

Lecture 07 (January 6, 2011)

15 Cocycles

Let \mathcal{M} be a closed model category such that

- 1) \mathcal{M} is right proper in the sense that weak equivalences pull back to weak equivalences along fibrations, and
- 2) the class of weak equivalences is closed under products: if $f : X \rightarrow Y$ is a weak equivalence, so is any map $f \times 1 : X \times Z \rightarrow Y \times Z$

Examples include any of the model structures on $s\text{Pre}$, $s\text{Shv}$, $s\text{Pre}_R$ or $s\text{Shv}_R$ that we've seen, where the weak equivalences are local weak equivalences. This can be verified by using Boolean localization arguments.

Suppose that X, Y are objects of \mathcal{M} , and write $H(X, Y)$ for the category whose objects are all pairs of maps (f, g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism

$$\alpha : (f, g) \rightarrow (f', g')$$

of $H(X, Y)$ is a commutative diagram

$$\begin{array}{ccc}
 & Z & \\
 f \swarrow & & \searrow g \\
 X & & Y \\
 f' \swarrow & & \searrow g' \\
 & Z' &
 \end{array}$$

I say that $H(X, Y)$ is the *category of cocycles*, or *cocycle category* from X to Y .

Example: Every set X has an associated (homotopically) trivial groupoid whose objects are the elements of X and whose morphisms are pairs of elements of X . Suppose that a presheaf map $U \rightarrow *$ is a local epimorphism. Then the canonical simplicial presheaf map $BC(U) \rightarrow *$ is a local weak equivalence (in fact, it's a local trivial fibration), and $BC(U)$ is called the *Čech resolution* associated to the covering $U \rightarrow *$.

Given a covering $U \rightarrow *$ and a (pre)sheaf of groups G , a normalized cocycle on U with values in G is, precisely, either a groupoid morphism $C(U) \rightarrow G$ or a simplicial presheaf map $BC(U) \rightarrow BG$. Such a map defines a cocycle

$$* \xleftarrow{\sim} BC(U) \rightarrow BG$$

in the sense described above. Normalized cocycles were the original examples of such cocycles.

Write $\pi_0 H(X, Y)$ for the class of path components of $H(X, Y)$. There is a function

$$\phi : \pi_0 H(X, Y) \rightarrow [X, Y]$$

defined by $(f, g) \mapsto g \cdot f^{-1}$.

Lemma 15.1. *Suppose that $\alpha : X \rightarrow X'$ and $\beta : Y \rightarrow Y'$ are weak equivalences. Then the function*

$$(\alpha, \beta)_* : \pi_0 H(X, Y) \rightarrow \pi_0 H(X', Y')$$

is a bijection.

Proof. An object (f, g) of $H(X', Y')$ is a map $(f, g) : Z \rightarrow X' \times Y'$ such that f is a weak equivalence. There is a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & W \\ & \searrow (f, g) & \downarrow (p_{X'}, p_{Y'}) \\ & & X' \times Y' \end{array}$$

such that j is a trivial cofibration and $(p_{X'}, p_{Y'})$ is a fibration. The map $p_{X'}$ is a weak equivalence. Form the pullback

$$\begin{array}{ccc} W_* & \xrightarrow{(\alpha \times \beta)_*} & W \\ (p_X^*, p_Y^*) \downarrow & & \downarrow (p_{X'}, p_{Y'}) \\ X \times Y & \xrightarrow{\alpha \times \beta} & X' \times Y' \end{array}$$

Then the map (p_X^*, p_Y^*) is a fibration and $(\alpha \times \beta)_*$ is a local weak equivalence (since $\alpha \times \beta$ is a weak equivalence, and by right properness). The map p_X^* is also a weak equivalence.

The assignment $(f, g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X', Y') \rightarrow \pi_0 H(X, Y)$$

which is inverse to $(\alpha, \beta)_*$. □

Lemma 15.2. *Suppose that Y is fibrant and X is cofibrant. Then the canonical map*

$$\phi : \pi_0 H(X, Y) \rightarrow [X, Y]$$

is a bijection.

Proof. The function $\pi(X, Y) \rightarrow [X, Y]$ relating naive homotopy classes to morphisms in the homotopy category is a bijection since X is cofibrant and Y is fibrant.

If $f, g : X \rightarrow Y$ are homotopic, there is a diagram

$$\begin{array}{ccccc}
 & & X & & \\
 & 1 \swarrow & \downarrow d_0 & \searrow f & \\
 X & \xleftarrow{s} & X \otimes I & \xrightarrow{h} & Y \\
 & \nwarrow 1 & \uparrow d_1 & \nearrow g & \\
 & & X & &
 \end{array}$$

where h is the homotopy. Thus, sending $f : X \rightarrow Y$ to the class of $(1_X, f)$ defines a function

$$\psi : \pi(X, Y) \rightarrow \pi_0 H(X, Y)$$

and there is a diagram

$$\begin{array}{ccc} \pi(X, Y) & \xrightarrow{\psi} & \pi_0 H(X, Y) \\ & \searrow \cong & \downarrow \phi \\ & & [X, Y] \end{array}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k .

The weak equivalence f has a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & V \\ & \searrow f & \downarrow p \\ & & X \end{array}$$

where j is a trivial cofibration and p is a trivial fibration. The object Y is fibrant, so the dotted arrow θ exists in the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow f & \downarrow j & \searrow g & \\ X & & V & & Y \\ & \swarrow p & \downarrow \theta & & \end{array}$$

Since X is cofibrant, the trivial fibration p has a

section σ , and so there is a commutative diagram

$$\begin{array}{ccccc}
 & & X & & \\
 & 1 & \swarrow & \theta\sigma & \\
 X & & & & Y \\
 & p & \searrow & \theta & \\
 & & V & &
 \end{array}$$

Then the composite $\theta\sigma$ is the required map k . \square

Theorem 15.3. *Suppose that the model category has the properties 1) and 2) listed above, and that X, Y are objects of \mathcal{M} . Then the canonical map*

$$\phi : \pi_0 H(X, Y) \rightarrow [X, Y]$$

is a bijection for all X and Y .

Proof. There are weak equivalences $\pi : X' \rightarrow X$ and $j : Y \rightarrow Y'$ such that X' and Y' are cofibrant and fibrant, respectively, and there is a commutative diagram

$$\begin{array}{ccc}
 \pi_0 H(X, Y) & \xrightarrow{\phi} & [X, Y] \\
 (1, j)_* \downarrow \cong & & \cong \downarrow j_* \\
 \pi_0 H(X, Y') & \xrightarrow{\phi} & [X, Y'] \\
 (\pi, 1)_* \uparrow \cong & & \cong \downarrow \pi^* \\
 \pi_0 H(X', Y') & \xrightarrow[\phi]{\cong} & [X', Y']
 \end{array}$$

The functions $(1, j)_*$ and $(\pi, 1)_*$ are bijections by the first Lemma, and the bottom map ϕ is a bijection by the second Lemma. \square

Remark 15.4. Cocycle categories have appeared before, in the context of Dwyer-Kan hammock localizations [3], [2]. One of the main results in the area, which holds for arbitrary model categories \mathcal{M} , says roughly that the nerve $BH(X, Y)$ is a model for the function space of maps from X to Y if Y is fibrant. This result implies Theorem 15.3 if the target object Y is fibrant. On the other hand, I shall demonstrate below that the most powerful applications of Theorem 15.3 involve target objects Y which are not fibrant in general.

16 Sheaf cohomology

Suppose that A is a sheaf of abelian groups, and let $A \rightarrow J$ be an injective resolution of A , thought of as a \mathbb{Z} -graded chain complex, concentrated in negative degrees.

Write $A[-n]$ for the chain complex consisting of A concentrated in degree n , and consider the chain map $A[-n] \rightarrow J[-n]$.

Recall that $K(A, n) = \Gamma A[-n]$ defines the Eilenberg-Mac Lane simplicial sheaf associated to A . Let

$$K(J, n) = \Gamma \mathrm{Tr}_0(J[-n])$$

where $\text{Tr}_0(J[-n])$ is the good truncation of $J[-n]$ in non-negative degrees.

Suppose that C is an ordinary chain complex and that I is an unbounded chain complex which is 0 in non-negative degrees. Form the bicomplex

$$\text{hom}(C, I)_{p,q} = \text{hom}(C_{-p}, I_q)$$

with the obvious induced differentials:

$$\partial' = \partial_C^* : \text{hom}(C_{-p}, I_q) \rightarrow \text{hom}(C_{-p-1}, I_q)$$

$$\partial'' = (-1)^p \partial_{I^*} : \text{hom}(C_{-p}, I_q) \rightarrow \text{hom}(C_{-p}, I_{q-1}).$$

Then $\text{hom}(C, I)$ is a third quadrant bicomplex with associated total complex

$$\begin{aligned} \text{Tot}_{-n} \text{hom}(C, I) &= \bigoplus_{p+q=-n} \text{hom}(C_{-p}, I_q) \\ &= \bigoplus_{0 \leq p \leq n} \text{hom}(C_p, I_{-n+p}), \end{aligned}$$

for $n \geq 0$, which is concentrated in negative degrees.

Exercise: Show that there are natural isomorphisms

$$\begin{aligned} H_{-n}(\text{Tot } \text{hom}(C, I)) &\cong \pi(C(0), I[-n]) \\ &\cong \pi(C, \text{Tr}_0 I[-n]), \end{aligned}$$

where $\pi(C(0), I[-n])$ denotes chain homotopy classes of maps from the unbounded complex $C(0)$ canonically associated to C to the shifted complex $I[-n]$, and $\pi(C, \text{Tr}_0 I[-n])$ is chain homotopy classes of maps in the bounded complex category.

Example: If $A \rightarrow J$ is an injective resolution of an abelian sheaf A , then the bicomplex $\text{hom}(C, I)$ determines a spectral sequence with

$$E_2^{p,q} = \text{Ext}^q(H_p(C), A) \Rightarrow \pi(C, \text{Tr}_0 J[-p-q]).$$

Lemma 16.1. *Every local weak equivalence $f : X \rightarrow Y$ induces an isomorphism*

$$\pi_{ch}(N\tilde{\mathbb{Z}}Y, \text{Tr}_0 J[-n]) \xrightarrow{\cong} \pi_{ch}(N\tilde{\mathbb{Z}}X, \text{Tr}_0 J[-n])$$

in chain homotopy classes for all $n \geq 0$.

Proof. The map f induces a homology sheaf isomorphism $N\tilde{\mathbb{Z}}X \rightarrow N\tilde{\mathbb{Z}}Y$, and then a comparison of spectral sequences

$$E_2^{p,q} = \text{Ext}^q(\tilde{H}_p(X), A) \Rightarrow \pi_{ch}(N\tilde{\mathbb{Z}}X, \text{Tr}_0 J[-p-q])$$

gives the desired result. \square

If two chain maps $f, g : N\tilde{\mathbb{Z}}X \rightarrow \text{Tr}_0 J[-n]$ are

chain homotopic, then there is a right homotopy

$$\begin{array}{ccc}
 & & Z \\
 & \nearrow & \downarrow p \\
 X & \xrightarrow{(f_*, g_*)} & K(J, n) \times K(J, n)
 \end{array}$$

for some path object Z over $K(J, n)$ in the projective model structure for \mathcal{C}^{op} -diagrams of simplicial sets. Choose a sectionwise trivial fibration $\pi : W \rightarrow X$ such that W is projective cofibrant. Then $f_*\pi$ is left homotopic to $g_*\pi$ for some choice of cylinder object $W \otimes I$ for W , again in the projective structure. This means that there is a diagram

$$\begin{array}{ccccc}
 & & W & \xrightarrow{\pi} & X \\
 & \swarrow 1 & \downarrow i_0 & & \searrow f_* \\
 W & \xleftarrow{s} & W \otimes I & \xrightarrow{h} & K(J, n) \\
 & \swarrow 1 & \uparrow i_1 & & \searrow g_* \\
 & & W & \xrightarrow{\pi} & X
 \end{array}$$

where the maps s, i_0, i_1 are all part of the cylinder object structure for $W \otimes I$, and are sectionwise weak equivalences. It follows that

$$(1, f_*) \sim (\pi, f_*\pi) \sim (\pi s, h) \sim (\pi, g_*\pi) \sim (1, g_*)$$

in $\pi_0 H(X, K(J, n))$. It follows that there is a well defined abelian group homomorphism

$$\phi : \pi_{ch}(N\tilde{Z}X, \text{Tr}_0 J[-n]) \rightarrow \pi_0 H(X, K(J, n)).$$

This map is natural in X .

Lemma 16.2. *The map*

$$\phi : \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n]) \rightarrow \pi_0 H(X, K(J, n)).$$

is an isomorphism.

Proof. Suppose that $X \xleftarrow{f} Z \xrightarrow{g} K(J, n)$ is an object of $H(X, K(J, n))$. Then there is a unique chain homotopy class $[v] : N\tilde{\mathbb{Z}}X \rightarrow J[-n]$ such that $[v_* f] = [g]$ since f is a local weak equivalence. This chain homotopy class $[v]$ is also independent of choice of representative for the component of (f, g) . We therefore have a well defined function

$$\psi : \pi_0 H(X, K(J, n)) \rightarrow \pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n]).$$

Then the composites $\psi \cdot \phi$ and $\phi \cdot \psi$ are identity morphisms. \square

We have proved

Theorem 16.3. *Suppose that A is a sheaf of abelian groups on \mathcal{C} , and let $A \rightarrow J$ be an injective resolution of A in the category of abelian sheaves. Let X be a simplicial presheaf on \mathcal{C} . Then there is an isomorphism*

$$\pi_{ch}(N\tilde{\mathbb{Z}}X, \mathrm{Tr}_0 J[-n]) \cong [X, K(A, n)].$$

This isomorphism is natural in X .

Suppose that A is an abelian (pre)sheaf on \mathcal{C} and that X is a simplicial presheaf. Write

$$H^n(X, A) = [X, K(A, n)],$$

and say that this group is the n^{th} *cohomology group* of X with coefficients in A .

The following basic result is then an immediate consequence of Theorem 16.3:

Corollary 16.4. *Suppose that $f : X \rightarrow Y$ induces an isomorphism*

$$\tilde{H}_*(X) \cong \tilde{H}_*(Y)$$

in all homology sheaves. Then the induced map in cohomology

$$H^*(Y, A) \rightarrow H^*(X, A)$$

is an isomorphism for all coefficient presheaves A .

Proof. Compare universal coefficients spectral sequences. □

There is also a torsion coefficients version:

Corollary 16.5. *If $f : X \rightarrow Y$ induces a homology sheaf isomorphism*

$$\tilde{H}_*(X, \mathbb{Z}/n) \cong \tilde{H}_*(Y, \mathbb{Z}/n)$$

then f induces an isomorphism

$$H^*(Y, A) \rightarrow H^*(X, A)$$

for all n -torsion presheaves A .

Proof. Use the universal coefficients spectral sequence in the category of n -torsion sheaves. \square

Remark 16.6. 1) The associated sheaf map

$$K(A, n) \rightarrow K(\tilde{A}, n)$$

is a local weak equivalence, so that

$$H^n(X, A) \cong H^n(X, \tilde{A}).$$

2) One can (and does) define sheaf cohomology $H^n(\mathcal{C}, A)$ for an abelian sheaf A on a site \mathcal{C} by

$$H^n(\mathcal{C}, A) = H_{-n}(\Gamma_* J)$$

where $A \rightarrow J$ is an injective resolution of A concentrated in negative degrees and Γ_* is global sections (ie. inverse limit). But $\Gamma_* Y = \text{hom}(*, Y)$ for any Y , and so

$$H^n(\mathcal{C}, A) \cong \pi_{ch}(\tilde{\mathbb{Z}}^*, \text{Tr}_0 J[-n]) \cong [*, K(A, n)]$$

by Theorem 16.3.

3) There is a *universal coefficients spectral sequence*

$$E_2^{p,q} = \text{Ext}^q(\tilde{H}_p(X), \tilde{A}) \Rightarrow H^{p+q}(X, A)$$

for abelian presheaves A and simplicial presheaves X . There is a corresponding spectral sequence

$$E_2^{p,q} = \text{Ext}^q(\tilde{H}_p(X, \mathbb{Z}/n), \tilde{A}) \Rightarrow H^{p+q}(X, A)$$

for n -torsion presheaves A .

Cup products

Suppose that

$$X \xleftarrow{\cong} X' \rightarrow K(A, n), \quad Y \xleftarrow{\cong} Y' \rightarrow K(B, m)$$

are cocycles. Then the adjoint simplicial abelian presheaf maps

$$\mathbb{Z}X' \rightarrow K(A, n), \quad \mathbb{Z}Y' \rightarrow K(B, m)$$

have a (simplicial abelian group) tensor product

$$\mathbb{Z}(X' \times Y') \cong \mathbb{Z}X' \otimes \mathbb{Z}Y' \rightarrow K(A, n) \otimes K(B, m)$$

and there is a natural weak equivalence

$$K(A, n) \otimes K(B, m) \simeq K(A \otimes B, n + m).$$

in simplicial abelian groups, hence in simplicial abelian presheaves (Exercise: suppose first that $A = B = \mathbb{Z}$). The adjoint

$$X \times Y \xleftarrow{\cong} X' \times Y' \rightarrow K(A \otimes B, n + m)$$

represents the external cup product of the classes represented by the two cocycles. We have defined

an *external cup product*

$$H^n(X, A) \times H^m(Y, B) \rightarrow H^{n+m}(X \times Y, A \otimes B).$$

If A happens to be a presheaf of rings this construction specializes to the cup product pairing

$$\begin{aligned} H^n(X, A) \times H^m(X, A) &\rightarrow H^{n+m}(X \times X, A) \\ &\xrightarrow{\Delta^*} H^{n+m}(X, A). \end{aligned}$$

where $\Delta : X \rightarrow X \times X$ is the diagonal map.

Cohomology operations

A *cohomology operation* is a map

$$K(A, n) \rightarrow K(B, m)$$

in the homotopy category.

The *Steenrod operation* Sq^i is a morphism

$$K(\mathbb{Z}/2, n) \rightarrow K(\mathbb{Z}/2, n + i)$$

in the ordinary homotopy category. The constant presheaf functor preserves weak equivalences, and so Sq^i induces a morphism

$$K(\Gamma^*\mathbb{Z}/2, n) \rightarrow K(\Gamma^*\mathbb{Z}/2, n + i)$$

in the homotopy category of simplicial presheaves on an arbitrary small site \mathcal{C} . It therefore induces a homomorphism

$$\text{Sq}^i : H^n(X, \mathbb{Z}/2) \rightarrow H^{n+i}(X, \mathbb{Z}/2)$$

which is natural in simplicial presheaves X . The collection of Steenrod operations $\{\mathrm{Sq}^i\}$ for simplicial presheaves has the same basic list of properties as the Steenrod operations for ordinary spaces.

Steenrod operations for mod 2 étale cohomology were first introduced by Breen [1]; the definition given here for mod 2 cohomology of arbitrary simplicial presheaves is a vast generalization. The first calculational application was in questions concerning Hasse-Witt classes for non-degenerate symmetric bilinear forms in the mod 2 Galois cohomology of fields — see [5] and [6].

That said, the definition of Steenrod operations which is given here has its uses, but it is now relatively naive. Voevodsky introduced and made very effective use of a much more sophisticated construction for motivic homotopy theory in his proof of the Milnor conjecture [9], [10].

17 Descent spectral sequences

Proposition 17.1. *Suppose that A is a presheaf of abelian groups, and that*

$$j : K(A, n) \rightarrow GK(A, n)$$

is an injective fibrant model of $K(A, n)$. Then there are isomorphisms

$$\pi_j GK(A, n)(U) \cong \begin{cases} H^{n-j}(\mathcal{C}/U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathcal{C}$.

Exercise 17.2. Suppose given a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & \searrow p & \swarrow p' \\ & & Y \end{array}$$

where p and p' are local fibrations and f is a local weak equivalence. Suppose that $Z \rightarrow Y$ is a map of simplicial presheaves. Show that the induced map

$$Z \times_Y X \xrightarrow{f_*} Z \times_Y X'$$

is a local weak equivalence — use Boolean localization.

Suppose that $U \in \mathcal{C}$ and write $X|_U$ for the restriction of X along the functor

$$\mathcal{C}/U \rightarrow \mathcal{C}.$$

Lemma 17.3. *The restriction functor $X \mapsto X|_U$ preserves injective fibrations and local weak*

equivalences, and therefore preserves injective fibrant models.

Proof. The restriction functor $X \mapsto X|_U$ has a left adjoint j_U^* where

$$j_U^*(Y)(V) = \bigsqcup_{V \rightarrow U} Y(V).$$

Then j_U^* clearly preserves cofibrations and sectionwise weak equivalences. The functor j_U^* also preserves local trivial fibrations (exercise) and therefore preserves local weak equivalences.

Restriction preserves sectionwise equivalences and local trivial fibrations, and therefore preserves local weak equivalences. \square

Proof of Proposition 17.1. There are isomorphisms

$$\begin{aligned} \pi_0 GK(A, n)(U) &\cong [*, GK(A, n)(U)] \\ &\cong [*, GK(A|_U, n)]_{\mathcal{C}/U} \\ &\cong H^n(\mathcal{C}/U, \tilde{A}|_U). \end{aligned}$$

Note that $GK(A, n)|_U$ is an injective fibrant model of $K(A|_U, n)$ by Lemma 17.3, giving the second and third isomorphisms.

Observe that the associated sheaf map

$$\eta : K(A, 0) \rightarrow K(\tilde{A}, 0)$$

is an injective fibrant model for the constant simplicial presheaf $K(A, 0)$, and

$$\pi_j K(\tilde{A}, 0)(U) = 0$$

for $j > 0$.

There is a sectionwise fibre sequence

$$\begin{aligned} K(A, n-1) &\rightarrow WK(A, n-1) \\ &\rightarrow \overline{WK}(A, n-1) = K(A, n) \end{aligned}$$

where $WK(A, n-1)$ is sectionwise contractible.

Take an injective fibrant model

$$\begin{array}{ccc} WK(A, n-1) & \xrightarrow{j} & GWK(A, n-1) \\ \downarrow & & \downarrow p \\ K(A, n) & \xrightarrow{j} & GK(A, n) \end{array}$$

where the maps labelled j are local weak equivalences, $GK(A, n)$ is injective fibrant and p is an injective fibration. Let $F = p^{-1}(0)$. Then F is injective fibrant and the induced map

$$K(A, n-1) \rightarrow F$$

is a local weak equivalence, by Exercise 17.2. Write $GK(A, n-1)$ for F .

We have sectionwise fibre sequences

$$\begin{aligned} GK(A, n-1)(U) &\rightarrow GWK(A, n-1)(U) \\ &\rightarrow GK(A, n)(U) \end{aligned}$$

for all $U \in \mathcal{C}$. The map

$$GWK(A, n - 1) \rightarrow *$$

is a trivial injective fibration, and is therefore a sectionwise trivial fibration. It follows that

$$\pi_j GK(A, n)(U) \cong \pi_{j-1} GK(A, n - 1)(U)$$

for $j \geq 1$, so that

$$\pi_j GK(A, n)(U) \cong H^{n-j}(\mathcal{C}/U, \tilde{A}|_U)$$

for $1 \leq j \leq n$ by induction on n . □

Example: Suppose \mathcal{C} is the big site $(Sch|_S)_{et}$ for a scheme S with the étale topology and that U is an S -scheme in this site. Then \mathcal{C}/U is isomorphic to the site $(Sch|_U)_{et}$. If A is a sheaf on the big étale site for S , and if $K(A, n) \rightarrow GK(A, n)$ is an injective fibrant model for $K(A, n)$, then the presheaves of homotopy groups for $GK(A, n)$ have the form

$$\pi_j GK(A, n)(U) \cong \begin{cases} H_{et}^{n-j}(U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathcal{C}$.

Similar statements obtain for all other geometric topologies on categories of S -schemes.

Suppose that X is a presheaf of pointed Kan complexes, and form the Postnikov tower

$$\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow \\
 & & P_2X & \xrightarrow{j} & GP_2X \\
 & \nearrow & \downarrow & & \downarrow p \\
 & & P_1X & \xrightarrow{j} & GP_1X \\
 & \nearrow & \downarrow & & \downarrow p \\
 X & \longrightarrow & P_0X & \xrightarrow{j} & GP_0X
 \end{array}$$

where all maps labelled j are injective fibrant models and the maps p are injective fibrations.

The fibre of $GP_nX \rightarrow GP_{n-1}X$ is sectionwise equivalent to $GK(\tilde{\pi}_nX, n)$, where

$$\tilde{\pi}_nX = \tilde{\pi}_n(X, *)$$

is the n^{th} homotopy group sheaf, based at the global base point.

Now take $U \in \mathcal{C}$ and consider the tower of fibrations

$$GP_0X(U) \leftarrow GP_1X(U) \leftarrow GP_2X(U) \leftarrow \dots$$

The fibre $GK(\tilde{\pi}_nX, n)(U)$ of the map

$$GP_nX(U) \rightarrow GP_{n-1}X(U)$$

has homotopy groups

$$\begin{aligned} \pi_j GK(\tilde{\pi}_n X, n)(U) \\ \cong \begin{cases} H^{n-j}(\mathcal{C}/U, \tilde{\pi}_n X|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases} \end{aligned}$$

and so the tower of fibrations spectral sequence (with the Thomason re-indexing trick [8, 5.54]) determines a spectral sequence with

$$E_2^{s,t}(U) = H^s(\mathcal{C}/U, \tilde{\pi}_s X|_U)$$

This is the (unstable) descent spectral sequence — it is actually a presheaf of spectral sequences. One sometimes sees this spectral sequence referred to as either a cohomological or topological descent spectral sequence.

There are two issues:

1) the spectral sequence might or might not converge to

$$\pi_{t-s} \varprojlim GP_n X(U)$$

2) it can be a bit of work to show that the map $X \rightarrow \varprojlim_n GP_n X$ is a local weak equivalence.

Both issues can be resolved (ie. the spectral sequence converges and the map of 2) is a local weak equivalence) if X is locally connected in the sense

that $\tilde{\pi}_0 X \cong *$ and there is a uniform bound on cohomological dimension for all sheaves $\tilde{\pi} X|_U$. See [4].

There are also “finite” descent spectral sequences, which are Bousfield-Kan spectral sequences arising from function complexes $\mathbf{hom}(U, Z)$, where $U \rightarrow *$ is a local weak equivalence and Z is injective fibrant.

Example: Suppose that L/k is a finite Galois extension of a field k with Galois group G . Then, by Galois theory, there is an isomorphism

$$G \times \mathrm{Sp}(L) \xrightarrow{\cong} \mathrm{Sp}(L) \times \mathrm{Sp}(L)$$

of k -schemes which induces an isomorphism

$$EG \times_G \mathrm{Sp}(L) \cong C(\mathrm{Sp}(L))$$

on simplicial sheaves (even simplicial schemes) on any of the étale sites for the field k . It follows that the canonical map

$$EG \times_G \mathrm{Sp}(L) \rightarrow *$$

is a local weak equivalence for the étale topology. Thus, if Z is an injective fibrant simplicial presheaf, then the induced map

$$Z(k) \cong \mathbf{hom}(*, Z) \rightarrow \mathbf{hom}(EG \times_G \mathrm{Sp}(L), Z)$$

is a weak equivalence of simplicial sets. At the same time, the Bousfield-Kan spectral sequence for the function complex on the right has the form

$$E_2^{s,t} = H^s(G, \pi_t Z(L)) \Rightarrow \pi_{t-s} Z(k).$$

This is a finite Galois descent spectral sequence for the homotopy groups of the global section $Z(k)$ of Z .

On the other hand, the full Galois (or étale) cohomological descent spectral sequence for Z has the form

$$E_2^{s,t} = H^s(\Omega, \tilde{\pi}_t Z) \Rightarrow \pi_{t-s} Z(k),$$

where Ω is the absolute Galois group of k .

One often says that a simplicial presheaf X on an étale site for k satisfies *finite descent* if the map

$$X(k) \cong \mathbf{hom}(*, X) \rightarrow \mathbf{hom}(EG \times_G \mathrm{Sp}(L), X)$$

is a weak equivalence for every finite Galois extension L/k . The question of whether a given simplicial presheaf X (like an algebraic K -theory presheaf) satisfies finite descent is also sometimes called the *homotopy fixed points problem*.

Warning: You might be tempted (many were) to say that finite descent for X implies that X

satisfies descent for the étale topology on k , but you would be wrong. Such claims hold only in very special cases — see [7], [8].

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