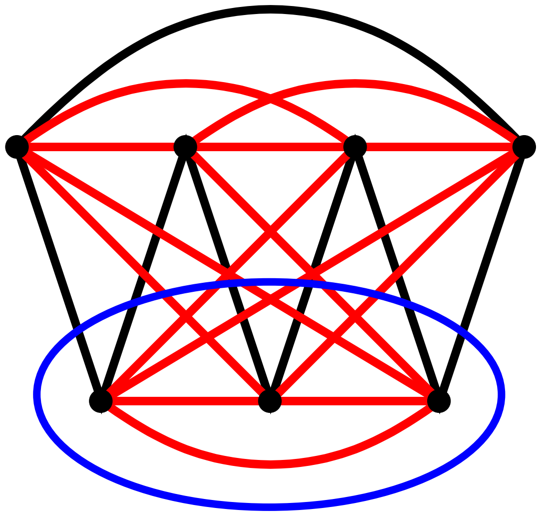
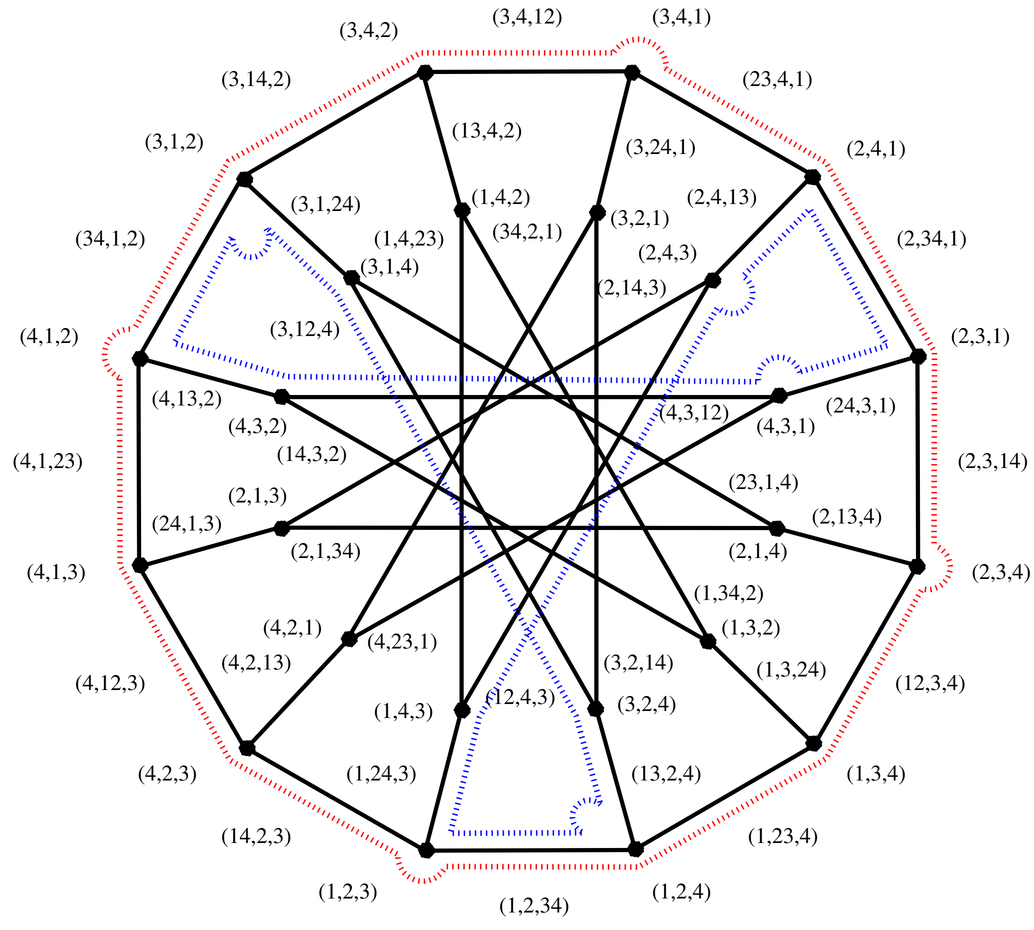


$\mathrm{Hom}(K_3, K_4)$ and $\mathrm{Hom}(\overline{C}_7, K_4)$:



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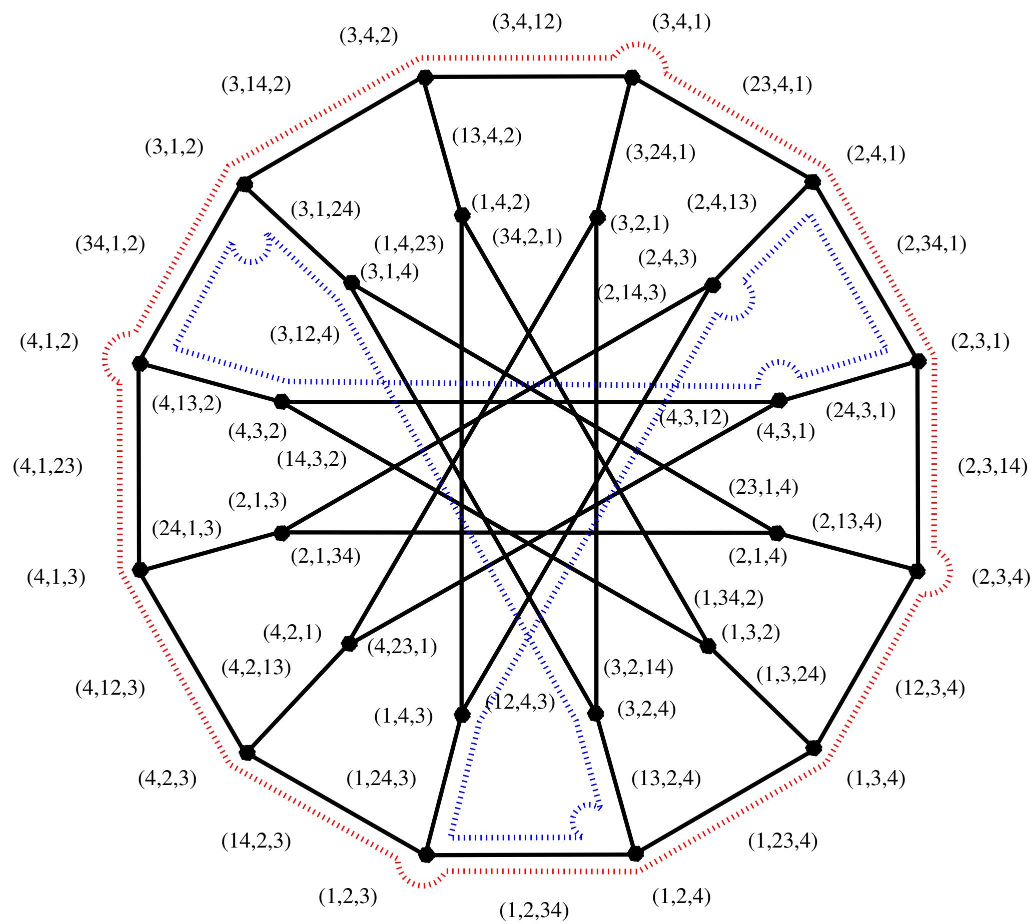


$\mathrm{Hom}(\mathrm{K}_\mathrm{r}, \mathrm{K}_{\mathrm{r}+1})$ is a graph with $(r+1)!$ vertices and $\frac{r}{2}(r+1)!$ edges.

**PROPOSITION** CSORBA & L. (2003)

$\mathrm{Hom}(\overline{\mathrm{C}}_{2r+1}, \mathrm{K}_{r+1})$ is the disjoint union of $r!$ circles with $(2r^2 + 3r + 1)$ vertices each.

**PROPOSITION** CSORBA & L. (2003)

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**CONJECTURE** L. (2003)

$H_*(\mathrm{Hom}(\overline{\mathrm{C}}_{2\mathrm{r}+1}, \mathrm{K}_{\mathrm{r}+2})) = (\mathbb{Z}, \mathbb{Z}^{2(\mathrm{r}!-2)} \oplus \mathbb{Z}_{\mathrm{r}!}, \mathbb{Z}^{2(\mathrm{r}!-2)}, \mathbb{Z})$.

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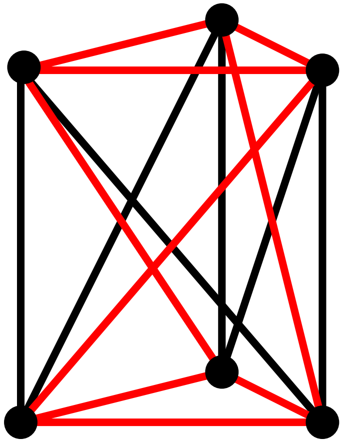
$H_*(\mathrm{Hom}(\overline{\mathrm{C}}_5, \mathrm{K}_4)) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z})$.

$H_*(\mathrm{Hom}(\overline{\mathrm{C}}_7, \mathrm{K}_5)) = (\mathbb{Z}, \mathbb{Z}^8 \oplus \mathbb{Z}_6, \mathbb{Z}^8, \mathbb{Z})$.

$C_6$ and $\overline{C}_6$:



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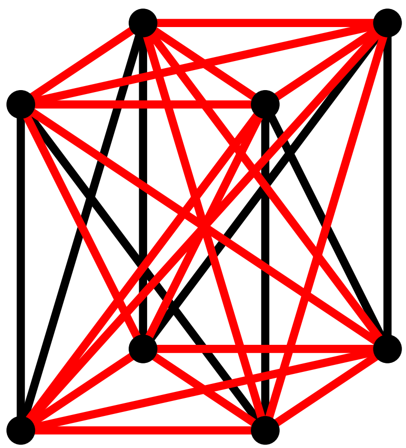

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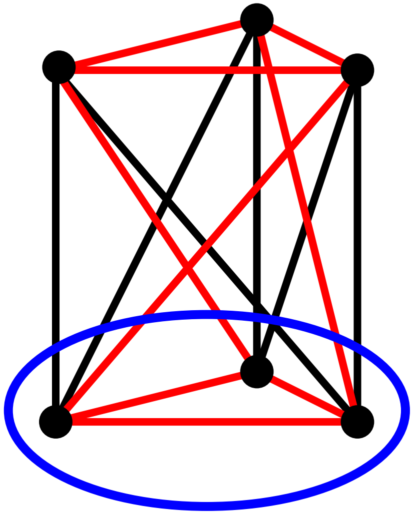

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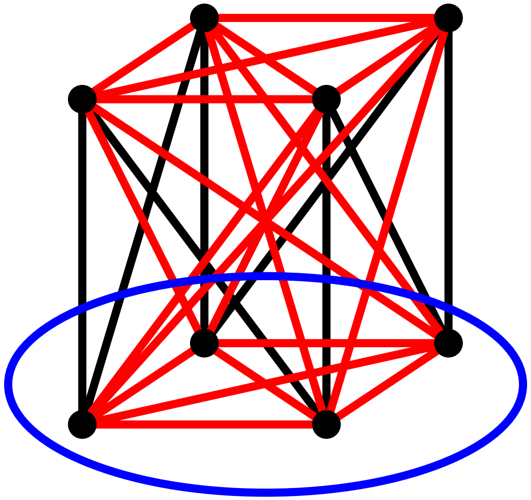

**THEOREM** BABSON & KOZLOV (2003)

$$\mathrm{Hom}(K_s, K_r) \simeq \bigvee_{f(s,r)} S^{r-s}.$$

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**PROPOSITION** BABSON & KOZLOV (2003)

The numbers $f(s,r)$ satisfy the recurrence relation

$$f(s,r) = sf(s-1,r-1) + (s-1)f(s,r-1),$$

for $r > s \geq 2$. $f(r,r) = r! - 1$, $f(1,r) = 0$ for $r \geq 1$,
and $f(s,r) = 0$ for $s > r$.

**PROPOSITION** CSORBA & L. (2004)

$\mathrm{Hom}(\overline{\mathrm{C}}_{2\mathrm{r}}, \mathrm{K}_{\mathrm{r}+1})$ is an orientable cubical surface of

of genus

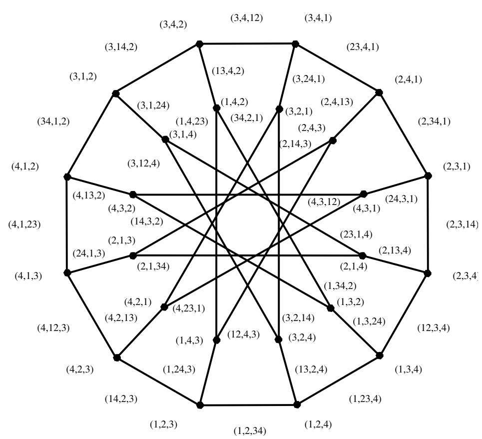
$$g(r) = f(r, r+1) = \binom{r}{2} r! - \sum_{m=1}^{r-1} m(m+2)(m+3)\cdots r$$

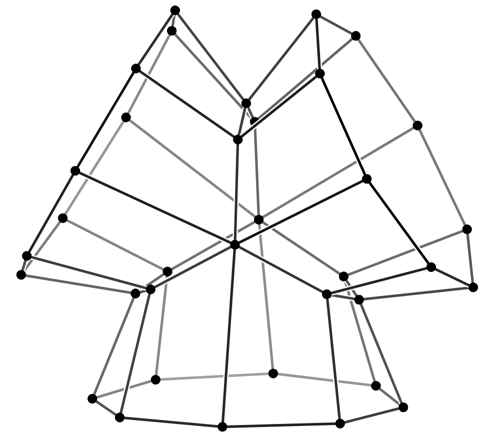
with

$n(r) := (2 + r^2) \cdot (r+1)!$ vertices,

$2(n(r) + 2g(r) - 2)$ edges,

$n(r) + 2g(r) - 2$ squares.





$$g(2) = 1,$$
$$g(3) = 13,$$
$$g(4) = 121,$$
$$g(5) = 1081,$$
$$g(6) = 10081,$$
$$g(7) = 100801,$$
$$g(8) = 1088641,$$
$$g(9) = 12700801,$$
$$g(10) = 159667201.$$

**CONJECTURE** L. (2004)

$\mathrm{Hom}(\overline{\mathrm{C}}_{2\mathrm{s}}, \mathrm{K}_\mathrm{r})$ is homeomorphic to the connected sum of $2f(s, r)$ copies of $S^{r-s} \times S^{r-s}$ for $s \geq 2$.

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CSORBA & L. (2004)

The local pieces are homeomorphic to $B^{r-s} \times S^{r-s}$.

$$H_*(\mathrm{Hom}(\overline{\mathrm{C}}_6, \mathrm{K}_4)) = (\mathbb{Z}, \mathbb{Z}^{26}, \mathbb{Z}).$$

$$H_*(\mathrm{Hom}(\overline{\mathrm{C}}_6, \mathrm{K}_5)) = (\mathbb{Z}, 0, \mathbb{Z}^{58}, 0, \mathbb{Z}).$$

**CONJECTURE** L. (2004)

$\mathrm{Hom}(\overline{\mathrm{C}}_{2\mathrm{s}}, \mathrm{K}_\mathrm{r})$ is homeomorphic to the connected sum of $2f(s, r)$ copies of $S^{r-s} \times S^{r-s}$ for $s \geq 2$.

**THEOREM** CSORBA & L. (2004)

The local pieces are homeomorphic to $B^{r-s} \times S^{r-s}$.

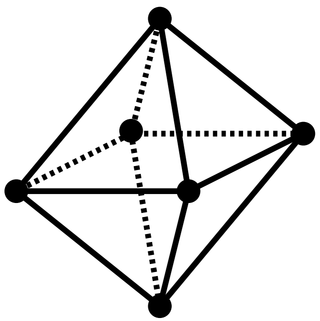
$H_*(\mathrm{Hom}(\overline{\mathrm{C}}_6, \mathrm{K}_4)) = (\mathbb{Z}, \mathbb{Z}^{26}, \mathbb{Z})$.

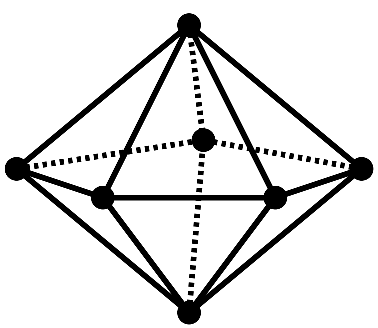
$H_*(\mathrm{Hom}(\overline{\mathrm{C}}_6, \mathrm{K}_5)) = (\mathbb{Z}, 0, \mathbb{Z}^{58}, 0, \mathbb{Z})$.

(If $\mathrm{Hom}(\overline{\mathrm{C}}_6, \mathrm{G})$ is $k$-connected, then $\chi(G) \geq k + 4$.)

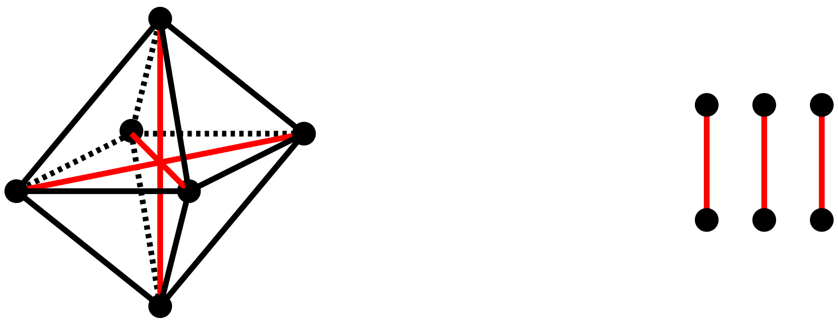
# 7. Two-dimensional flag spheres

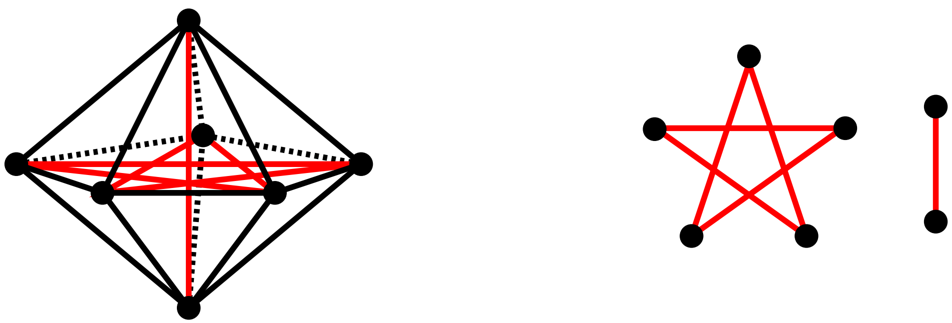
| # vertices | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| # manifolds: | 3 | 9 | 43 | 655 | 42426 |
| # spheres: | 2 | 5 | 14 | 50 | 233 |
| # flag spheres | 1 | 1 | 2 | 4 | 10 |

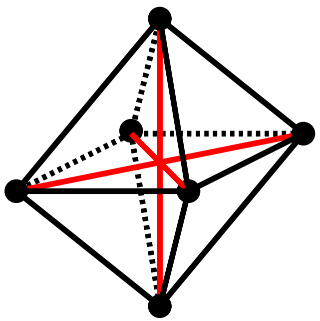


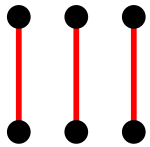


## flag 2-spheres ($n = 6, 7$)

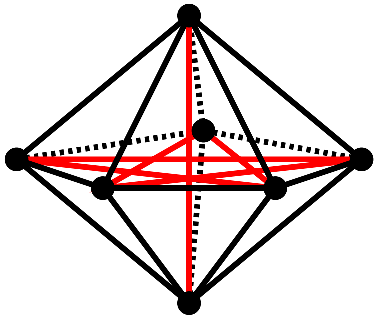


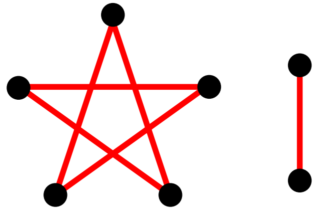




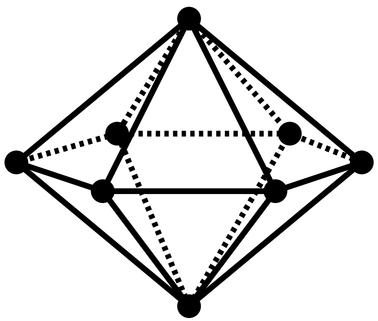


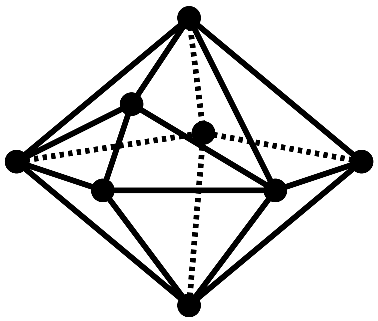
$$\mathrm{Hom}(\overline{\mathrm{SK}_1(\partial C_3^{\Delta})}, K_r) = \mathrm{Hom}(K_2, K_r) \times \mathrm{Hom}(K_2, K_r) \times \mathrm{Hom}(K_2, K_r)$$



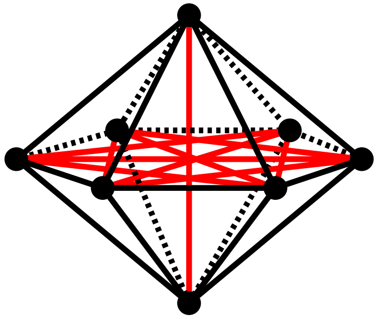


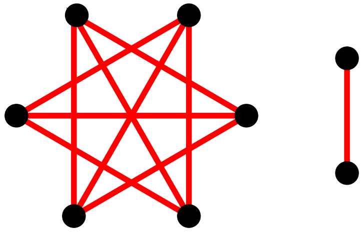
$$\mathrm{Hom}(\overline{\mathrm{SK}_1(C_5 * S^0)}, K_r) = \mathrm{Hom}(\overline{C}_5, K_r) \times \mathrm{Hom}(K_2, K_r)$$

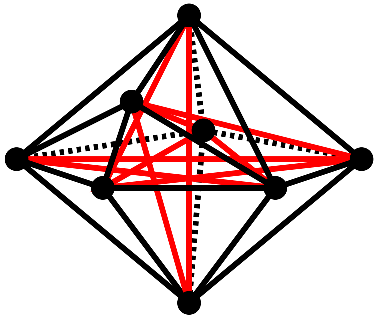


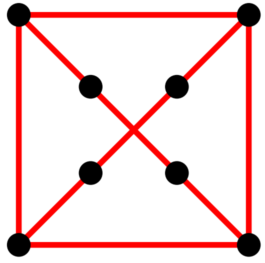


# flag 2-spheres ($n = 8$)

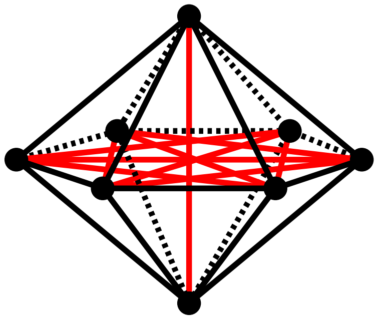


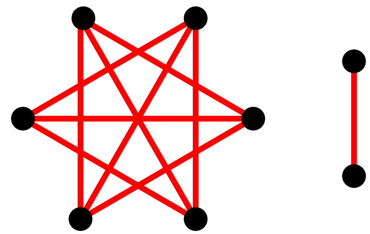




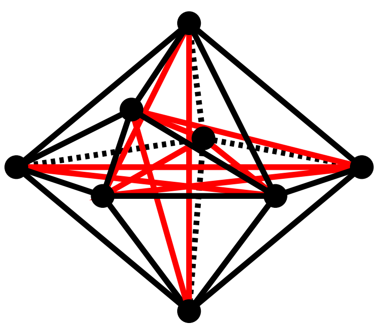


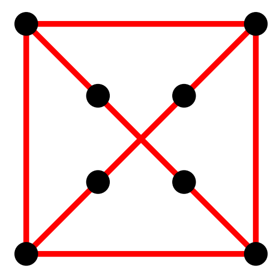
# flag 2-spheres ($n = 8$)





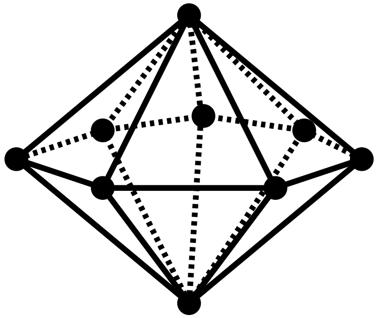
$$\mathrm{Hom}(\overline{\mathrm{SK}_1(C_6 * S^0)}, K_r) = \mathrm{Hom}(\overline{C}_6, K_r) \times \mathrm{Hom}(K_2, K_r)$$

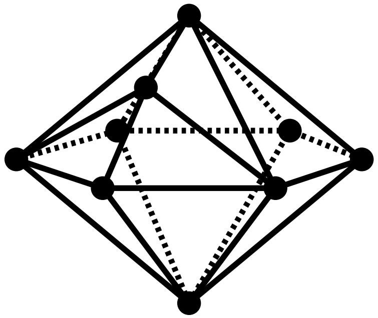




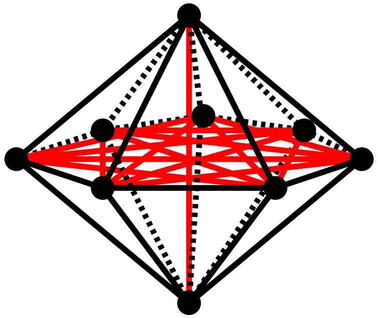
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 8_{43})}, K_3)$: $96$ vertices, $4$ circles

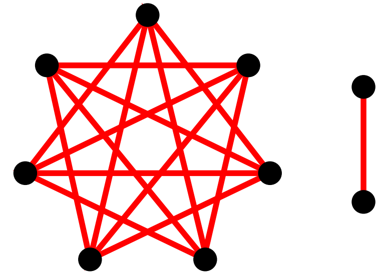
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 8_{43})}, K_4)$: $3624$ vertices, $H_* = (\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z})$

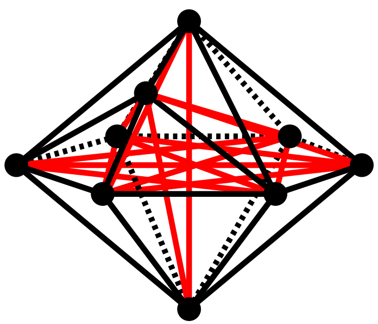


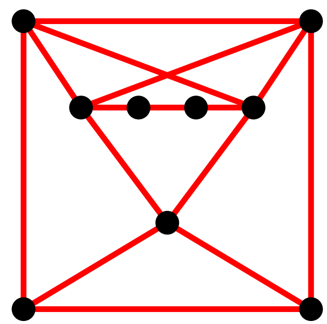


# flag 2-spheres ($n = 9$)

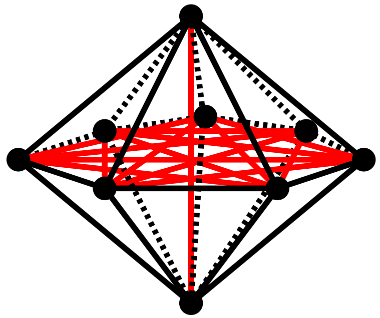


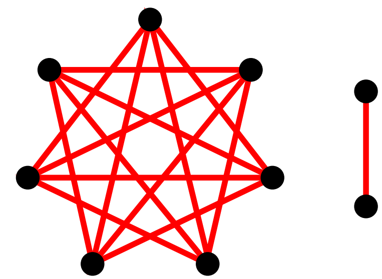




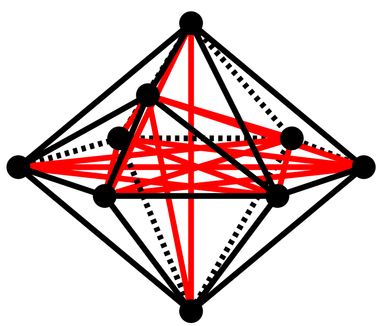


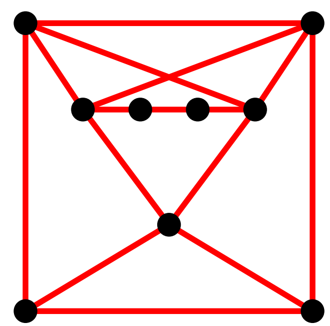
# flag 2-spheres ($n = 9$)





$$\mathrm{Hom}(\overline{\mathrm{SK}_1(C_7 * S^0)}, K_r) = \mathrm{Hom}(\overline{C}_7, K_r) \times \mathrm{Hom}(K_2, K_r)$$

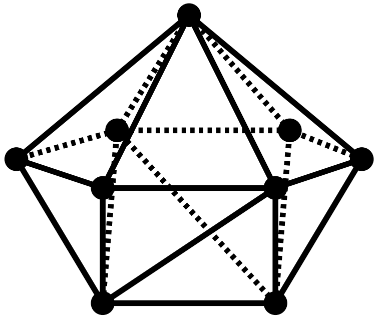


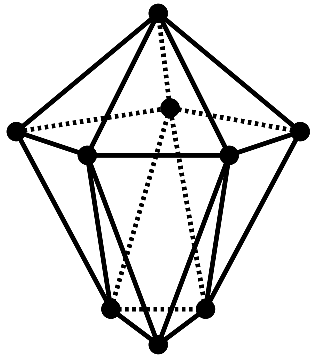


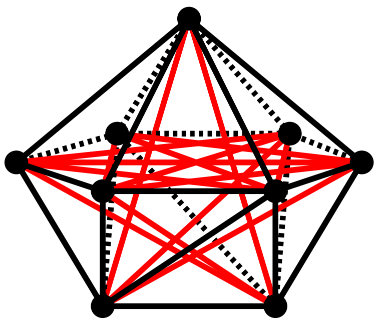
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{651})}, K_3)$:   $24$ vertices

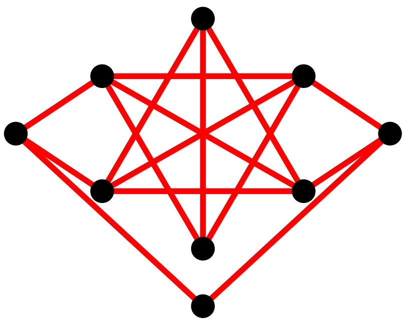
$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{651})}, K_4)$:   $2928$ vertices, $H_* = (\mathbb{Z}, \mathbb{Z}^{27}, \mathbb{Z}^{27}, \mathbb{Z})$

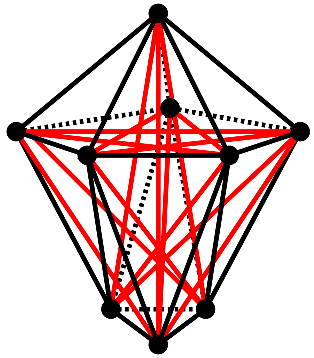
# flag 2-spheres ($n = 9$)

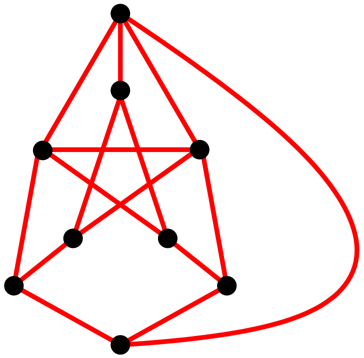


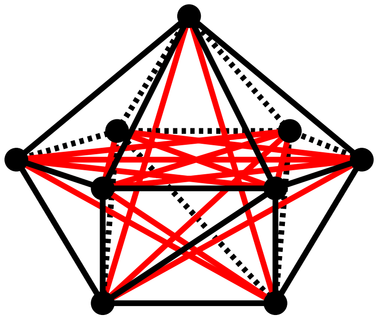


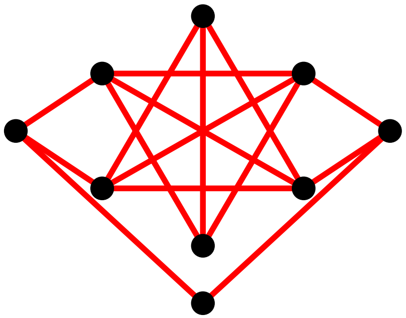






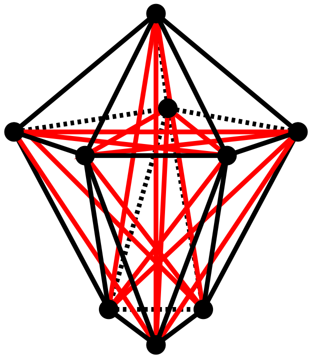


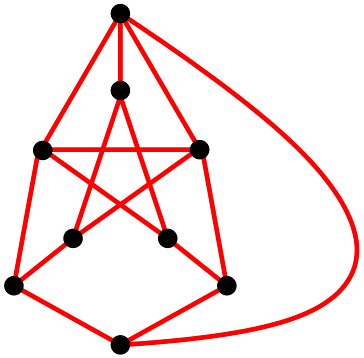




$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{652})}, K_3)$: 24 vertices

$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{652})}, K_4)$: 3120 vertices, $H_* = (\mathbb{Z}, \mathbb{Z}^{27}, \mathbb{Z}^{27}, \mathbb{Z})$





$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{655})}, K_3)$: 12 vertices

$\mathrm{Hom}(\overline{\mathrm{SK}_1({}^2 9_{655})}, K_4)$: 3096 vertices, $H_* = (\mathbb{Z}, \mathbb{Z}^{13}, \mathbb{Z}^{13}, \mathbb{Z})$

# questions

Let $S_n^d$ be a flag simplicial $d$-sphere with $n$ vertices.

**QUESTION**

What are good candidates for $S_n^d$ such that $\mathrm{Hom}(\overline{\mathrm{SK}_1(S_n^d)}, K_r)$ is highly connected?

Let $S_n^d$ be a flag simplicial $d$-sphere with $n$ vertices.

**QUESTION**

What are good candidates for $S_n^d$ such that $\mathrm{Hom}(\overline{\mathrm{SK}_1(S_n^d)}, K_r)$ is highly connected?

**QUESTION**

General properties of graph coloring manifolds?