

Lecture 12 (March 27, 2009)

25 T -spectra

Suppose that T is a pointed simplicial presheaf on a small site \mathcal{C} .

A T -spectrum X is a collection of pointed simplicial presheaves X^n , $n \geq 0$, with pointed maps $\sigma : T \wedge X^n \rightarrow X^{n+1}$. A map $f : X \rightarrow Y$ of T -spectra consists of pointed simplicial presheaf maps $f : X^n \rightarrow Y^n$ which respect structure in the sense that the diagrams

$$\begin{array}{ccc} T \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ T \wedge f \downarrow & & \downarrow f \\ T \wedge Y^n & \xrightarrow{\sigma} & Y^{n+1} \end{array}$$

commute. Write $\text{Spt}_T(\mathcal{C})$ for the category of T -spectra.

Say that a map $f : X \rightarrow Y$ of T -spectra is a *strict weak equivalence* (respectively *strict fibration*) if all maps $f : X^n \rightarrow Y^n$ are local weak equivalences (respectively injective fibrations) of pointed simplicial presheaves on \mathcal{C} .

A *cofibration* of T -spectra is a map $i : A \rightarrow B$ such that

- $i : A^0 \rightarrow B^0$ is a cofibration of simplicial presheaves, and
- all maps

$$(T \wedge B^n) \cup_{(T \wedge A^n)} A^{n+1} \rightarrow B^{n+1}$$

are cofibrations of simplicial presheaves.

If K is a pointed simplicial presheaf and X is a T -spectrum, then $X \wedge K$ has the obvious meaning:

$$(X \wedge K)^n = X^n \wedge K.$$

The *function complex* $\mathbf{hom}(X, Y)$ for T -spectra X and Y is the pointed simplicial set with

$$\mathbf{hom}(X, Y)_n = \{ X \wedge \Delta_+^n \rightarrow Y \}.$$

Lemma 25.1. *With these definitions, the category of $\mathrm{Spt}_T(\mathcal{C})$ of T -spectra on \mathcal{C} satisfies the definitions for a proper closed simplicial model category.*

The proof is the usual thing.

Suspensions and shifts work in $\mathrm{Spt}_T(\mathcal{C})$ just like for ordinary spectra:

- Given a pointed simplicial presheaf K , the *suspension spectrum* $\Sigma_T^\infty K$ is the T -spectrum

$$K, T \wedge K, T^2 \wedge K, \dots$$

with $T^n = T \wedge \cdots \wedge T$ (n -fold smash power). The functor $K \mapsto \Sigma_T^\infty K$ is left adjoint to the 0-level functor $X \mapsto X^0$.

The suspension spectrum $\Sigma_T^\infty S^0$ is also denoted by S_T and is called the T -sphere spectrum.

- Given a T -spectrum X , $n \in \mathbb{Z}$,

$$X[n]^k = \begin{cases} X^{n+k} & n+k \geq 0 \\ * & n+k < 0 \end{cases}$$

Lemma 25.2. *Suppose given the diagram*

$$\begin{array}{ccc} A \cap X & \longrightarrow & X \\ j_* \downarrow & & \downarrow j \\ A & \xrightarrow{i} & Y \end{array}$$

in spectra, where j is a cofibration and i is a levelwise cofibration. Then the induced map $j_ : A \cap X \rightarrow A$ is a cofibration.*

Proof. The proof is set theoretic. □

What now follows is a general set of tricks that applies to any set S of cofibrations $i : A \rightarrow B$ of $\text{Spt}_T(\mathcal{C})$.

Suppose that α is a cardinal such that $\alpha > |\text{Mor}(\mathcal{C})|$. Suppose also that $\alpha > |B|$ for all morphisms $i :$

$A \rightarrow B$ appearing in the set S and that $\alpha > |S|$.
Choose a cardinal λ such that $\lambda > 2^\alpha$.

Suppose that $f : X \rightarrow Y$ is a morphism of $\text{Spt}_T(\mathcal{C})$.
Define a functorial system of factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_s} & E_s(f) \\ & \searrow f & \downarrow f_s \\ & & Y \end{array}$$

of the map f indexed on all ordinal numbers $s < \lambda$
as follows:

- 1) Given the factorization (f_s, i_s) define the factorization (f_{s+1}, i_{s+1}) by requiring that the diagram

$$\begin{array}{ccc} \vee_{\mathbf{D}} A & \xrightarrow{(\alpha_{\mathbf{D}})} & E_s(f) \\ \vee i \downarrow & & \downarrow \\ \vee_{\mathbf{D}} B & \rightarrow & E_{s+1}(f) \end{array}$$

is a pushout, where the wedge is indexed over all diagrams \mathbf{D} of the form

$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathbf{D}}} & E_s(f) \\ i \downarrow & & \downarrow f_s \\ B & \xrightarrow{\beta_{\mathbf{D}}} & Y \end{array}$$

with $i : A \rightarrow B$ in the set S . Then the map i_{s+1} is the composite

$$X \xrightarrow{i_s} E_s(f) \xrightarrow{g_*} E_{s+1}(f)$$

2) If s is a limit ordinal, set $E_s(f) = \varinjlim_{t < s} E_t(f)$.
Set $E_\lambda(f) = \varinjlim_{s < \lambda} E_s(f)$. Then there is an induced factorization

$$\begin{array}{ccc} X & \xrightarrow{i_\lambda} & E_\lambda(f) \\ & \searrow f & \downarrow f_\lambda \\ & & Y \end{array}$$

of the map f . Then i_λ is a cofibration. The map f_λ has the right lifting property with respect to the cofibrations $i : A \rightarrow B$ in S by a standard argument, since any map $\alpha : A \rightarrow E_\lambda(f)$ must factor through some $E_s(f)$ by the choice of cardinal λ .

Write $L(X) = E_\lambda(c)$ for the result of this construction when applied to the canonical map $c : X \rightarrow *$. Then we have the following:

Lemma 25.3. *1) Suppose that $t \mapsto X_t$ is a diagram of level cofibrations indexed by any cardinal $\gamma > 2^\alpha$. Then the natural map*

$$\varinjlim_{t < \gamma} L(X_t) \rightarrow L(\varinjlim_{t < \gamma} X_t)$$

is an isomorphism.

2) The functor $X \mapsto L(X)$ preserves level cofibrations.

3) Suppose that ζ is a cardinal with $\zeta > \alpha$, and let $\mathcal{F}_\zeta(X)$ denote the filtered system of

subobjects of X having cardinality less than ζ . Then the natural map

$$\varinjlim_{Y \in \mathcal{F}_\zeta(X)} L(Y) \rightarrow L(X)$$

is an isomorphism.

4) If $|X| < 2^\omega$ where $\omega \geq \alpha$ then $|L(X)| < 2^\omega$.

5) Suppose that U, V are subobjects of a presheaf of T -spectra X . Then the natural map

$$L(U \cap V) \rightarrow L(U) \cap L(V)$$

is an isomorphism.

Proof. The argument is the same as for Lemma 22.4. \square

Basic Assumptions: Suppose that S is a set of cofibrations such that

- 1) A is cofibrant for all $i : A \rightarrow B$ in S ,
- 2) S includes the set I of generating maps

$$\Sigma_T^\infty C[-n] \rightarrow \Sigma_T^\infty D[-n], \quad n \geq 0,$$

for the strict trivial cofibrations of $\text{Spt}_T(\mathcal{C})$, which are induced by the α -bounded trivial cofibrations $C \rightarrow D$ of pointed simplicial presheaves, and

3) S includes all cofibrations

$$(A \wedge D) \cup (B \wedge C) \rightarrow B \wedge D, \quad m \geq 0,$$

for $A \rightarrow B$ in S and all α -bounded pointed cofibrations $C \rightarrow D$ of simplicial presheaves.

A map $p : X \rightarrow Y$ is said to be *injective* if it has the right lifting property with respect to all maps of S . An object X is injective if the map $X \rightarrow *$ is injective. By construction, LX is injective for every object X . Every injective object is strictly fibrant.

Say that a map $f : X \rightarrow Y$ of $\text{Spt}(\mathcal{C})$ is an *L-equivalence* if it induces a bijection

$$f^* : [Y, Z] \xrightarrow{\cong} [X, Z]$$

in morphisms in the strict homotopy category for every injective object Z .

Every strict equivalence $X \rightarrow Y$ is an *L-equivalence*.

Lemma 25.4. *Suppose that $i : A \rightarrow B$ is a cofibration with A cofibrant. Then i is an L-equivalence if*

1) *i induces a trivial fibration*

$$i^* : \mathbf{hom}(B, Z) \rightarrow \mathbf{hom}(A, Z)$$

for all injective Z , or

2) all injective Z have the right lifting property with respect to i and with respect to the cofibration

$$(A \wedge \Delta_+^1) \cup (B \wedge \partial\Delta_+^1) \rightarrow B \wedge \Delta_+^1.$$

Proof. The first claim is trivial.

The second claim is almost as easy: we must show that the induced function

$$i^* : \pi(B, Z) \rightarrow \pi(A, Z)$$

in naive homotopy classes is a bijection for all injective Z . This suffices, because A and B are cofibrant and Z is strictly fibrant.

Every morphism $A \rightarrow Z$ extends to a morphism $B \rightarrow Z$ because $Z \rightarrow *$ has the right lifting property with respect to i . It follows that i^* is surjective.

Given $f, g : B \rightarrow Z$, if there is a homotopy $h : A \wedge \Delta_+^1 \rightarrow Z$ from $f|_A$ to $g|_A$, then there is a diagram

$$\begin{array}{ccc} (B \wedge \partial\Delta_+^1) \cup (A \wedge \Delta_+^1) & \xrightarrow{((f,g),h)} & Z \\ \downarrow & \nearrow \text{dotted} & \\ B \wedge \Delta_+^1 & & \end{array}$$

where the indicated lifting exists because Z is injective and the vertical map is a member of S . But then f and g are homotopic, so that i^* is injective. \square

Corollary 25.5. *All cofibrations appearing in the set S are L -equivalences.*

Proof. Every cofibration $i : A \rightarrow B$ appearing in the set S induces a trivial fibration

$$i^* : \mathbf{hom}(B, Z) \rightarrow \mathbf{hom}(A, Z)$$

by construction. \square

Note that a map $f : Z \rightarrow W$ between injective objects is an L -equivalence if and only if it is a strict equivalence. In effect, the requirement that f is an L -equivalence forces f to be an isomorphism in the strict homotopy category, and hence a strict equivalence.

A *cofibrant replacement* for a map $f : X \rightarrow Y$ is a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ \pi_X \downarrow & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array}$$

in which the maps π_X and π_Y are trivial strict fibrations, \tilde{X} is cofibrant and j is a cofibration. Any

two cofibrant replacements for a fixed map f are strictly equivalent, by a standard argument. The map f is an L -equivalence if and only if it has a cofibrant replacement j which is an L -equivalence.

Note that if some cofibrant replacement j for f induces a trivial fibration

$$j^* : \mathbf{hom}(\tilde{Y}, Z) \rightarrow \mathbf{hom}(\tilde{X}, Z)$$

for all injective objects Z , then all cofibrant replacements for f have this property.

Lemma 25.6. *All cofibrations in the saturation of the set S are L -equivalences.*

Proof. The saturation of the set S is the family of cofibrations which has the left lifting property with respect to all injective maps $X \rightarrow Y$.

If the cofibration $j : C \rightarrow D$ is coproduct of members of S (hence with C and D cofibrant), then

$$j^* : \mathbf{hom}(D, Z) \rightarrow \mathbf{hom}(C, Z)$$

is a product of trivial fibrations and is therefore a trivial fibration.

Suppose given a pushout diagram

$$\begin{array}{ccc} C & \longrightarrow & C' \\ j \downarrow & & \downarrow j' \\ D & \longrightarrow & D' \end{array}$$

where j is a coproduct of members of S and C' is cofibrant. Then from the pullback diagram

$$\begin{array}{ccc} \mathbf{hom}(D', Z) & \longrightarrow & \mathbf{hom}(D, Z) \\ j'^* \downarrow & & \downarrow j^* \\ \mathbf{hom}(C', Z) & \longrightarrow & \mathbf{hom}(C, Z) \end{array}$$

we see that j'^* is a trivial fibration for all injective Z .

Suppose given a pushout diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & E \\ j \downarrow & & \downarrow \\ D & \longrightarrow & D \cup_C E \end{array}$$

with j as above and E arbitrary. Then there is a factorization

$$\begin{array}{ccc} C & \xrightarrow{i} & \tilde{E} \\ & \searrow \alpha & \downarrow \pi \\ & & E \end{array}$$

of α with π a strictly trivial fibration and i a cofibration, and there is an induced commutative di-

agram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{j}_*} & D \cup_C \tilde{E} \\ \pi \downarrow & & \downarrow \pi_* \\ E & \xrightarrow{j_*} & D \cup_C E \end{array}$$

The map π is a strict equivalence, so that π_* is a strict equivalence by properness. The map \tilde{j}_* induces a trivial fibration

$$(\tilde{j}_*)^* : \mathbf{hom}(D \cup_C \tilde{E}, Z) \rightarrow \mathbf{hom}(\tilde{E}, Z)$$

for all injective Z , by the previous paragraph. It follows that some cofibrant replacement of the map

$$j_* : E \rightarrow D \cup_C E$$

induces a corresponding function complex weak equivalence.

Suppose given a string of morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \rightarrow \dots$$

such that each f_i is an L -equivalence. Take a “cofibrant replacement”

$$\begin{array}{ccccccc} A_0 & \xrightarrow{i_1} & A_1 & \xrightarrow{i_2} & A_2 & \longrightarrow & \dots \\ \pi_0 \downarrow & & \pi_1 \downarrow & & \downarrow \pi_2 & & \\ X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \longrightarrow & \dots \end{array}$$

in which A_0 is cofibrant, all i_k are cofibrations and all π_j are trivial strict fibrations. Then all maps i_k induce trivial fibrations

$$i_k^* : \mathbf{hom}(A_k, Z) \rightarrow \mathbf{hom}(A_{k-1}, Z)$$

for all injective Z , so the cofibration $A_0 \rightarrow \varinjlim_i A_i$ induces a trivial fibration

$$\mathbf{hom}(\varinjlim_i A_i, Z) \rightarrow \mathbf{hom}(A_0, Z).$$

for all injective Z . The map

$$\varinjlim_i A_i \rightarrow \varinjlim_i X_i$$

is a (sectionwise) weak equivalence, and it follows that some cofibrant replacement for the map $X_0 \rightarrow \varinjlim_i X_i$ induces a trivial fibration in all function complexes taking values in injective objects Z .

It follows that every member $i : A \rightarrow B$ of the saturation of S has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ & \searrow i & \downarrow \pi \\ & & B \end{array}$$

such that π is injective and j is a member of the saturation of S which is also an L -equivalence. The map i has the left lifting property with re-

spect to all injective maps such as π , so that i is a retract of j . \square

Corollary 25.7. 1) *The natural map $j : X \rightarrow LX$ is an L -equivalence.*

2) *A map $f : X \rightarrow Y$ is an L -equivalence if and only if the induced map $Lf : LX \rightarrow LY$ is a strict equivalence.*

Lemma 25.8. *Suppose that $\gamma \geq \alpha$. Suppose further that $i : X \rightarrow Y$ is a level cofibration and a strict equivalence and that $A \subset Y$ is an γ -bounded subobject. Then there is a γ -bounded subobject $B \subset Y$ with $A \subset B$ such that the level cofibration $B \cap X \rightarrow B$ is a strict equivalence.*

Proof. First of all, consider the diagram of cofibrations

$$\begin{array}{ccc} & & X^0 \\ & & \downarrow i \\ A^0 & \longrightarrow & Y^0 \end{array}$$

Then by Lemma 11.1 (the bounded cofibration condition for simplicial presheaves) there is a subobject $B^0 \subset Y^0$ such that B^0 is γ -bounded, $A^0 \subset B^0$ and $B^0 \cap X^0 \rightarrow B^0$ is a local weak equivalence.

Form the diagram

$$\begin{array}{ccccc} T \wedge A^0 & \longrightarrow & T \wedge B^0 & \longrightarrow & T \wedge Y^0 \\ \sigma \downarrow & & & & \downarrow \sigma \\ A^1 & \longrightarrow & & \longrightarrow & Y^1 \end{array}$$

Then the induced map

$$A^1 \cup_{T \wedge A^0} T \wedge B^0 \rightarrow Y^1$$

factors through a γ -bounded subobject $C^1 \subset Y^1$. There is a γ -bounded subobject $B^1 \subset Y^1$ such that $C^1 \subset B^1$ and $B^1 \cap X^1 \rightarrow B^1$ is a local weak equivalence. The composite

$$T \wedge B^0 \rightarrow A^1 \cup_{T \wedge A^0} T \wedge B^0 \rightarrow C^1 \subset B^1$$

is the bonding map up to level 1 for the object B .

Construct the remaining objects B^n , $n \geq 1$, inductively according to this recipe. \square

Lemma 25.9. *Suppose given a cofibration $i : X \rightarrow Y$ which is an L -equivalence, and suppose that $A \subset Y$ is a 2^λ -bounded subobject, where λ is chosen as above. Then there is a 2^λ -bounded subobject $B \subset Y$ with $A \subset B$ and such that the cofibration $B \cap X \rightarrow B$ is an L -equivalence.*

Proof. Write $B_0 = A$, and set $\kappa = 2^\lambda$.

Consider the diagram

$$\begin{array}{ccc} & LX & \\ & \downarrow & \\ LB_0 & \longrightarrow & LY \end{array}$$

Then the maps are level cofibrations (Lemma 25.3.2) and $LX \rightarrow LY$ is a strict equivalence by assumption. The object LB_0 is κ -bounded by Lemma 25.3.4, so there is a κ -bounded subobject $C_1 \subset LY$ with $LB_0 \subset C_1$ such that $C_1 \cap LX \rightarrow C_1$ is a strict equivalence, by Lemma 25.8. Since C_1 is κ -bounded there is a κ -bounded subobject $B_1 \subset Y$ with $B_0 \subset B_1$ such that $C_1 \subset LB_1$ (Lemma 25.3.3). Proceeding inductively we find κ -bounded subobjects

$$C_1 \subset C_2 \subset \dots$$

of LY and κ -bounded subobjects

$$B_0 \subset B_1 \subset B_2 \subset \dots$$

indexed by $i < \kappa$, such that C_s and B_s are defined at limit ordinals s by colimits, and

$$LB_i \subset C_{i+1} \subset LB_{i+1}$$

and $C_i \cap LX \rightarrow C_i$ is a level weak equivalence.

Write $B = \varinjlim_{i < \kappa} B_i$. Then B is κ -bounded, and

$$L(B) = \varinjlim_{i < \kappa} L(B_i) = \varinjlim_{i < \kappa} C_i$$

by Lemma 25.3.1 and construction. Also

$$\begin{aligned} L(B \cap X) &= L(B) \cap L(X) = \varinjlim_{i < \kappa} L(B_i) \cap L(X) \\ &\cong \varinjlim_{i < \kappa} C_i \cap L(X) \end{aligned}$$

by Lemma 25.3.1 and 25.3.5 and construction. It follows that the map

$$B \cap X \rightarrow B$$

is an L -equivalence. \square

Say that a cofibration is L -trivial if it is an L -equivalence.

Lemma 25.10. *The set of κ -bounded L -trivial cofibrations is a generating set for the class of L -trivial cofibrations.*

Proof. Run the solution set argument of Lemma 23.5 using Lemma 25.9 for the set of κ -bounded cofibrations. Recall that the κ -bounded cofibrations generate the class of cofibrations. \square

Say that a map $p : X \rightarrow Y$ is an L -fibration if it has the right lifting property with respect to all L -trivial cofibrations. Observe that every L -fibration is a strict fibration, since S contains a generating set for the class of strict trivial cofibrations.

Lemma 25.11. *A map $p : X \rightarrow Y$ is an L -fibration and an L -equivalence if and only if p is a trivial strict fibration.*

Proof. We need only show that p is a trivial strict fibration if it is an L -fibration and an L -equivalence, but this is the usual proof: find a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & \searrow p & \downarrow \pi \\ & & Y \end{array}$$

where j is a cofibration and π is a trivial strict fibration. But then j is an L -equivalence so the lifting exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ j \downarrow & \nearrow & \downarrow p \\ W & \xrightarrow{\pi} & Y \end{array}$$

so that p is a retract of π . □

Theorem 25.12. *Suppose that S is a set of cofibrations which satisfies the list of basic assumptions above. Let the L -equivalences and L -fibrations be defined relative to the set S . Then with these definitions the category $\text{Spt}_T(\mathcal{C})$ satisfies the axioms for a closed simplicial model category.*

Proof. Every map $f : X \rightarrow Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & \searrow f & \downarrow p \\ & & Y \end{array}$$

such that p is an L -fibration and j is a cofibration and an L -equivalence, by Lemma 25.6 and Lemma 25.10.

Every map $f : X \rightarrow Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & \searrow f & \downarrow q \\ & & Y \end{array}$$

such that i is a cofibration and q is a strictly trivial fibration. But then q is an L -fibration and an L -equivalence.

The rest of the closed model axioms are trivial to verify.

For the closed simplicial model structure, we need to show that if $i : A \rightarrow B$ is a cofibration and an L -equivalence, then all maps

$$i \wedge \partial\Delta_+^n : A \wedge \partial\Delta_+^n \rightarrow B \wedge \partial\Delta_+^n$$

are L -equivalences. By replacing by a cofibrant model if necessary, it is enough to assume that

A is cofibrant. Then one uses the usual patching argument for the category of cofibrant objects in the L -model structure for $\text{Spt}_T(\mathcal{C})$ to compare pushouts of the form

$$\begin{array}{ccc} A \wedge \partial\Delta_+^{n-1} & \longrightarrow & A \wedge \Lambda_{k+}^n \\ \downarrow & & \downarrow \\ A \wedge \Delta_+^{n-1} & \longrightarrow & A \wedge \partial\Delta_+^n \end{array}$$

to show inductively that the question reduces to showing that the map

$$i \vee i : A \vee A \rightarrow B \vee B$$

is an L -equivalence. But $i \vee i$ has the left lifting property with respect to all L -fibrations, and must therefore be an L -trivial cofibration. \square

Lemma 25.13. *The L -structure on $\text{Spt}_T(\mathcal{C})$ is left proper: given a pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ i \downarrow & & \downarrow \\ B & \xrightarrow{f_*} & D \end{array}$$

in which i is a cofibration, if f is an L -equivalence then f_ is an L -equivalence.*

Proof. The original diagram may be replaced up

to strict weak equivalence by a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f'} & C' \\ i \downarrow & & \downarrow \\ B & \xrightarrow{f'_*} & D' \end{array}$$

in which f' is a cofibration and an L -equivalence. But then f'_* is also an L -trivial cofibration and is in particular an L -equivalence. \square

Lemma 25.14. *Every injective object is L -fibrant, so that the L -fibrant T -spectra coincide with the injective T -spectra.*

Proof. Suppose that X is injective, and suppose given a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ i \downarrow & & \\ B & & \end{array}$$

where the morphism i is a cofibration and an L -equivalence. Then $\alpha = \alpha' \cdot j$ for some map $\alpha' : LA \rightarrow X$ since X is injective, and so there is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{j} & LA & \xrightarrow{\alpha'} & X \\ i \downarrow & & \downarrow Li & & \\ B & \xrightarrow{j} & LB & & \end{array}$$

which factorizes the original. The map Li is a strict equivalence by Corollary 25.7.

One finishes the argument in the usual way: Li has a factorization

$$\begin{array}{ccc} LA & \xrightarrow{i'} & W \\ & \searrow Li & \downarrow p \\ & & LB \end{array}$$

where i' is a cofibration, p is a strict fibration and both maps are strict weak equivalences. Then X is strictly fibrant so there is a map $\sigma : W \rightarrow X$ such that $\sigma \cdot i' = \alpha'$, and there is a map $\theta : B \rightarrow W$ such that $p \cdot \theta = j$ and $\theta \cdot i = i' \cdot j$. \square

Now we can go further, to give a general recognition principle for L -fibrations. The most complete statement (Theorem 25.17 below) depends on right properness for the L -structure, which will be addressed in a subsequent section.

Lemma 25.15. *Suppose that $p : X \rightarrow Y$ is a strict fibration between L -fibrant T -spectra. Then p is an L -fibration.*

Proof. Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array} \tag{25.1}$$

where i is a cofibration and an L -equivalence. Then the induced map $i_* : LA \rightarrow LB$ is a strict equivalence, as are the L -fibrant model maps $j : X \rightarrow LX$ and $j : Y \rightarrow LY$. The induced diagram

$$\begin{array}{ccc} LA & \longrightarrow & LX \\ i_* \downarrow & & \downarrow p_* \\ LB & \longrightarrow & LY \end{array}$$

has a factorization

$$\begin{array}{ccccc} LA & \xrightarrow{j_A} & V_X & \xrightarrow{p_X} & LX \\ i_* \downarrow & & \downarrow i' & & \downarrow p_* \\ LB & \xrightarrow{j_B} & V_Y & \xrightarrow{q_Y} & LY \end{array}$$

such that j_A and j_B are strict trivial cofibrations and p_X and p_Y are strict fibrations. In the pullback diagram

$$\begin{array}{ccc} V_X \times_{LX} X & \longrightarrow & X \\ j_{X*} \downarrow & & \downarrow j_X \\ V_X & \xrightarrow{p_X} & LX \end{array}$$

the map j_{X*} is a strict equivalence. The corresponding map j_{Y*} in the diagram

$$\begin{array}{ccccc} LA & \xrightarrow{j_A} & V_X & \xleftarrow{j_{X*}} & V_X \times_{LX} X \\ \downarrow & & \downarrow & & \downarrow \\ LB & \xrightarrow{j_B} & V_Y & \xleftarrow{j_{Y*}} & V_Y \times_{LY} Y \end{array}$$

is also a strict equivalence. It follows that the induced map

$$V_X \times_{LX} X \rightarrow V_Y \times_{LY} Y$$

is a strict equivalence, and that the diagram (25.1) has a factorization

$$\begin{array}{ccccc} A & \longrightarrow & V_X \times_{LX} X & \longrightarrow & X \\ i \downarrow & & \downarrow \simeq & & \downarrow p \\ B & \longrightarrow & V_Y \times_{LY} Y & \longrightarrow & Y \end{array}$$

in which the middle vertical map is a strict equivalence. The result follows by a standard argument: one factorizes the middle vertical map as a trivial strict cofibration followed by a trivial strict fibration. \square

Proposition 25.16. *Suppose that $p : X \rightarrow Y$ is a strict fibration. Then p is an L -fibration if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & LX \\ p \downarrow & & \downarrow Lp \\ Y & \xrightarrow{i} & LY \end{array} \quad (25.2)$$

is strictly homotopy cartesian.

Proof. Suppose that the diagram (25.2) is strictly

homotopy cartesian. There is a factorization

$$\begin{array}{ccc} LX & \xrightarrow{j} & Z \\ & \searrow^{Lp} & \downarrow q \\ & & LY \end{array}$$

of LP such that j is a stable equivalence and q is an injective fibration. But then Z is injective, hence L -fibrant, so that j is a strict equivalence. It also follows from Lemma 25.15 that q is an L -fibration. By pulling back q along i , we see from the hypothesis that the induced map

$$X \rightarrow Y \times_{LY} Z$$

is a strict equivalence. Every trivial strict fibration is an L -fibration, and it follows that p is a retract of an L -fibration, and hence is itself an L -fibration. \square

Theorem 25.17. *Suppose that the L -structure of Theorem 25.12 is right proper. Suppose that $p : X \rightarrow Y$ is a strict fibration. Then p is an L -fibration if and only if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & LX \\ p \downarrow & & \downarrow Lp \\ Y & \xrightarrow{i} & LY \end{array} \quad (25.3)$$

is strictly homotopy cartesian.

Proof. We already have Proposition 25.16.

Suppose that the map $p : X \rightarrow Y$ is a stable fibration, and take a factorization

$$\begin{array}{ccc} LX & \xrightarrow{j} & Z \\ & \searrow^{Lp} & \downarrow q \\ & & LY \end{array}$$

of the map Lp such that q is a stable fibration and j is a stably trivial cofibration. Then j is an L -equivalence between L -fibrant T -spectra, so that j is a strict equivalence on account of Lemma 25.11.

The induced map $i_* : Y \times_{LY} Z \rightarrow Z$ is an L -equivalence by the right properness assumption, so that the canonical map $\theta : X \rightarrow Y \times_{LY} Z$ is a stable equivalence, and the map

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \times_{LY} Z \\ & \searrow p & \swarrow q_* \\ & & Y \end{array}$$

is an equivalence of fibrant objects for the model structure on $\text{Spt}_T(\mathcal{C})/Y$ which is induced by the

L -structure on $\text{Spt}_T(\mathcal{C})$. Form the diagram

$$\begin{array}{ccc}
 V_1 & \xrightarrow{\tilde{\theta}} & V_2 \\
 \pi_1 \downarrow & & \downarrow \pi_2 \\
 X & \xrightarrow{\theta} & Y \times_{LY} Z \\
 p \searrow & & \swarrow q_* \\
 & Y &
 \end{array}$$

where π_1 and π_2 are trivial strict fibrations and V_1 and V_2 are cofibrant. Then $\tilde{\theta}$ is a weak equivalence between objects of $\text{Spt}_T(\mathcal{C})/T$ which are both fibrant and cofibrant, and is therefore a (fibrewise) homotopy equivalence, and hence a strict weak equivalence. \square