

A Homotopic Approach to Mesh Generation

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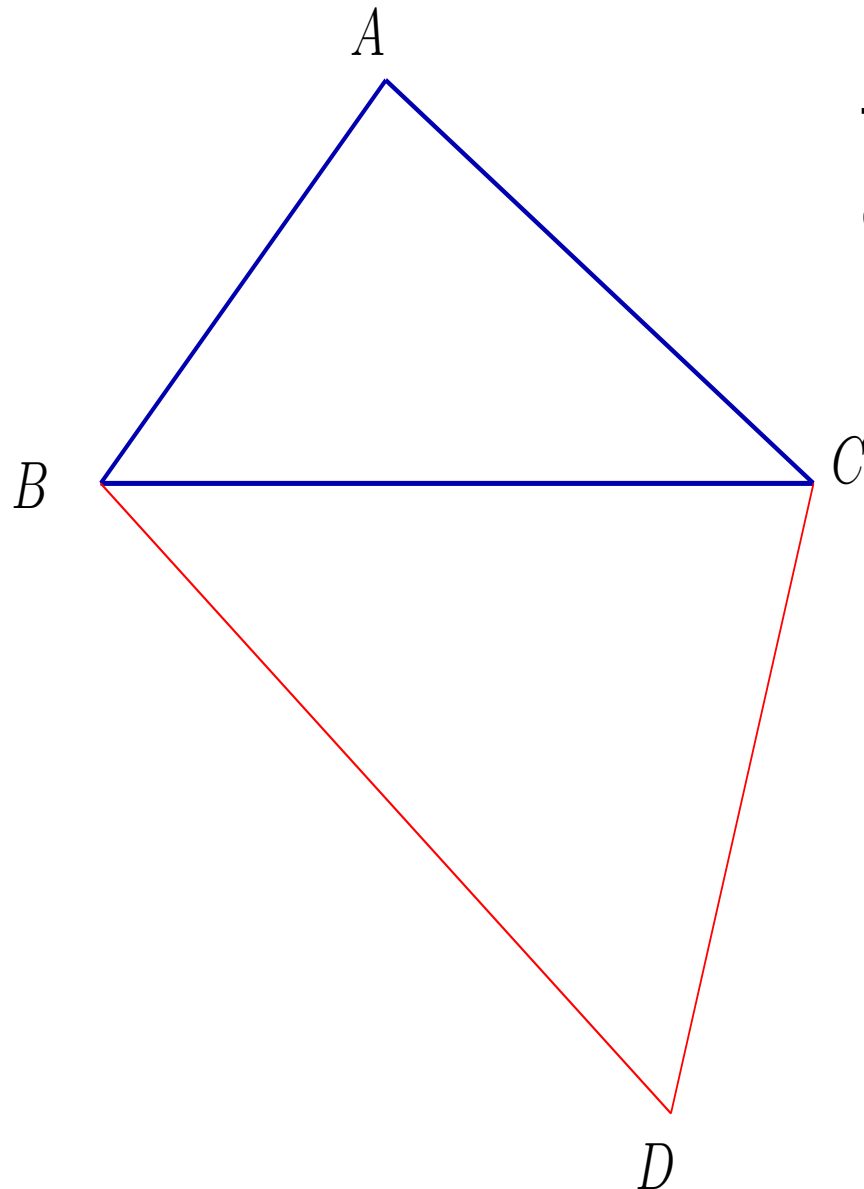
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Delaunay Triangulations

A triangulation of a region Ω (possibly non-convex and/or non-simply connected) in \mathbb{R}^2 is called **Delaunay** if the circumcircle of any triangle does not contain any other vertices in its interior.

- The dual of a Voronoi Diagram is a Delaunay Triangulation.
- Given a set of points in Ω , there is a **unique** Delaunay Triangulation having the points as vertices.
- Among all triangulations having the same set of vertices, the Delaunay Triangulation is the one with the maximum minimum angle.
- A triangulation of a convex region Ω is Delaunay if and only if it is locally Delaunay.



Then D is not in the circumdisc of $\triangle ABC$ iff:

$$\begin{vmatrix} 1 & a_1 & a_2 & a_1^2 + a_2^2 \\ 1 & b_1 & b_2 & b_1^2 + b_2^2 \\ 1 & c_1 & c_2 & c_1^2 + c_2^2 \\ 1 & d_1 & d_2 & d_1^2 + d_2^2 \end{vmatrix} > 0$$

Alternatively,

$$\angle BDC < \pi - \angle BAC.$$

Parallel Mesh Generation

Requirements for a parallel mesh generation algorithm:

- Efficiency and Scalability:
 - Small communication, synchronization, and load imbalancing costs;
- Stability:
 - distributed meshes should retain the **same quality of the elements** and partition properties as sequentially generated and partitioned meshes;
- Simple Domain decomposition:
 - Inexpensive and no new "smaller" artificial features;
- Re-use Scalar Codes:
 - It takes 5-10 years to develop sequential algorithms and libraries. Improvements are open-ended.

Overview

- A priori meshing of subproblem interfaces (Lohner89, Galtier97, Said97)
Mesh the interfaces of the subproblems and then mesh, in parallel, the individual subproblems.
- A posteriori meshing interfaces (Shephard99, Lohner99)
First solve the meshing problem in each of the subproblems in parallel, and then mesh the interfaces so that the global mesh is consistent.
- Simultaneous meshing and partitioning
of subproblems and interfaces
(Chrisochoides97 and '99)
— Objective: stability

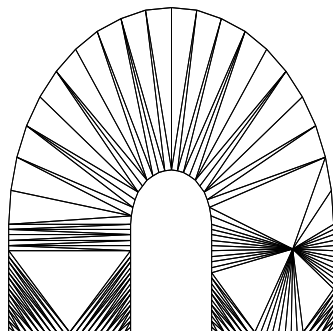
Algorithms

For parallel mesh generation, the most appropriate algorithms are **incremental algorithms**. These are algorithms that the refinement is done by adding points in the original triangulation (usually one at the time) and re-triangulating the result. The algorithm of choice is the **Bowyer–Watson Algorithm**:

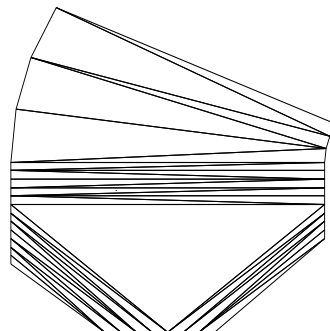
Bowyer-Watson Delaunay Kernel

- point creation: a new point is created using an appropriate spatial distribution technique
- cavity computation: triangles that violate the *empty sphere criterion* are removed
- element creation: new triangles are built by properly connecting the newly created point with boundary points of the cavity.

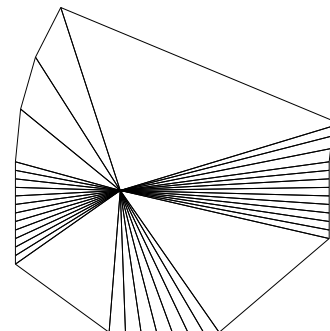
M_i Mesh



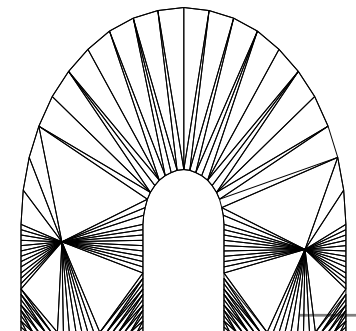
Point Insertion &
comput. of C_i



Triang. of C_i



New Mesh M_{i+1}

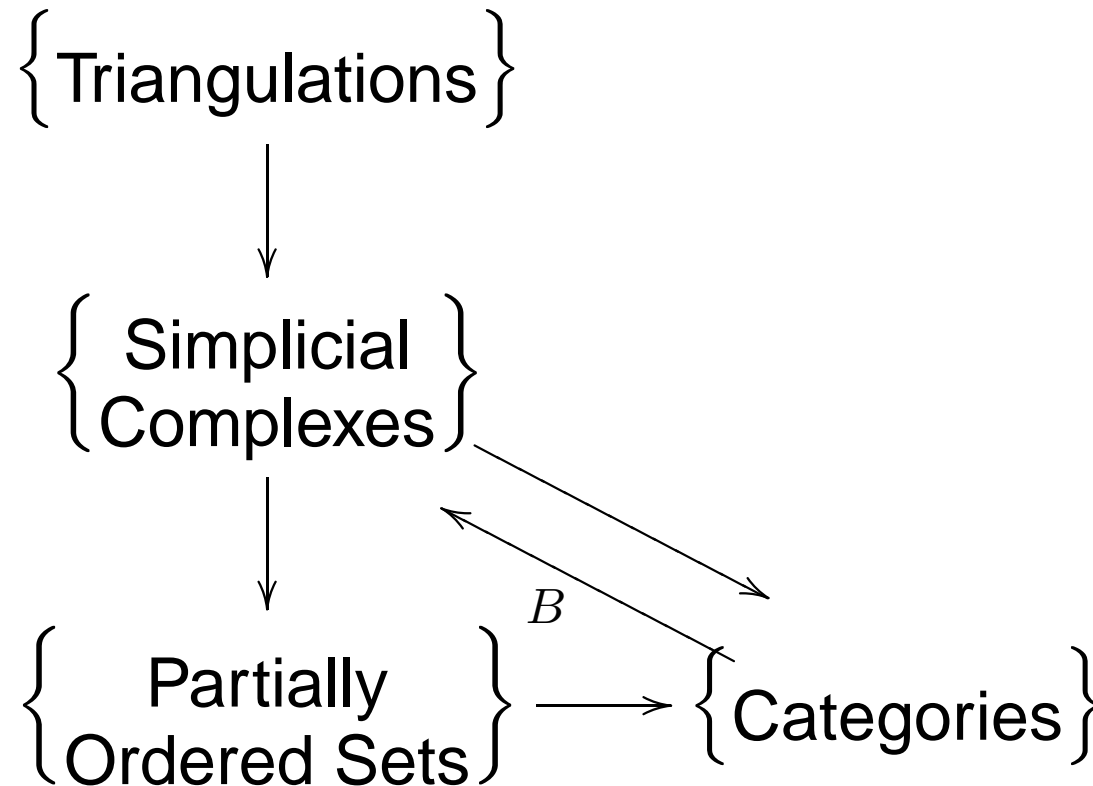


Parallel Guaranteed Quality Meshing

- Is it possible to prove termination and correctness in the context of concurrency (i.e., partial order)?
- Need mathematical framework to argue about quality and concurrency at the same time.

Categorization of Meshes

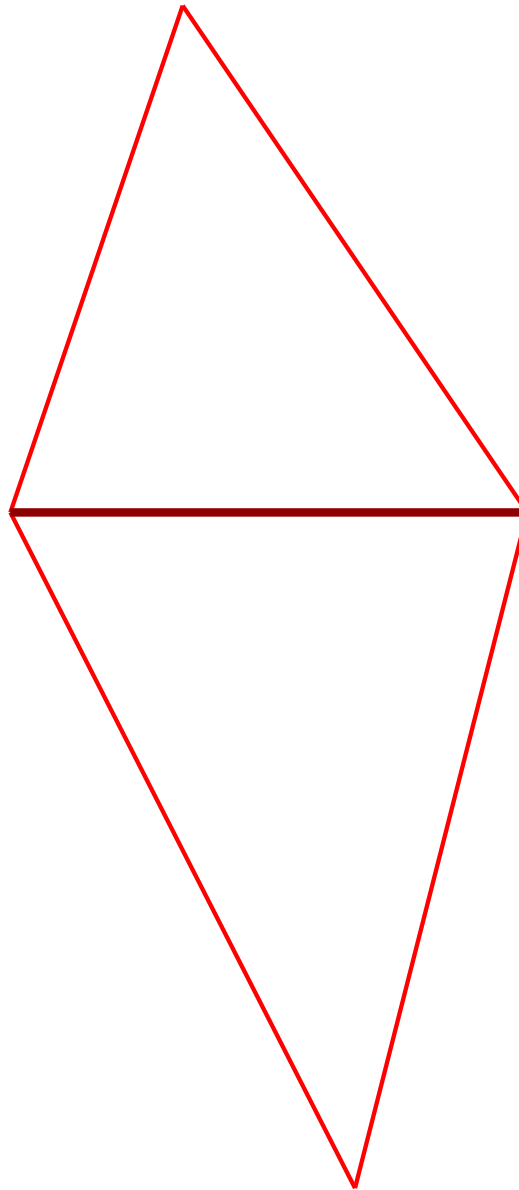
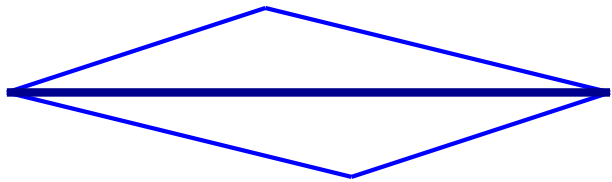
There is a naive correspondence



The partial order structure in a triangulation is given by the obvious:

$$\text{vertex} \leq \text{edge} \leq \text{face}$$

The problem is that such a simplistic approach does not capture the geometric properties of the triangulation. It captures only its topological properties.



Let \mathcal{IsoRep} be the category of pairs (G, ρ) where

- G is a group,
- $\rho : G \rightarrow \text{Iso}(\mathbb{R}^2)$ a faithful representation of G into the group of the Euclidean Isometries of \mathbb{R}^2 .

We write $\mathcal{FP}\mathcal{IsoRep}$ the category of finite products of objects of \mathcal{IsoRep} .

Let \mathcal{P} be the partially ordered set of the elements of a triangulations, considered as a category.

A **mesh generation** is a functor:

$$\mathbb{F} : \mathcal{P}^{op} \rightarrow \mathcal{FPIsoRep}$$

such that:

- For each $v \in V$ of degree d_v , $\mathbb{F}(v) = (G, \rho)$, where

$$G = \prod_{i=1}^{d_v-1} G_i, \quad \rho = \prod_{i=1}^{d_v-1} \rho_i$$

and each G_i is a dihedral group and $\text{Im}(\rho_i)$ is the dihedral group generated by two reflections. All the axes of the reflections intersect at the same point.

- For each $e \in E$, $\mathbb{F}(e) = (\mathbb{Z}/2\mathbb{Z}, \rho)$, where ρ maps the non-trivial element to a reflection about an axis ℓ_x .
- The image of a face under \mathbb{F} is the trivial group.

On the **morphisms**:

If v is an vertex of an edge e , then $\mathbb{F}(e \geq v)$ is induced by the inclusion map.

With the extra structure given on \mathcal{P} , the functor \mathbb{F} induces a map:

$$\phi : |\mathcal{P}^{op}| = \mathbf{hocolim}_{\mathcal{P}^{op}*} \rightarrow \mathbb{R}^2$$

that maps vertices to points. The image of the map is a triangulation.

Conversely, given a pair $(\mathcal{P}, \mathbb{F})$ so that \mathcal{P} is a partially ordered set such that

- The geometric realization $|\mathcal{P}|$ is a **triangulation** of a planar region.
- The triangles determined by \mathbb{F} are **Euclidean**.

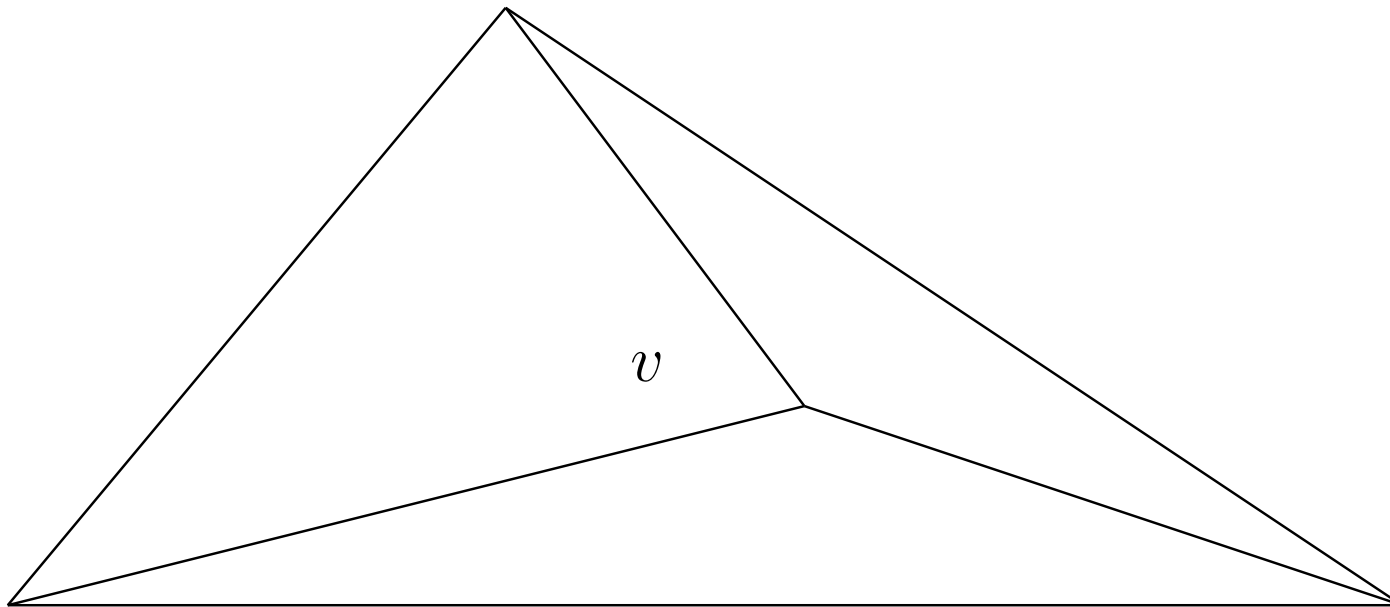
Categorical Description of the B–W

Let $(\mathcal{P}_0, \mathbb{F}_0)$ be a Delaunay Mesh.

(1) **Insert a point.** Change

$$V(\mathcal{P}) \longrightarrow V(\mathcal{P}) \cup \{v\}$$

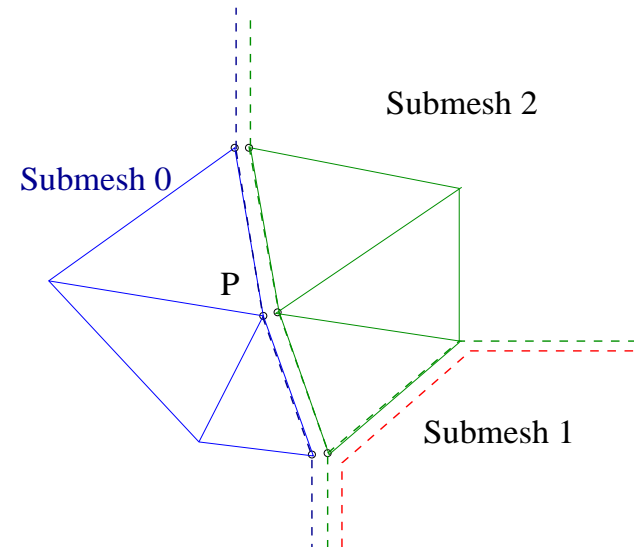
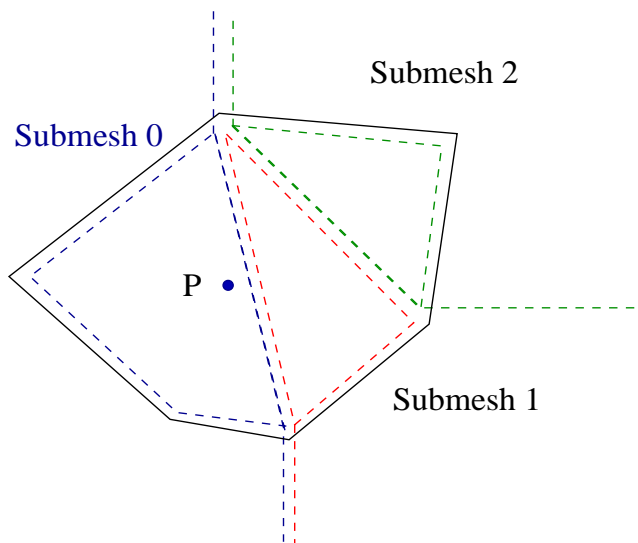
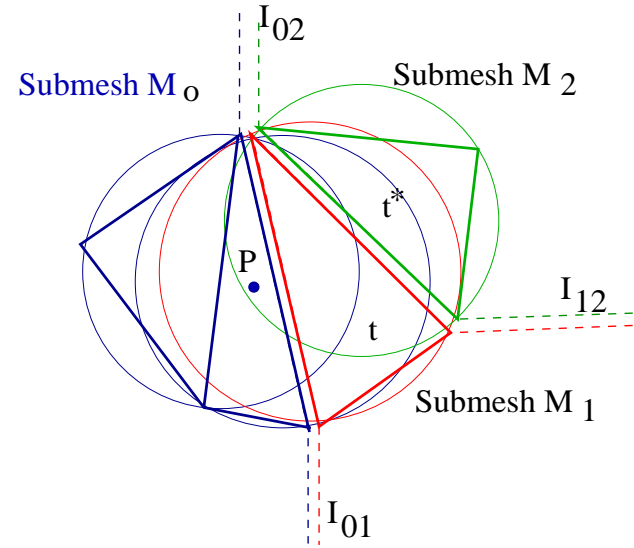
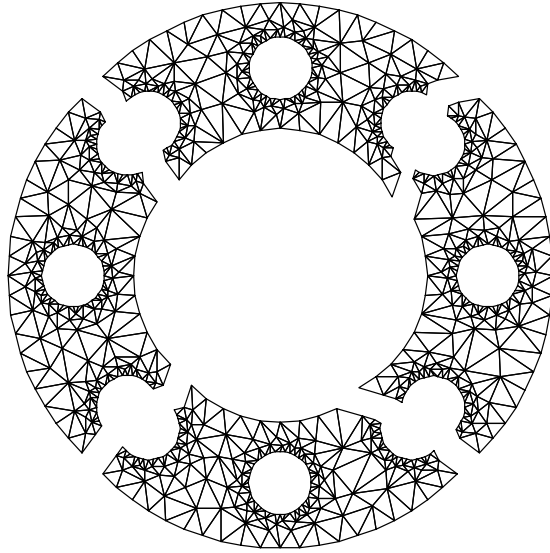
That is determined by a **triangle** and **two dihedral groups**



- (2) **Determine the cavity C_v** . That is done using the angle criterion to determine the triangles of the cavity.
- (3) **“Remove” the cavity**. The result is a partially ordered set \mathcal{P}' and a functor \mathbb{F}' defined on all the vertices and edges, with $\mathbb{F}'(v)$ the trivial group.
- (4) **“Re-triangulate” the ball of v** . That is done by **adding the edges** in the cavity of v i.e., determining the corresponding dihedral angles. That determines the next step $(\mathcal{P}_1, \mathbb{F}_1)$.

All the calculations involve solving systems of linear equations.

SMGP



Extension to Dim. 3

In this case, $(\mathcal{P}, \mathbb{F})$ is a pair such that:

- The geometric realization $|\mathcal{P}|$ is a tetrahedration of a body in \mathbb{R}^3 .
- The functor \mathbb{F} assigns:

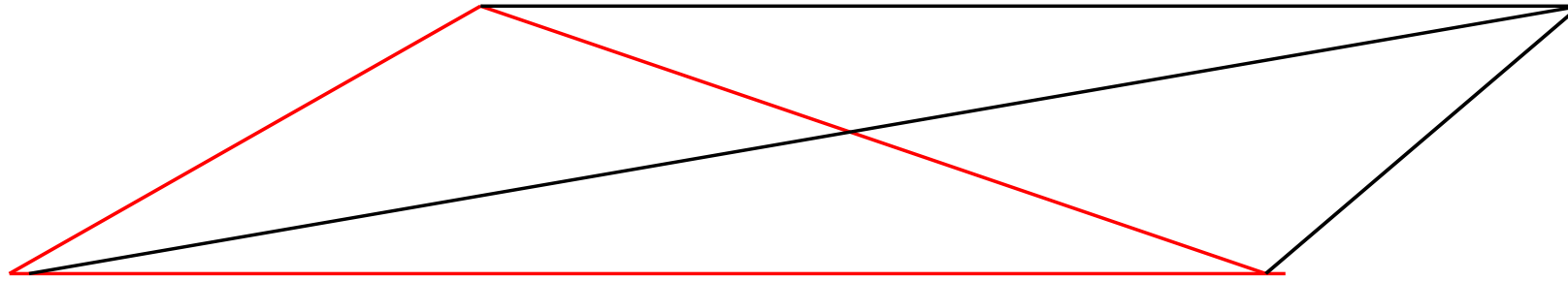
vertices \longrightarrow products of dihedral groups,

edges \longrightarrow dihedral group,

faces $\longrightarrow \mathbb{Z}/2\mathbb{Z}$,

tetrahedra \longrightarrow trivial group.

Slivers

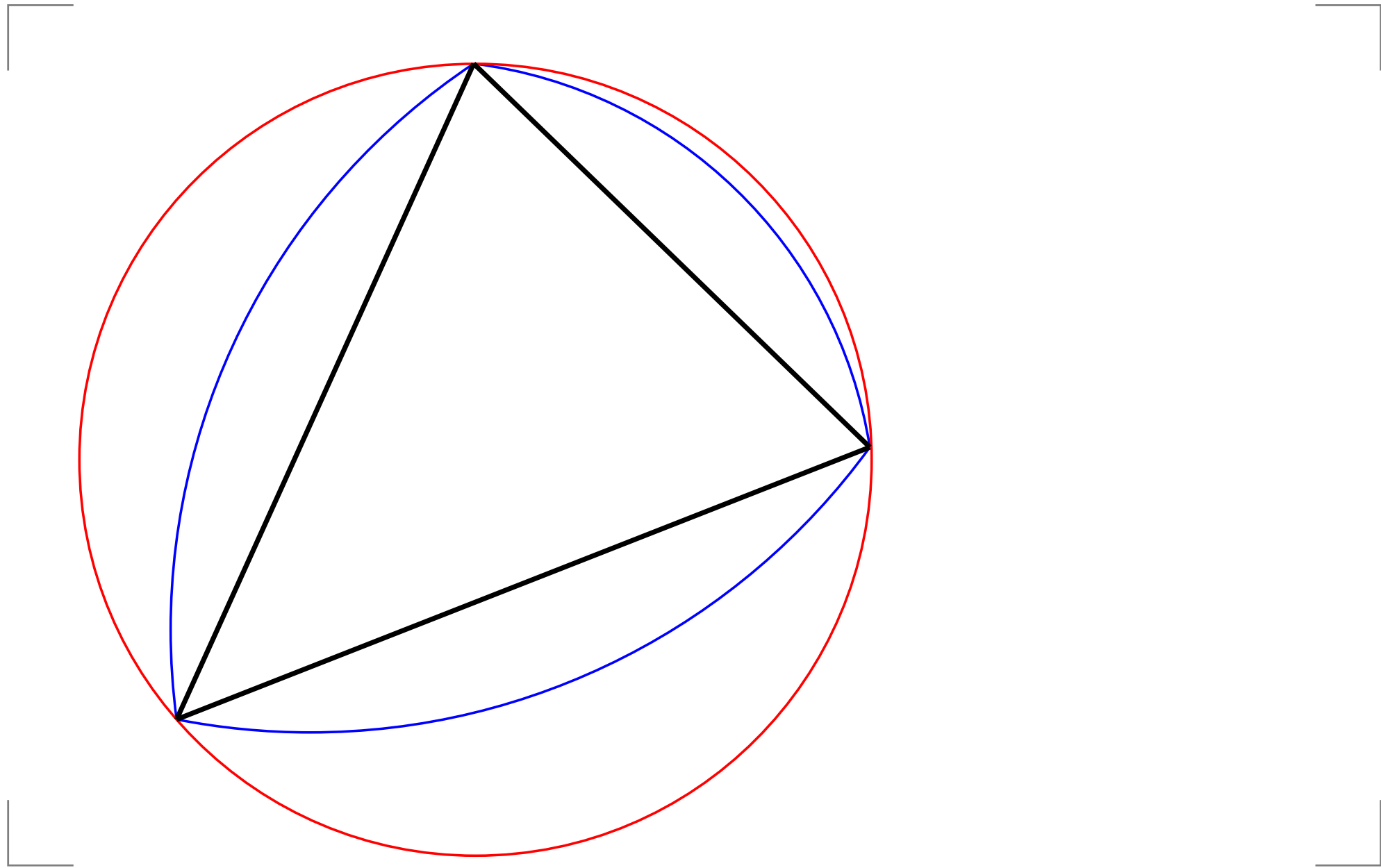


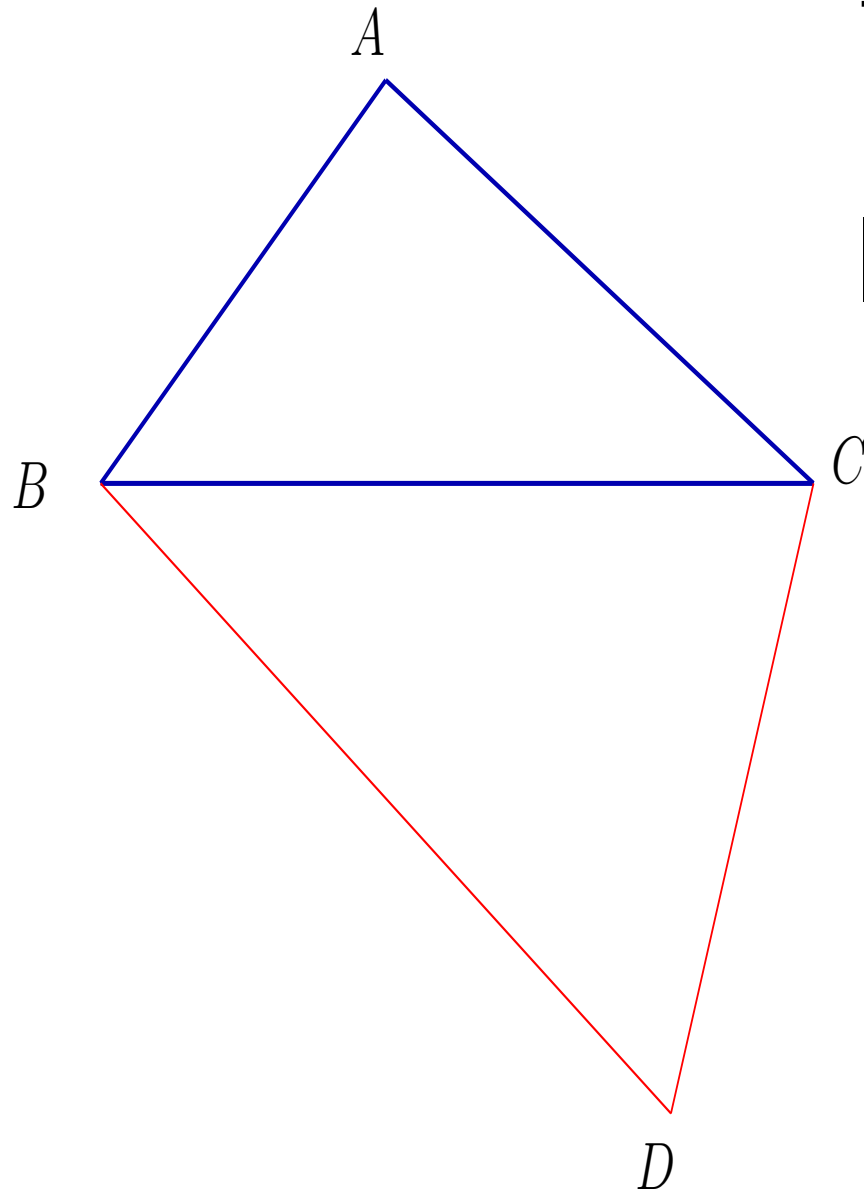
- Distance from a vertex to the opposite plane is small.
- The **radius-edge ratio** and the **volume-cube of edge ratio** are both small. The **aspect ratio** (circumradius over inradius) is large.
- One **dihedral angle** is very large.

Floating Point Issues

Fortune suggested a modification to deal with floating point arithmetic. He suggested the notion of α -pseudosphere of a triangle for some $\alpha > 0$ and he relaxed the Delaunay condition to:

The α -pseudosphere of every triangle is empty.





Then D is not in the α -pseudodisc of $\triangle ABC$ iff:

$$\angle BDC < \pi - \angle BAC + \alpha.$$

Single Space Description

Alternatively, we can “describe” the triangulation as a functor:

$$\mathbb{F} : \mathcal{P}^{op} \rightarrow \mathcal{FPSub}(O(2))$$

to the category of finite products of subgroups of the group of linear isometries of \mathbb{R}^2 .

Composing with a *classifying* space functor:

$$\mathcal{P}^{op} \xrightarrow{\mathbb{F}} \mathcal{FPSub}(O(2)) \hookrightarrow \mathit{Groups} \xrightarrow{\mathcal{B}} \mathit{Spaces}$$

where $\mathcal{B}(G)$ is the **metric space** \mathbb{R}^2/G with the induced metric.

The mesh space is defined to be the **metric space**:

$$\mathcal{T}(\mathcal{P}, \mathbb{F}) = \text{hocolim}_{\mathcal{P}^{op}} (\mathcal{B} \circ \mathbb{F}).$$

For a functor:

$$\Phi : \mathcal{P}^{op} \rightarrow \mathcal{S}paces,$$

the homotopy colimit is defined as:

$$\text{hocolim}_{\mathcal{P}^{op}} \Phi = \coprod_{\sigma \in |\mathcal{P}^{op}|} \sigma \times \Phi(\sigma) / \sim, \quad (t, x) \sim (t, \Phi(x)),$$

for $t \in \tau$, $x \in \Phi(\sigma)$, and $\sigma \geq \tau$,

Similarity Structures

Rivin used angle assignments to study the **Realization Problem**:

When is a angle assignment to a triangulation of a singular surface can be realized by a **similarity structure** on the surface?

He showed that if an angle assignment satisfies certain obvious necessary conditions and it is **Delaunay** then it is realizable.

Furthermore, similar methods were used in calculations of **moduli spaces** of such structures.

In this case, the functor \mathbb{F} takes values in the category of **similarities** of the Euclidean Space.

Hyperbolic Triangulations

Thurston used **disc patterns** to study uniform structures on surfaces (**Andreev–Thurston Theorem**).

Leibon (extending Thurston's work) studied the discrete analogue and:

- Characterized the uniform structure of hyperbolic surfaces using **Delaunay Triangulations**.
- Proved the **Discrete Andreev–Thurston Theorem**.
- Proved the **Gauss–Bonnet Theorem**.
- Derived the continuous theorems as limits of the discrete ones.

In this case, the functor \mathbb{F} takes values in the category of **hyperbolic isometries**.