

Lecture 09 (February 27, 2009)

21 The Verdier hypercovering theorem

Suppose that \mathcal{C} is a small Grothendieck site. As before, write $s\text{Pre}(\mathcal{C})$ for the category of simplicial presheaves on the site \mathcal{C} . The discussion that follows will be confined to simplicial presheaves. It has an exact analog for simplicial sheaves.

Let A be a fixed choice of simplicial presheaf. The slice category $A/s\text{Pre}(\mathcal{C})$ has all morphisms $x : A \rightarrow X$ as objects, and all diagrams

$$\begin{array}{ccc} & A & \\ x \swarrow & & \searrow y \\ X & \xrightarrow{f} & Y \end{array}$$

as morphisms.

The intuition, in applications, is that $x : A \rightarrow X$ is a “base point” of X (geometric points for the étale topology are good examples to keep in mind) even though A could be non-trivial homotopically.

Recall that the category $A/s\text{Pre}(\mathcal{C})$ inherits a local model structure from $s\text{Pre}(\mathcal{C})$, in that a mor-

phism $f : x \rightarrow y$ as above is a local weak equivalence (respectively cofibration, fibration) if and only if the underlying map $f : X \rightarrow Y$ is a local weak equivalence (respectively cofibration, fibration) of simplicial presheaves.

Remark 21.1. 1) Not all objects of the slice category are cofibrant: the identity morphism $1 : A \rightarrow A$ is initial, and so an object $x : A \rightarrow X$ is cofibrant if and only if the map x is a cofibration of simplicial presheaves.

2) The unique map $A \rightarrow *$ taking values in the terminal simplicial presheaf $*$ is the terminal object of $A/s\text{Pre}(\mathcal{C})$, and it follows that an object $x : A \rightarrow X$ is fibrant if and only if X is an injective fibrant simplicial presheaf.

Say that a map $f : x \rightarrow y$ in the slice category $A/s\text{Pre}(\mathcal{C})$ is a *hypercover* if the underlying simplicial presheaf map $f : X \rightarrow Y$ is a hypercover (or a local trivial fibration). More generally, $f : x \rightarrow y$ is a *local fibration* if the map $f : X \rightarrow Y$ is a local fibration of simplicial presheaves. In particular, $x : A \rightarrow X$ is locally fibrant if X is locally fibrant.

The theory of cocycle categories of [3] applies with-

out change to the model category $A/s\text{Pre}(\mathcal{C})$. Explicitly, a cocycle (g, f) from x to y is a diagram

$$x \xleftarrow[\simeq]{g} z \xrightarrow{f} y$$

in the slice category, or equivalently a commutative diagram of simplicial presheaf maps

$$\begin{array}{ccc} & A & \\ x \swarrow & \downarrow z & \searrow y \\ X & \xleftarrow[\simeq]{g} Z & \xrightarrow{f} Y \end{array}$$

for which the map g is a weak equivalence. These cocycles are the objects of a category $H(x, y)$ which has morphisms $\theta : (g, f) \rightarrow (g', f')$ given by the commutative diagrams

$$\begin{array}{ccc} & z & \\ x \xleftarrow{g} & \downarrow \theta & \xrightarrow{f} y \\ & z' & \\ & \xleftarrow{g'} & \xrightarrow{f'} \end{array}$$

The category $H(x, y)$ is the category of cocycles from x to y .

The model structure on $A/s\text{Pre}(\mathcal{C})$ is right proper, and weak equivalences in this structure are closed under finite products, because these properties both hold for the category of simplicial presheaves. Thus, Theorem 1 of [3] implies the following:

Lemma 21.2. *The function*

$$\phi : \pi_0 H(x, y) \rightarrow [x, y],$$

which is defined by $(g, f) \mapsto f \cdot g^{-1}$ for a cocycle (g, f) in the slice category $A/s\text{Pre}(\mathcal{C})$, is a bijection.

Suppose that $f, g : x \rightarrow y$ are morphisms of the slice category $A/s\text{Pre}(\mathcal{C})$. A (naive) pointed homotopy from f to g is a commutative diagram

$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr} & A \\ x \times \Delta^1 \downarrow & & \downarrow y \\ X \times \Delta^1 & \xrightarrow{h} & Y \end{array}$$

such that h is a simplicial homotopy from f to g in the usual sense. Here, the projection map $pr : A \times \Delta^1 \rightarrow A$ onto A defines the constant homotopy on A .

Equivalently, such a pointed homotopy is a map

$$h : (X \times \Delta^1) \cup_{A \times \Delta^1} A \rightarrow Y.$$

In the pushout diagram

$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr} & A \\ x \times \Delta^1 \downarrow & & \downarrow \\ X \times \Delta^1 & \xrightarrow{pr_*} & (X \times \Delta^1) \cup_{A \times \Delta^1} A \end{array}$$

the map pr_* is a weak equivalence if the map $x : A \rightarrow X$ is a cofibration, or if x is a cofibrant object of $A/s\text{Pre}(\mathcal{C})$. In that case, the pushout object is a cylinder for x in the slice category.

Every object $x : A \rightarrow X$ has a cofibrant model, meaning a diagram

$$\begin{array}{ccc} A & \xrightarrow{v} & Z \\ & \searrow x & \downarrow p \\ & & X \end{array}$$

such that v is a cofibration and p is a weak equivalence. If the maps $f, g : x \rightarrow y$ are pointed homotopic and $p : v \rightarrow x$ is a cofibrant model of x , then the composites fp and gp are pointed homotopic and therefore represent the same map in the homotopy category since v is cofibrant. But then p is an isomorphism in that category, so that $f = g$ in the homotopy category.

The objects of the category Triv/x are the pointed homotopy classes of maps $[p] : z \rightarrow x$ which are represented by hypercovers $p : z \rightarrow x$. The morphisms of this category are commutative triangles of pointed homotopy classes of maps in the obvious sense.

There is a contravariant set-valued functor which

takes an object $[p] : z \rightarrow x$ of $Triv/x$ to the set $\pi(z, y)$ of pointed homotopy classes of maps between z and y . There is a function

$$\phi_h : \varinjlim_{[p]:z \rightarrow x} \pi(z, y) \rightarrow [x, y]$$

which is defined by sending the diagram of pointed homotopy classes

$$x \xleftarrow{[p]} z \xrightarrow{[f]} y$$

to the morphism $f \cdot p^{-1}$ in the homotopy category.

The colimit

$$\varinjlim_{[p]:z \rightarrow x} \pi(z, y)$$

is the set of path components of a category $H_h(x, y)$ whose objects are the pictures of pointed homotopy classes

$$x \xleftarrow{[p]} z \xrightarrow{[f]} y,$$

such that $p : z \rightarrow x$ is a hypercover, and whose morphisms are the commutative diagrams

$$\begin{array}{ccccc} & & z & & \\ & [p] & \swarrow & [f] & \\ x & & & & y \\ & [p'] & \searrow & [f'] & \\ & & z' & & \end{array} \quad (21.1)$$

in pointed homotopy classes of maps. The map ϕ_h therefore has the form

$$\phi_h : \pi_0 H_h(x, y) \rightarrow [x, y]$$

The following result is a generalized Verdier hypercovering theorem:

Theorem 21.3. *The function*

$$\phi_h : \pi_0 H_h(x, y) \rightarrow [x, y]$$

is a bijection if y is locally fibrant.

Remark 21.4. Theorem 21.3 specializes to a generalization of the standard form of the Verdier hypercovering theorem [1, p.425], [2] if $A = \emptyset$, for the unique map $x : \emptyset \rightarrow X$. The object X is not required to be locally fibrant.

There are multiple variations of the category $H_h(x, y)$:

1) Write $H'_h(x, y)$ for the category whose objects are pictures

$$x \xleftarrow{p} z \xrightarrow{[f]} y$$

where p is a hypercover and $[f]$ is a pointed homotopy class of maps. The morphisms of $H'_h(x, y)$

are diagrams

$$\begin{array}{ccccc}
 & & z & \xrightarrow{[f]} & y \\
 & p \swarrow & \downarrow [\theta] & \searrow & \\
 x & & & & \\
 & p' \swarrow & z' & \xrightarrow{[f']} & y \\
 & & & &
 \end{array} \tag{21.2}$$

such that $[\theta]$ is a fibrewise pointed homotopy class of maps over x , and $[f'][\theta] = [f]$ as pointed homotopy classes. There is a functor

$$\omega : H'_h(x, y) \rightarrow H_h(x, y)$$

which is defined by the assignment $(p, [f]) \mapsto ([p], [f])$, and which sends the morphism (21.2) to the morphism (21.1).

2) Write $H''_h(x, y)$ for the category whose objects are the pictures

$$x \xleftarrow{p} z \xrightarrow{[f]} y$$

where p is a hypercover and $[f]$ is a pointed simplicial homotopy class of maps (as before). The morphisms of $H''_h(X, Z)$ are commutative diagrams

$$\begin{array}{ccc}
 & z & \\
 & \swarrow p & \downarrow \theta \\
 x & & \\
 & \swarrow p' & z'
 \end{array}$$

such that $[f' \cdot \theta] = [f]$. There is a canonical functor

$$H''_h(x, y) \xrightarrow{\omega'} H'_h(x, y)$$

which is the identity on objects, and takes morphisms θ to their associated fibrewise pointed homotopy classes.

3) Let $H_{hyp}(x, y)$ be the full subcategory of $H(x, y)$ whose objects are the cocycles

$$x \xleftarrow{p} z \xrightarrow{f} y$$

with p a hypercover. There is a functor

$$\omega'' : H_{hyp}(x, y) \rightarrow H_h''(x, y)$$

which takes a cocycle (p, f) to the object $(p, [f])$.

Lemma 21.5. *Suppose that y is locally fibrant. Then the inclusion functor $i : H_{hyp}(x, y) \subset H(x, y)$ is a homotopy equivalence.*

Proof. Objects of the cocycle category $H(x, y)$ can be identified with maps $(g, f) : z \rightarrow x \times y$ such that the morphism g is a weak equivalence, and morphisms of $H(x, y)$ are commutative triangles in the obvious way. Maps of the form (g, f) have functorial factorizations

$$\begin{array}{ccc} z & \xrightarrow{j} & v \\ & \searrow (g,f) & \downarrow (p,g') \\ & & x \times y \end{array} \quad (21.3)$$

such that j is a pointwise trivial cofibration and (p, g') is a pointwise Kan fibration. It follows that

(p, g') is a local fibration and the map p , or rather the composite

$$z \xrightarrow{(p, g')} x \times y \xrightarrow{pr} x,$$

is a local weak equivalence. The projection map pr is a local fibration since y is locally fibrant, so the map p is also a local fibration, and hence a hypercover.

It follows that the assignment $(u, g) \mapsto (p, g')$ defines a functor

$$\psi' : H(x, y) \rightarrow H_h(x, y).$$

The weak equivalences j of the diagram (21.3) define homotopies $p' \cdot i \simeq 1$ and $i \cdot \psi' \simeq 1$. \square

Proof of Theorem 21.3. The composite

$$H(x, y) \xrightarrow{\psi'} H_{hyp}(x, y) \xrightarrow{\omega''} H_h''(x, y) \xrightarrow{\omega'} H_h'(x, y) \xrightarrow{\omega} H_h(x, y)$$

is the functor ψ , and the composite

$$\begin{aligned} \pi_0 H(x, y) &\xrightarrow{\psi'_*} \pi_0 H_{hyp}(x, y) \xrightarrow{\omega''_*} \pi_0 H_h''(x, y) \xrightarrow{\omega'_*} \pi_0 H_h'(x, y) \\ &\xrightarrow{\omega_*} \pi_0 H_h(x, y) \xrightarrow{\phi_h} [x, y] \end{aligned} \quad (21.4)$$

is the bijection ϕ of Lemma 21.2. The function ψ'_* is a bijection by Lemma 21.5, and the functions ω''_* , ω'_* and ω_* are surjective, as is the function ϕ_h .

The functions which make up the string (21.4) are therefore all bijections. \square

The following is a corollary of the proof of Theorem 21.3 which deserves independent mention:

Corollary 21.6. *Suppose that the object $y : A \rightarrow Y$ of $A/s\text{Pre}(\mathcal{C})$ is locally fibrant. Then the induced functions*

$$\pi_0 H_{hyp}(x, y) \xrightarrow{\omega''_*} \pi_0 H''_h(x, y) \xrightarrow{\omega'_*} \pi_0 H'_h(x, y) \xrightarrow{\omega_*} \pi_0 H_h(x, y)$$

are bijections, and all of these sets are isomorphic to the set $[x, y]$ of morphisms $x \rightarrow y$ in the homotopy category $\text{Ho}(s/\text{Pre}(\mathcal{C}))$.

The bijections of the path component objects in the statement of Corollary 21.6 with the set $[x, y]$ all represent specific variants of the Verdier hypercovering theorem.

References

- [1] Kenneth S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1974.
- [2] J. F. Jardine. Simplicial objects in a Grothendieck topos. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, pages 193–239. Amer. Math. Soc., Providence, RI, 1986.
- [3] J. F. Jardine. Cocycle categories. Preprint, to appear in *Abel Symposium Proceedings*, <http://www.math.uwo.ca/~jardine/papers>, 2007.