

## Lecture 02 (January 18, 2009)

### 5 Grothendieck topologies

A *Grothendieck site* is a small category  $\mathcal{C}$  equipped with a topology  $\mathcal{T}$ .

A *Grothendieck topology*  $\mathcal{T}$  consists of a collection of subfunctors

$$R \subset \text{hom}(\_, U), \quad U \in \mathcal{C},$$

called *covering sieves*, such that the following axioms hold:

- 1) (base change) If  $R \subset \text{hom}(\_, U)$  is covering and  $\phi : V \rightarrow U$  is a morphism of  $\mathcal{C}$ , then the subfunctor

$$\phi^{-1}(R) = \{\gamma : W \rightarrow V \mid \phi \cdot \gamma \in R\}$$

is covering for  $V$ .

- 2) (local character) Suppose that  $R, R' \subset \text{hom}(\_, U)$  are subfunctors and  $R$  is covering. If  $\phi^{-1}(R')$  is covering for all  $\phi : V \rightarrow U$  in  $R$ , then  $R'$  is covering.
- 3)  $\text{hom}(\_, U)$  is covering for all  $U \in \mathcal{C}$

Typically Grothendieck topologies arise from covering families in sites  $\mathcal{C}$  having pullbacks. Covering families are sets of functors which generate covering sieves.

Suppose that  $\mathcal{C}$  has pullbacks. A topology  $\mathcal{T}$  on  $\mathcal{C}$  consists of families of sets of morphisms

$$\{\phi_\alpha : U_\alpha \rightarrow U\}, \quad U \in \mathcal{C},$$

called *covering families*, such that the following axioms hold:

- 1) Suppose that  $\phi_\alpha : U_\alpha \rightarrow U$  is a covering family and that  $\psi : V \rightarrow U$  is a morphism of  $\mathcal{C}$ . Then the collection  $V \times_U U_\alpha \rightarrow V$  is a covering family for  $V$ .
- 2) If  $\{\phi_\alpha : U_\alpha \rightarrow V\}$  is covering, and  $\{\gamma_{\alpha,\beta} : W_{\alpha,\beta} \rightarrow U_\alpha\}$  is covering for all  $\alpha$ , then the family of composites

$$W_{\alpha,\beta} \xrightarrow{\gamma_{\alpha,\beta}} U_\alpha \xrightarrow{\phi_\alpha} U$$

is covering.

- 3) The family  $\{1 : U \rightarrow U\}$  is covering for all  $U \in \mathcal{C}$ .

## Examples:

- 1)  $X =$  topological space.  $\text{op}|_X$  is the poset of open subsets  $U \subset X$ . A covering family for an open subset  $U$  is an open cover  $V_\alpha \subset U$ .
- 2)  $X =$  topological space.  $\text{loc}|_X$  is the category of all maps  $f : Y \rightarrow X$  which are local homeomorphisms.  $f$  is a local homeomorphism if each  $x \in Y$  has a neighbourhood  $U$  such that  $f(U)$  is open in  $X$  and the restricted map  $U \rightarrow f(U)$  is a homeomorphism. A morphism of  $\text{loc}|_X$  is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ & \searrow f & \swarrow f' \\ & X & \end{array}$$

where  $f$  and  $f'$  are local homeomorphisms. A family  $\{\phi_\alpha : Y_\alpha \rightarrow Y\}$  of local homeomorphisms (over  $X$ ) is covering if  $X = \cup \phi_\alpha(Y_\alpha)$ .

- 3)  $X =$  a scheme (topological space with sheaf of rings locally isomorphic to affine schemes  $\text{Sp}(R)$ ). The underlying topology on  $X$  is the Zariski topology.  $\text{Zar}|_X$  is the poset with objects all open subschemes  $U \subset X$ . A family  $V_\alpha \subset U$  is covering if  $\cup V_\alpha = U$  (as sets).

A scheme homomorphism  $\phi : Y \rightarrow X$  is *étale at*  $y \in Y$  if

- a)  $\mathcal{O}_y$  is a flat  $\mathcal{O}_{f(y)}$ -module ( $\phi$  is flat at  $y$ ).
- b)  $\phi$  is unramified at  $y$ :  $\mathcal{O}_y/\mathcal{M}_{f(y)}\mathcal{O}_y$  is a finite separable field extension of  $k(f(y))$ .

Say that a map  $\phi : Y \rightarrow X$  is *étale* if it is étale at every  $y \in Y$  (and locally of finite type).

- 4)  $S =$  scheme. The étale site  $et|_S$  has as objects all étale maps  $\phi : V \rightarrow S$  and all diagrams

$$\begin{array}{ccc} V & \longrightarrow & V' \\ & \searrow \phi & \swarrow \phi' \\ & & S \end{array}$$

for morphisms (with  $\phi, \phi'$  étale). A covering family for the étale site is a collection of étale morphisms  $\phi_\alpha : V_\alpha \rightarrow V$  such that  $V = \bigcup \phi_\alpha(V_\alpha)$  as a set. Equivalently every morphism  $\text{Sp}(\Omega) \rightarrow V$  lifts to some  $V_\alpha$  if  $\Omega$  is a separably closed field.

- 5) The Nisnevich site  $Nis|_S$  has the same underlying category as the étale site, namely all étale maps  $V \rightarrow S$  and morphisms between them. A Nisnevich cover is a family of étale maps  $V_\alpha \rightarrow V$  such that every morphism  $\text{Sp}(K) \rightarrow V$  lifts to some  $V_\alpha$  where  $K$  is any field.

- 6) A flat covering family of a scheme  $S$  is a set of flat morphisms  $\phi_\alpha : S_\alpha \rightarrow S$  (ie. morphisms which are flat at each point) such that  $S = \bigcup \phi_\alpha(S_\alpha)$  as a set (equivalently  $\bigsqcup S_\alpha \rightarrow S$  is faithfully flat).  $(Sch|_S)_{fl}$  is the “big” flat site. Pick a large cardinal  $\kappa$ ; then  $(Sch|_S)$  is the category of  $S$ -schemes  $X \rightarrow S$  such that the cardinality of both the underlying point set of  $X$  and all sections  $\mathcal{O}_X(U)$  of its sheaf of rings are bounded above by  $\kappa$ .
- 7) There are corresponding big sites  $(Sch|_S)_{Zar}$ ,  $(Sch|_S)_{et}$ ,  $(Sch|_S)_{Nis}$ , ... and you can play similar games with topological spaces.
- 8) Suppose that  $G = \{G_i\}$  is profinite group such that all  $G_j \rightarrow G_i$  are surjective group homomorphisms. Write also  $G = \varprojlim G_i$ . A discrete  $G$ -set is a set  $X$  with  $G$ -action which factors through an action of  $G_i$  for some  $i$ . Write  $G - \mathbf{Set}_{df}$  for the category of  $G$ -sets which are both discrete and finite. A family  $U_\alpha \rightarrow X$  in this category is covering if and only if  $\bigsqcup U_\alpha \rightarrow X$  is surjective.
- 9) Suppose that  $\mathcal{C}$  is any small category. Say that  $R \subset \text{hom}(\_, x)$  is covering if and only if  $1_x \in R$ .

This is the *chaotic topology* on  $\mathcal{C}$ .

- 10) Suppose that  $\mathcal{C}$  is a site and that  $U \in \mathcal{C}$ . Then the slice category  $\mathcal{C}/U$  inherits a topology from  $\mathcal{C}$ : a collection of maps  $V_\alpha \rightarrow V \rightarrow U$  is covering if and only if the family  $V_\alpha \rightarrow V$  covers  $V$ .

**Definitions:** Suppose that  $\mathcal{C}$  is a Grothendieck site.

- 1) A *presheaf* (of sets) on  $\mathcal{C}$  is a functor  $\mathcal{C}^{op} \rightarrow \mathbf{Set}$ . If  $\mathcal{A}$  is a category, an  $\mathcal{A}$ -valued presheaf on  $\mathcal{C}$  is a functor  $\mathcal{C}^{op} \rightarrow \mathcal{A}$ .

The set-valued presheaves on  $\mathcal{C}$  form a category (morphisms are natural transformation), written  $\text{Pre}(\mathcal{C})$ . One can talk about presheaves taking values in any category: I write  $s\text{Pre}(\mathcal{C})$  for presheaves on  $\mathcal{C}$  taking values in simplicial sets — this is the category of simplicial presheaves on  $\mathcal{C}$ .

- 2) A *sheaf* (of sets) on  $\mathcal{C}$  is a presheaf  $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$  such that the canonical map

$$F(U) \rightarrow \varprojlim_{V \rightarrow U \in R} F(V)$$

is an isomorphism for each covering sieve  $R \subset \text{hom}(\_, U)$ .

Morphisms of sheaves are natural transformations: write  $\text{Shv}(\mathcal{C})$  for the corresponding category. The sheaf category  $\text{Shv}(\mathcal{C})$  is a full subcategory of  $\text{Pre}(\mathcal{C})$ . One can also speak of sheaves in any complete category, such as simplicial sets:  $s\text{Shv}(\mathcal{C})$  denotes the category of simplicial sheaves on the site  $\mathcal{C}$ .

**Exercise:** If the topology on  $\mathcal{C}$  is defined by a pretopology (so that  $\mathcal{C}$  has all pullbacks), then  $F$  is a sheaf if and only if all pictures

$$F(U) \rightarrow \prod_{\alpha} F(U_{\alpha}) \rightrightarrows \prod_{\alpha, \beta} F(U_{\alpha} \times_U U_{\beta})$$

arising from covering families  $U_{\alpha} \rightarrow U$  are equalizers.

**Lemma 5.1.** 1) If  $R \subset R' \subset \text{hom}(\_, U)$  and  $R$  is covering then  $R'$  is covering.

2) If  $R, R' \subset \text{hom}(\_, U)$  are covering then  $R \cap R'$  is covering.

*Proof.* 1)  $\phi^{-1}(R) = \phi^{-1}(R')$  for all  $\phi \in R$ .

2)  $\phi^{-1}(R \cap R') = \phi^{-1}(R')$  for all  $\phi \in R$ .  $\square$

Suppose that  $R \subset \text{hom}(\_, U)$  is a sieve, and  $F$  is a presheaf on  $\mathcal{C}$ . Write

$$F(U)_R = \varprojlim_{V \rightarrow U \in R} F(V)$$

If  $S \subset R$  then there is an obvious map

$$F(U)_R \rightarrow F(U)_S$$

Write

$$LF(U) = \varinjlim_R F(U)_R$$

where the colimit is indexed over the filtering diagram of all covering sieves  $R \subset \text{hom}(\cdot, U)$ . Then  $x \mapsto LF(U)$  is a presheaf and there is a natural presheaf map

$$\eta : F \rightarrow LF$$

Say that a presheaf  $G$  is *separated* if (equivalently)

- 1) the map  $\eta : G \rightarrow LG$  is monic in each section, ie. all functions  $G(U) \rightarrow LG(U)$  are injective, or
- 2) Given  $x, y \in G(U)$ , if there is a covering sieve  $R \subset \text{hom}(\cdot, U)$  such that  $\phi^*(x) = \phi^*(y)$  for all  $\phi \in R$ , then  $x = y$ .

**Lemma 5.2.** 1)  $LF$  is separated, for all presheaves  $F$ .

2) If  $G$  is separated then  $LG$  is a sheaf.

3) If  $f : F \rightarrow G$  is a presheaf map and  $G$  is a sheaf, then  $f$  factors uniquely through a presheaf map  $f_* : LF \rightarrow G$ .

The object  $L^2F$  is a sheaf for every presheaf  $F$ , and the functor  $F \mapsto L^2F$  is left adjoint to the inclusion  $\text{Shv}(\mathcal{C}) \subset \text{Pre}(\mathcal{C})$ . The unit of the adjunction is the composite

$$F \xrightarrow{\eta} LF \xrightarrow{\eta} L^2F$$

One often writes  $\eta : F \rightarrow L^2F = \tilde{F}$  for this composite.

## 6 Exactness properties

**Lemma 6.1.** *1) The associated sheaf functor preserves all finite limits.*

*2)  $\text{Shv}(\mathcal{C})$  is complete and co-complete. Limits are formed sectionwise.*

*3) Every monic is an equalizer.*

*4) If  $\theta : F \rightarrow G$  in  $\text{Shv}(\mathcal{C})$  is both monic and epi, then  $\theta$  is an isomorphism.*

*Proof.* 1)  $LF$  is defined by filtered colimits, and finite limits commute with filtered colimits.

2) If  $X : I \rightarrow \text{Shv}(\mathcal{C})$  is a diagram of sheaves, then the colimit in the sheaf category is  $L^2(\varinjlim X)$ , where  $\varinjlim X$  is the presheaf colimit.

3) If  $A \subset X$  is a subset, then there is an equalizer

$$A \longrightarrow X \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{*} \end{array} X/A$$

The same holds for subobjects  $A \subset X$  of presheaves, and hence for subobjects of sheaves, since  $L^2$  is exact.

4) The map  $\theta$  appears in an equalizer

$$F \xrightarrow{\theta} G \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} K$$

since  $\theta$  is monic.  $\theta$  is an epi, so  $f = g$ . But then  $1_G : G \rightarrow G$  factors through  $\theta$ , giving a section  $\sigma : G \rightarrow F$ . Finally,  $\theta\sigma\theta = \theta$  and  $\theta$  is monic, so  $\sigma\theta = 1$ . □

### Definitions:

1) A presheaf map  $f : F \rightarrow G$  is a *local epimorphism* if for each  $\alpha \in G(x)$  there is a covering  $R \subset \text{hom}(\_, x)$  such that  $\phi^*(\alpha) = f(y_\phi)$  for all  $\phi \in R$ .

2)  $f : F \rightarrow G$  is a *local monic* if given  $\alpha, \beta \in F(x)$  such that  $f(\alpha) = f(\beta)$ , there is a covering  $R \subset \text{hom}(\_, x)$  such that  $\phi^*(\alpha) = \phi^*(\beta)$  for all  $\phi \in R$ .

3) A presheaf map  $f : F \rightarrow G$  which is both a local epi and a local monic is a *local isomorphism*.

**Lemma 6.2.** 1) *The natural map  $\eta : F \rightarrow LF$  is a local monomorphism and a local epimorphism.*

2) *Suppose that  $f : F \rightarrow G$  is a presheaf morphism. Then  $f$  induces an isomorphism of associated sheaves if and only if  $f$  is both a local epi and a local monic.*

*Proof.* For 2) observe that, given a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{g} & F' \\ & \searrow h & \downarrow f \\ & & F'' \end{array}$$

of presheaf morphisms, if any two of  $f$ ,  $g$  and  $h$  are local isomorphisms, then so is the third.

A sheaf map  $g : E \rightarrow E'$  is a monic (respectively epi) if and only if it is a local monic (respectively local epi).  $\square$

A *Grothendieck topos* is a category  $\mathcal{E}$  which is equivalent to a sheaf category  $\text{Shv}(\mathcal{C})$  on some Grothendieck site  $\mathcal{C}$ .

Grothendieck toposes are characterized by exactness properties:

**Theorem 6.3** (Giraud). *A category  $\mathcal{E}$  having all finite limits is a Grothendieck topos if and only if it has the following properties:*

- 1)  $\mathcal{E}$  has all small coproducts; they are disjoint and stable under pullback
- 2) every epimorphism of  $\mathcal{E}$  is a coequalizer
- 3) every equivalence relation  $R \rightrightarrows E$  in  $\mathcal{E}$  is a kernel pair and has a quotient
- 4) every coequalizer  $R \rightrightarrows E \rightarrow Q$  is stably exact
- 5) there is a (small) set of objects which generates  $\mathcal{E}$ .

A sketch proof of Giraud's Theorem appears below, but the result is proved in many places — see, for example, [2], [3], [1].

Here are the definitions of the terms appearing in the statement of the Theorem:

- 1) The coproduct  $\sqcup_i A_i$  is *disjoint* if all diagrams

$$\begin{array}{ccc} \emptyset & \longrightarrow & A_j \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & \sqcup_i A_i \end{array}$$

are pullbacks for  $i \neq j$ .  $\sqcup_i A_i$  is *stable under pullback* if all diagrams

$$\begin{array}{ccc} \sqcup_i B' \times_B A_i & \longrightarrow & \sqcup_i A_i \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

are pullbacks.

- 3) An *equivalence relation* is a monomorphism  $m = (m_0, m_1) : R \rightarrow E \times E$  such that
- the diagonal  $\Delta : E \rightarrow E \times E$  factors through  $m$  (ie.  $a \sim a$ )
  - the composite  $R \xrightarrow{m} E \times E \xrightarrow{\tau} E \times E$  factors through  $m$  (ie.  $a \sim b \Rightarrow b \sim a$ ).
  - the map

$$(m_0 m_{0*}, m_1 m_{1*}) : R \times_E R \rightarrow R \times R$$

factors through  $m$  (this is transitivity) where the pullback is defined by

$$\begin{array}{ccc} R \times_E R & \xrightarrow{m_{1*}} & R \\ m_{0*} \downarrow & & \downarrow m_0 \\ R & \xrightarrow{m_1} & E \end{array}$$

The *kernel pair* of a morphism  $u : E \rightarrow D$  is

a pullback

$$\begin{array}{ccc} R & \xrightarrow{m_1} & E \\ m_0 \downarrow & & \downarrow u \\ E & \xrightarrow{u} & D \end{array}$$

(Exercise: every kernel pair is an equivalence relation).

A *quotient* for an equivalence relation  $(m_0, m_1) : R \rightarrow E \times E$  is a coequalizer

$$R \begin{array}{c} \xrightarrow{m_0} \\ \xrightarrow{m_1} \end{array} E \longrightarrow E/R$$

- 4) A coequalizer  $R \rightrightarrows E \rightarrow Q$  is *stably exact* if the diagram

$$R \times_Q Q' \rightrightarrows E \times_Q Q' \rightarrow Q'$$

is a coequalizer for all morphisms  $Q' \rightarrow Q$ .

- 5) A *generating set* is a set  $\{A_i\}$  which detects non-trivial monomorphisms: if a monomorphism  $m : P \rightarrow Q$  induces bijections  $\text{hom}(A_i, P) \rightarrow \text{hom}(A_i, Q)$  for all  $i$ , then  $m$  is an isomorphism.

**Exercise:** Show that any category  $\text{Shv}(\mathcal{C})$  on a site  $\mathcal{C}$  satisfies the conditions of Giraud's theorem. The family  $L^2 \text{hom}(\_, U)$ ,  $U \in \mathcal{C}$  is a set of generators.

*Sketch proof of Giraud's Theorem.* The key is to show that the category  $\mathcal{E}$  is cocomplete — see [2], [1].

If  $A$  is the set of generators for  $\mathcal{E}$  prescribed by Giraud's theorem, let  $\mathcal{C}$  be the full subcategory of  $\mathcal{E}$  on the set of objects  $A$ . A subfunctor  $R \subset \text{hom}(\_, x)$  on  $\mathcal{C}$  is covering if the map

$$\bigsqcup_{y \rightarrow x \in R} y \rightarrow x$$

is an epimorphism of  $\mathcal{E}$ .

Every object  $E \in \mathcal{E}$  represents a sheaf  $\text{hom}(\_, E)$  on  $\mathcal{C}$ , and a sheaf  $F$  on  $\mathcal{C}$  determines an object

$$\lim_{\text{hom}(\_, y) \rightarrow F} y$$

of  $\mathcal{E}$ .

The adjunction

$$\text{hom}\left(\lim_{\text{hom}(\_, y) \rightarrow F} y, E\right) \cong \text{hom}(F, \text{hom}(\_, E))$$

determines an adjoint equivalence between  $\mathcal{E}$  and  $\text{Shv}(\mathcal{C})$ .  $\square$

The proof of Giraud's Theorem is arguably more important than the statement of the Theorem itself. Here are some examples of the use of the basic ideas:

1) Suppose that  $G$  is a sheaf of groups, and let  $G - \text{Shv}(\mathcal{C})$  denote the category of all sheaves  $X$  admitting  $G$ -action, with equivariant maps between them.  $G - \text{Shv}(\mathcal{C})$  is a Grothendieck topos, called the *classifying topos* for  $G$ , by Giraud's theorem. The objects  $G \times \text{hom}(, x)$  form a generating set.

2) If  $G = \{G_i\}$  is a profinite group with all transition maps  $G_i \rightarrow G_j$  epi, then the category  $G - \mathbf{Set}_d$  of discrete  $G$ -sets is a Grothendieck topos. The finite discrete  $G$ -sets form a generating set for this topos, and the site of finite discrete  $G$ -sets is "the" site prescribed by Giraud's theorem.

## 7 Geometric morphisms

Suppose that  $\mathcal{C}$  and  $\mathcal{D}$  are Grothendieck sites. A *geometric morphism*  $f : \text{Shv}(\mathcal{C}) \rightarrow \text{Shv}(\mathcal{D})$  consists of functors  $f_* : \text{Shv}(\mathcal{C}) \rightarrow \text{Shv}(\mathcal{D})$  and  $f^* : \text{Shv}(\mathcal{D}) \rightarrow \text{Shv}(\mathcal{C})$  such that  $f^*$  is left adjoint to  $f_*$  and  $f^*$  preserves finite limits.

The left adjoint  $f^*$  is called the *inverse image* functor, while  $f_*$  is called the *direct image*. The functor  $f^*$  is left and right exact in the sense that it preserves all finite limits and colimits;  $f_*$  is usually

not left exact (does not preserve finite colimits), and hence has higher derived functors.

### Examples

1) Suppose  $f : X \rightarrow Y$  is a continuous map of topological spaces. Pullback along  $f$  induces a functor  $\text{op}|_Y \rightarrow \text{op}|_X: U \subset Y \mapsto f^{-1}(U)$ . Open covers pull back to open covers, so if  $F$  is a sheaf on  $X$  then composition with the pullback gives a sheaf  $f_*F$  on  $Y$  with  $f_*F(U) = F(f^{-1}(U))$ . The resulting functor  $f_* : \text{Shv}(\text{op}|_X) \rightarrow \text{Shv}(\text{op}|_Y)$  is the direct image

The left Kan extension  $f^p : \text{Pre}(\text{op}|_Y) \rightarrow \text{Pre}(\text{op}|_X)$  is defined by

$$f^p G(V) = \varinjlim G(U)$$

where the colimit is indexed over all diagrams

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

The category  $\text{op}|_Y$  has all products (ie. intersections), so the colimit is filtered. The functor  $G \mapsto f^p G$  therefore commutes with finite limits. The inverse image functor

$$f^* : \text{Shv}(\text{op}|_Y) \rightarrow \text{Shv}(\text{op}|_X)$$

is defined by  $f^*(G) = L^2 f^p(G)$ . The resulting pair of functors forms a geometric morphism  $f : \text{Shv}(\text{op}|_X) \rightarrow \text{Shv}(\text{op}|_Y)$ .

2) Suppose that  $f : X \rightarrow Y$  is a morphism of schemes. Etale maps (resp. covers) are stable under pullback, and so there is a functor  $\text{et}|_Y \rightarrow \text{et}|_X$  defined by pullback, and if  $F$  is a sheaf on  $\text{et}|_X$  then there is a sheaf  $f_*F$  on  $\text{et}|_Y$  defined by  $f_*F(V \rightarrow Y) = f(X \times_Y V \rightarrow X)$

The restriction functor  $f_* : \text{Pre}(\text{et}|_X) \rightarrow \text{Pre}(\text{et}|_Y)$  has a left adjoint  $f^p$  defined by

$$f^p G(U \rightarrow X) = \varinjlim G(V)$$

where the colimit is indexed over all diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where both vertical maps are étale. The colimit is filtered, essentially because étale maps are stable under pullback and composition. The inverse image functor

$$f^* : \text{Shv}(\text{et}|_Y) \rightarrow \text{Shv}(\text{et}|_X)$$

is defined by  $f^*F = L^2 f^p F$ , and so  $f$  induces a

geometric morphism  $f : \mathrm{Shv}(\mathrm{et} |_X) \rightarrow \mathrm{Shv}(\mathrm{et} |_Y)$ .

A morphism of schemes  $f : X \rightarrow Y$  induces a geometric morphism  $f : \mathrm{Shv}(\cdot |_X) \rightarrow \mathrm{Shv}(\cdot |_Y)$  and/or  $f : (\mathrm{Sch}|_X)_? \rightarrow (\mathrm{Sch}|_Y)_?$  for all of the geometric topologies (eg. Zariski, flat, Nisnevich, qfh, ...), by similar arguments.

3) A *point* of  $\mathrm{Shv}(\mathcal{C})$  is a geometric morphism  $\mathbf{Set} \rightarrow \mathrm{Shv}(\mathcal{C})$ . Every point  $x \in X$  of a topological space  $X$  determines a continuous map  $\{x\} \subset X$  and hence a geometric morphism

$$\mathbf{Set} \cong \mathrm{Shv}(\mathrm{op} |_{\{x\}}) \xrightarrow{x} \mathrm{Shv}(\mathrm{op} |_X)$$

The set

$$x^*F = \varinjlim_{x \in U} F(U)$$

is the *stalk* of  $F$  at  $x$

4) Suppose that  $k$  is a field. Any scheme map  $x : \mathrm{Sp}(k) \rightarrow X$  induces a geometric morphism

$$\mathrm{Shv}(\mathrm{et}|_k) \rightarrow \mathrm{Shv}(\mathrm{et}|_X)$$

If  $k$  happens to be separably closed, then there is an equivalence  $\mathrm{Shv}(\mathrm{et}|_k) \simeq \mathbf{Set}$  and the resulting geometric morphism  $x : \mathbf{Set} \rightarrow \mathrm{Shv}(\mathrm{et} |_X)$  is called a geometric point of  $X$ . The inverse image

functor

$$F \mapsto f^* F = \varinjlim_{U \rightarrow X} F(U)$$

$$\begin{array}{ccc} & & U \\ & \nearrow & \downarrow \\ \mathrm{Sp}(k) & \xrightarrow{x} & X \end{array}$$

is the stalk of  $F$  at  $x$ .

5) Suppose that  $S$  and  $T$  are topologies on a site  $\mathcal{C}$  so that  $S \subset T$ . In other words,  $T$  has more covers than  $S$  and hence refines  $S$ . Then every sheaf for  $T$  is a sheaf for  $S$ ; write

$$\pi_* : \mathrm{Shv}(\mathcal{C}, T) \subset \mathrm{Shv}(\mathcal{C}, S)$$

for the corresponding inclusion. The associated sheaf functor for the topology  $T$  gives a left adjoint  $\pi^*$  for the inclusion functor  $\pi_*$ , and of course  $\pi^*$  preserves finite limits.

Here's an example: there is a geometric morphism

$$\mathrm{Shv}(\mathcal{C}) \rightarrow \mathrm{Pre}(\mathcal{C})$$

determined by the inclusion of the sheaf category in the presheaf category and the associated sheaf functor.

## 8 Points

Say that a Grothendieck topos  $\mathrm{Shv}(\mathcal{C})$  has *enough points* if there is a set of geometric morphisms  $x_i : \mathbf{Set} \rightarrow \mathrm{Shv}(\mathcal{C})$  such that the induced morphism

$$\mathrm{Shv}(\mathcal{C}) \xrightarrow{(x_i^*)} \prod_i \mathbf{Set}$$

is faithful.

**Lemma 8.1.** *Suppose that  $f : \mathrm{Shv}(\mathcal{D}) \rightarrow \mathrm{Shv}(\mathcal{C})$  is a geometric morphism. Then the following are equivalent:*

- a)  $f^* : \mathrm{Shv}(\mathcal{C}) \rightarrow \mathrm{Shv}(\mathcal{D})$  is faithful.
- b)  $f^*$  reflects isomorphisms
- c)  $f^*$  reflects epimorphisms
- d)  $f^*$  reflects monomorphisms

*Proof.* Suppose that  $f^*$  is faithful, ie. that  $f^*(g_1) = f^*(g_2)$  implies that  $g_1 = g_2$ . Suppose that  $m : F \rightarrow G$  is a morphism of  $\mathrm{Shv}(\mathcal{C})$  such that  $f^*(m)$  is monic. If  $m \cdot f_1 = m \cdot f_2$  then  $f^*(f_1) = f^*(f_2)$  so  $f_1 = f_2$ . The map  $m$  is therefore monic. Similarly  $f^*$  reflects epimorphisms and hence isomorphisms.

Suppose that  $f^*$  reflects epimorphisms and suppose given  $g_1, g_2 : F \rightarrow G$  such that  $f^*(g_1) = f^*(g_2)$ .  $g_1 = g_2$  if and only if their equalizer  $e : E \rightarrow F$  is an epimorphism. But  $f^*$  preserves equalizers and reflects epimorphisms, so  $e$  is an epi and  $g_1 = g_2$ . The other arguments are similar.  $\square$

Here are some basic definitions:

1) A *lattice*  $L$  is a partially ordered set which has all finite coproducts  $x \vee y$  and all finite products  $x \wedge y$ .

2) A lattice  $L$  has 0 and 1 if it has an initial and terminal object, respectively.

3) A lattice  $L$  is said to be *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all  $x, y, z$ .

4) A *complement* for  $x$  is a lattice  $L$  with 0 and 1 is an element  $a$  such that  $x \vee a = 1$  and  $x \wedge a = 0$ .

If  $L$  is also distributive the complement, if it exists, is unique: if  $b$  is another complement for  $x$ , then

$$\begin{aligned} b &= b \wedge 1 = b \wedge (x \vee a) = (b \wedge x) \vee (b \wedge a) \\ &= (x \wedge a) \vee (b \wedge a) = (x \vee b) \wedge a = a \end{aligned}$$

One usually writes  $\neg x$  for the complement of  $x$ .

5) A *Boolean algebra*  $\mathcal{B}$  is a distributive lattice with 0 and 1 in which every element has a complement.

6) A lattice  $L$  is said to be *complete* if it has all small limits and colimits (aka. all small meets and joins).

7) A *frame*  $P$  is a lattice which has all small joins (and all finite meets) and which satisfies an infinite distributive law

$$U \wedge \left( \bigvee_i V_i \right) = \bigvee_i (U \wedge V_i)$$

### **Examples:**

- 1) The poset  $\mathcal{O}(T)$  of open subsets of a topological space  $T$  is a frame. Every continuous map  $f : S \rightarrow T$  induces a morphism of frames  $f^{-1} : \mathcal{O}(T) \rightarrow \mathcal{O}(S)$ , defined by  $U \mapsto f^{-1}(U)$ .
- 2) The power set  $\mathcal{P}(I)$  of a set  $I$  is a complete Boolean algebra.
- 3) Every complete Boolean algebra  $\mathcal{B}$  is a frame. In effect, every join is a filtered colimit of finite joins.

Every frame  $A$  has a canonical Grothendieck topology: a family  $y_i \leq x$  is covering if  $\bigvee_i y_i = x$ . Write  $\text{Shv}(A)$  for the corresponding sheaf category. Every complete Boolean algebra  $\mathcal{B}$  is a frame, and therefore has an associated sheaf category  $\text{Shv}(\mathcal{B})$ .

**Example:** Suppose that  $I$  is a set. Then there is an equivalence

$$\text{Shv}(\mathcal{P}(I)) \simeq \prod_{i \in I} \mathbf{Set}$$

Any set  $I$  of points  $x_j : \mathbf{Set} \rightarrow \text{Shv}(\mathcal{C})$  assembles to give a geometric morphism

$$x : \text{Shv}(\mathcal{P}(I)) \rightarrow \text{Shv}(\mathcal{C}).$$

**Lemma 8.2.** *Suppose that  $F$  is a sheaf of sets on a complete Boolean algebra  $\mathcal{B}$ . Then the poset  $\text{Sub}(F)$  of subobjects of  $F$  is a complete Boolean algebra.*

*Proof.*  $\text{Sub}(F)$  is a frame, by an argument on the presheaf level. It remains to show that every object  $G \in \text{Sub}(F)$  is complemented. The obvious candidate for  $\neg G$  is

$$\neg G = \bigvee_{H \wedge G = \emptyset} H$$

and we need to show that  $G \vee \neg G = F$ .

Every  $K \leq \text{hom}(\_, A)$  is representable: in effect,

$$K = \varinjlim_{\text{hom}(\_, B) \rightarrow K} \text{hom}(\_, B) = \text{hom}(\_, C)$$

where

$$C = \bigvee_{\text{hom}(\_, B) \rightarrow K} B \in \mathcal{B}.$$

It follows that  $\text{Sub}(\text{hom}(\_, A)) \cong \text{Sub}(A)$  is a complete Boolean algebra.

Consider all diagrams

$$\begin{array}{ccc} \phi^{-1}(G) & \longrightarrow & G \\ \downarrow & & \downarrow \\ \text{hom}(\_, A) & \xrightarrow{\phi} & F \end{array}$$

There is an induced pullback

$$\begin{array}{ccc} \phi^{-1}(G) \vee \neg\phi^{-1}(G) & \longrightarrow & G \vee \neg G \\ \cong \downarrow & & \downarrow \\ \text{hom}(\_, A) & \xrightarrow{\phi} & F \end{array}$$

$F$  is a union of its representables (all  $\phi$  are monic since all  $\text{hom}(\_, A)$  are subobjects of the terminal sheaf), so  $G \vee \neg G = F$ .  $\square$

**Lemma 8.3.** *Suppose that  $\mathcal{B}$  is a complete Boolean algebra. Then every epimorphism  $\pi : F \rightarrow G$  in  $\text{Shv}(\mathcal{B})$  has a section.*

**Remark 8.4.** Lemma 8.3 asserts that the sheaf category on a complete Boolean algebra satisfies the Axiom of Choice.

*Proof.* Consider the family of lifts

$$\begin{array}{ccc} & & F \\ & \nearrow & \downarrow \pi \\ N & \xrightarrow{\leq} & G \end{array}$$

This family is non-empty, because every  $x \in G(1)$  restricts along some covering  $B \leq 1$  to a family of elements  $x_B$  which lift to  $F(B)$ .

All maps  $\text{hom}(\_, B) \rightarrow G$  are monic, since all maps  $\text{hom}(\_, B) \rightarrow \text{hom}(\_, 1) = *$  are monic. Thus, all such morphisms represent objects of  $\text{Sub}(G)$ , which is a complete Boolean algebra by Lemma 8.2.

Zorn's Lemma implies that the family of lifts has maximal elements. Suppose that  $N$  is maximal and that  $\neg N \neq \emptyset$ . Then there is an  $x \in \neg N(C)$  for some  $C$ , and there is a covering  $B' \leq C$  such that  $x_{B'} \in N(B')$  lifts to  $F(B')$  for all members of the cover. Then  $N \wedge \text{hom}(\_, B') = \emptyset$  so the lift extends to a lift on  $N \vee \text{hom}(\_, B')$ , contradicting the maximality of  $N$ .  $\square$

A *Boolean localization* for  $\mathrm{Shv}(\mathcal{C})$  is a geometric morphism  $p : \mathrm{Shv}(\mathcal{B}) \rightarrow \mathrm{Shv}(\mathcal{C})$  such that  $\mathcal{B}$  is a complete Boolean algebra and  $p^*$  is faithful.

**Theorem 8.5** (Barr). *Boolean localizations exist for every Grothendieck topos  $\mathrm{Shv}(\mathcal{C})$ .*

Theorem 8.5 is one of the big results of topos theory, and is proved in multiple places — see [2], for example. There is a simple proof in [1].

A Grothendieck topos  $\mathrm{Shv}(\mathcal{C})$  does not have enough points, in general (eg. sheaves on the flat site for a scheme), but the result asserts that every Grothendieck topos has a “fat point” given by a Boolean localization. This is of fundamental importance in setting up the general local homotopy theory of simplicial sheaves and presheaves.

## References

- [1] J. F. Jardine. Lectures on Simplicial Presheaves. <http://www.math.uwo.ca/~jardine/papers/sPre>, 2005.
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