Lecture 001 (September 25, 2005)

Simplicial sets

Simplicial sets are contravariant functors $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$, and simplicial maps are natural transformations. $\mathbf{\Delta} =$ finite ordinal numbers $\mathbf{n} = \{0, 1, \dots, n\}$, with order-preserving functions between them. The category of simplicial sets will be denoted by **sSet**.

Examples:

- The *standard n-simplex* Δ^n is the simplicial set represented by the ordinal number $\mathbf{n}$.

  $$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n}).$$

  For every n-simplex $x \in X_n = X(\mathbf{n})$ there is a unique simp. set map $\sigma_x : \Delta^n \to X$ such that $\sigma_x(1_{\mathbf{n}}) = x$ (Yoneda lemma). Notation $\iota_n = 1_{\mathbf{n}}$.

- $d^i : \mathbf{n-1} \to \mathbf{n}$ is the string

  $$0 \to 1 \to \dots i-1 \to i+1 \to \dots n$$

  of length $n-1$ in $\mathbf{n}$. $d^i \in \Delta^n_{n-1}$ and one writes $d_i(\iota_n) = d^i$.

  More generally $d_i(x) = (d^i)^*(x) \in X_{n-1}$. $d_i(x)$ is the i^{th} *face* of x.

$\partial\Delta^n \subset \Delta^n$ (*boundary* of Δ^n) is the subcomplex generated by the $d_i(\iota_n)$. The k^{th} *horn* $\Lambda^n_k \subset \Delta^n$ is the subcomplex generated by $d_i(\iota_n), i \neq k$.

- $s^j : \mathbf{n+1} \to \mathbf{n}$ is the string

  $$0 \to 1 \to \cdots \to j \xrightarrow{1} j \to j+1 \to \cdots \to n$$

  s^j induces $s_j : X_n \to X_{n+1}$ via $s_j(x) = (s^j)^*(x)$. $s^j x$ is the j^{th} *degeneracy* of x. A simplex y is said to be *degenerate* if $y = s_j(x)$ for some x; otherwise y is *non-degenerate*.

- C = small category. $BC_n = \hom(\mathbf{n}, C)$, and the simplicial structure is induced by precomposition. BC is called the *classifying space* or *nerve* of C. If G is a group, the realization $|BG|$ of BG is the classifying space for G in old-time algebraic topology.

- There is a covariant functor $\mathbf{\Delta} \to \mathbf{Top}$ with $n \mapsto |\Delta^n|$.

  $$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$$

  Y = space: the *singular complex* $S(Y)$ for Y is defined by

  $$S(Y)_n = \hom(|\Delta^n|, Y)$$

with simplicial structure defined by precomposition.

The *singular functor* $S : \mathbf{Top} \to \mathbf{sSet}$ has a left adjoint $| \ | : \mathbf{sSet} \to \mathbf{Top}$, called *realization*.

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|.$$

- X = simplicial set. The *simplex category*

$$\mathbf{\Delta}/X = i_{\mathbf{\Delta}} X$$

is the category with objects all simplices $\Delta^n \to X$, and morphisms given by all incidence relations

$$\begin{array}{ccc} \Delta^n & & \\ \downarrow & \searrow & \\ & & X \\ & \nearrow & \\ \Delta^m & & \end{array}$$

Exercise: Show that there is an isomorphism

$$\varinjlim_{\Delta^n \to X} \Delta^n \cong X.$$

In other words, show that X is a colimit of its simplices.

- If X, Y are simplicial sets the *function complex* $\mathbf{hom}(X, Y)$ is the simplicial set with

$$\mathbf{hom}(X, Y)_n = \hom_{\mathbf{sSet}}(X \times \Delta^n, Y).$$

The functor $(X, Y) \mapsto \mathbf{hom}(X, Y)$ is contravariant in X and covariant in Y. There is a natural bijection

$$\hom(X, \mathbf{hom}(Z, Y)) \cong \hom(X \times Z, Y)$$

which is induced by the exponential map $\mathbf{hom}(Z, Y) \times Y \to Z$ defined by

$$(f : Z \times \Delta^n \to Y, z) \mapsto f(z, \iota_n).$$

Recall **sSet** = simplicial sets has a *closed model structure* such that

- A map $f : X \to Y$ is a *weak equivalence* iff the induced map $|X| \to |Y|$ of top. realizations is a homotopy equivalence (of CW-complexes).
- A *cofibration* is $A \to B$ of simplicial sets is a monomorphism.
- A *fibration* is a map which has the RLP wrt all trivial cofibrations (aka. maps which are cofibrations and weak equivalences).

Equivalently (big theorem): $p : X \to Y$ is a fibration iff p has the RLP wrt all inclusions $\Lambda^n_k \subset \Delta^n$. This is the Kan fibration condition, and the theorem says that the fibrations are the Kan fibrations.

Theorem: With these definitions, **sSet** satisfies the conditions for a proper closed simplicial model category.

The "simplicial" in *simplicial model category* means that if $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration, then the induced map

$$\mathbf{hom}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, Y)$$

is a fibration which is trivial if either i or p is trivial (ie. a weak equivalence).

To say that the model structure is *proper* means two things:

- Given a pullback diagram

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{f_*} & X \\ \downarrow & & \downarrow p \\ Z & \xrightarrow[f]{} & Y \end{array}$$

  with p a fibration, if f is a weak equivalence so is f_*. *Right properness*: weak equivalences are stable under base change along fibrations.

- Given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ i \downarrow & & \downarrow \\ B & \xrightarrow[g_*]{} & B \cup_A X \end{array}$$

with i a cofibration, if g is a weak equivalence, so is g_*. *Left properness*: weak equivalences are stable under cobase change along cofibrations.

Theorem: The adjoint pair of functors

$$| \ | : \mathbf{sSet} \leftrightarrows \mathbf{Top} : S$$

induces an equivalence of homotopy categories

$$\mathrm{Ho}(\mathbf{sSet}) \simeq \mathrm{Ho}(\mathbf{Top}),$$

for the standard model structure on **Top**.

The adjunction of the Theorem induces an adjoint equivalence of homotopy categories, also called a *Quillen equivalence*: S preserves fibrations and trivial fibrations, $| \ |$ preserves cofibrations and trivial cofibrations (Quillen adjunction), and $|X| \to Y$ is a weak equivalence if and only if $X \to S(Y)$ is a weak equivalence (NB: all simplicial sets are cofibrant and all spaces are fibrant).

Pointed simplicial sets

$\Delta^0 = \hom(\ , \mathbf{0})$ has one element in each simplicial degree. It is the terminal object for simplicial sets, and one often writes $* = \Delta^0$.

A *pointed simplicial set* is a map $* \to X$, or equivalently a pair (X, x) consisting of a simplicial set X with distinguished choice of vertex $x \in X_0$. A *map of pointed simplicial sets* is the obvious thing, namely a commutative triangle

$$\begin{array}{ccc} & & X \\ & \overset{x}{\nearrow} & \\ * & & \downarrow f \\ & \underset{y}{\searrow} & \\ & & Y \end{array}$$

or equivalently a simplicial set map $f : X \to Y$ which preserves base points in the sense that $f(x) = y$.

Write $\mathbf{sSet}_* = */\mathbf{sSet}$ for the category of pointed simplicial sets.

Fact: Suppose that $\mathbf{M}$ is a closed model category, and take $X \in \mathbf{M}$. Then $X/\mathbf{M}$ has a model structure for which a map

$$\begin{array}{ccc} & & Y \\ & \nearrow & \\ X & & \downarrow f \\ & \searrow & \\ & & Z \end{array}$$

is a fibration (respectively cofibration, weak equivalence) if and only if the map $f : Y \to Z$ is a fibration (respectively cofibration, weak equivalence) of $\mathbf{M}$.

Exercise: formulate and prove a corresponding statement for $\mathbf{M}/Y$.

Examples:

- $\mathbf{sSet}_*$ has a model structure for which a map $f : (X, x) \to (Y, y)$ of pointed simplicial sets is a weak equivalence (resp. fibration, cofibration) if and only if the underlying map $f : X \to Y$ is a weak equivalence (resp. fibration, cofibration) of simplicial sets.
- The category $\mathbf{Top}_*$ of pointed topological spaces has a model structure for which a map $f : (X, x) \to (Y, y)$ of pointed spaces is a weak equivalence (resp. fibration, cofibration) if and only if the underlying map $f : X \to Y$ is a weak equivalence (resp. fibration, cofibration) of topological spaces.

NB: $f : X \to Y$ is a weak equivalence of spaces iff f induces a bijection $\pi_0 X \cong \pi_0 Y$ of path components and induces isomorphisms $\pi_i(X, z) \cong \pi_i(Y, f(z))$

for all $i \geq 1$ and all $z \in X$. The condition on homotopy groups **is not** equivalent to the assertion that the morphisms $\pi_i(X, x) \to \pi_i(Y, y)$ are isomorphisms for a map $f : (X, x) \to (Y, y)$ of pointed spaces, unless X is connected.

Here are some basic constructions for pointed objects:

- Given $X \in \mathbf{Set}$, $X_+ = \{*\} \sqcup X$ is a pointed "space" (ie. simplicial set), pointed by the disjoint base point. The functor $X \mapsto X_+$ is left adjoint to the functor $\mathbf{sSet}_* \to \mathbf{sSet}$ which forgets the base point.

- Given X, Y in $\mathbf{sSet}_*$, the wedge $X \vee Y$ is defined by the pushout

$$\begin{array}{ccc} * & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \vee Y \end{array}$$

  $(X, Y) \mapsto X \vee Y$ is the coproduct in $\mathbf{sSet}_*$. $X \mapsto X \vee Y$ preserves weak equivalences, by properness of simplicial sets.

- There is a canonical inclusion $X \vee Y \subset X \times Y$, defined by $x \mapsto (x, *)$ and $y \mapsto (*, y)$. The *smash product* $X \wedge Y$ is defined by

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

On the level of pointed sets,

$$K \wedge L = \bigvee_{x \in L - \{*\}} K,$$

so that $X \mapsto X \wedge Y$ preserves monomorphisms, ie. preserves cofibrations. $X \to X \times Y$ preserves weak equivalences, since $|X \times Y| \cong |X| \times |Y|$. It follows that $X \mapsto X \wedge Y$ preserves weak equivalences.

- The simplicial circle S^1 is defined by $S^1 = \Delta^1 / \partial\Delta^1$, with the canonical choice of base point. Note that $|S^1|$ is a circle in the usual sense.

- Given $X \in \mathbf{sSet}_*$, the *suspension* of X is the smash product $S^1 \wedge X$. There are natural isomorphisms

  $$|S^1 \wedge X| \cong |S^1| \wedge |X| = \Sigma|X|.$$

- The pointed function complex $\mathbf{hom}_*(X, Y)$ is the pointed simplicial set with

  $$\mathbf{hom}_*(X, Y)_n = \hom_{\mathbf{sSet}_*}(X \wedge \Delta^n_+, Y).$$

  where the base point is the unique map factoring through the base point $y \in Y$. There is a natural bijection

  $$\hom(X \wedge Z, Y) \cong \hom(X, \mathbf{hom}_*(Z, Y))$$

NB: $X \wedge K_+ \cong (X \times K)/(* \times K) := X \ltimes K$
$X \ltimes K$ is sometimes called a *half smash product.*

Theorem: With the definitions given above the category $\mathbf{sSet}_*$ of pointed simplicial sets satisfies the axioms for a proper closed simplicial model category.

Spectra

A *spectrum* X consists of pointed (level) simplicial sets X^n, $n \geq 0$ together with *bonding maps* $\sigma : S^1 \wedge X^n \to X^{n+1}$.
A *map* $f : X \to Y$ *of spectra* consists of pointed maps $f : X^n \to Y^n$ which respect structure in the sense that the diagrams

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle S^1\wedge f}\downarrow & & \downarrow{\scriptstyle f} \\ S^1 \wedge Y^n & \xrightarrow[\sigma]{} & Y^{n+1} \end{array}$$

commute. The corresponding category will be denoted by **Spt**. This category is complete and cocomplete.

Examples:

- Suppose that Y is a pointed simplicial set. The *suspension spectrum* $\Sigma^\infty Y$ consists of the pointed sets

$$Y,\ S^1 \wedge Y,\ S^1 \wedge S^1 \wedge Y,\ \dots\ , S^n \wedge Y, \dots$$

where $S^n = S^1 \wedge \cdots \wedge S^1$ (n-fold smash power). The bonding maps of $\Sigma^\infty Y$ are the canonical isomorphisms

$$S^1 \wedge S^n \wedge Y \cong S^{n+1} \wedge Y.$$

There is a natural bijection

$$\hom(\Sigma^\infty Y, X) \cong \hom(X, Y^0),$$

so that the suspension spectrum functor is left adjoint to the "level 0" functor $X \mapsto X^0$.

- $S = \Sigma^\infty S^0$ is the *sphere spectrum*.
- $X =$ spectrum and $K =$ pointed simplicial set: there is a spectrum $X \wedge K$ with

  $$(X \wedge K)^n = X^n \wedge K$$

  and having bonding maps

  $$\sigma \wedge K : S^1 \wedge X^n \wedge K \to X^{n+1} \wedge K.$$

  $\Sigma^\infty K \cong S \wedge K$.
- The *suspension* of a spectrum X is the spectrum $X \wedge S^1$. The *fake suspension* ΣX of X has level spaces $S^1 \wedge X^n$ and bonding maps

  $$S^1 \wedge \sigma : S^1 \wedge S^1 \wedge X^n \to S^1 \wedge X^{n+1}.$$

Remark: There is a commutative diagram

$$\begin{array}{ccc}
S^1 \wedge S^1 \wedge X^n & \xrightarrow{S^1 \wedge \sigma} & S^1 \wedge X^{n+1} \\
{\scriptstyle \tau \wedge X^n}\downarrow & & \\
S^1 \wedge S^1 \wedge X^n & \cdots\cdots\nearrow & \cong \downarrow \tau \\
{\scriptstyle S^1 \wedge \tau}\downarrow \cong & & \\
S^1 \wedge X^n \wedge S^1 & \xrightarrow[\sigma \wedge S^1]{} & X^{n+1} \wedge S^1
\end{array}$$

where τ flips adjacent smash factors: $\tau(x \wedge y) = y \wedge x$. The dotted arrow (bonding map induced by $\sigma \wedge S^1$) differs from $S^1 \wedge \sigma$ by precomposition by $\tau \wedge X^n$. The flip $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$ is non-trivial: it is multiplication by -1 in $H_2(S^2)$. This observation was a source of difficulty in stable homotopy theory for many years.

- X = simplicial set: write $\tilde{\mathbb{Z}}(X)$ for the kernel of the induced map $\mathbb{Z}(X) \to \mathbb{Z}(*)$. Then $H_n(X, \mathbb{Z}) = \pi_n(\mathbb{Z}(X), 0)$ and $\tilde{H}_n(X, \mathbb{Z}) = \pi_n(\tilde{\mathbb{Z}}(X), 0)$ (reduced homology). If X is pointed, then there is a natural isomorphism $\tilde{\mathbb{Z}}(X) = \mathbb{Z}(X)/\mathbb{Z}(*)$, and there is a natural pointed map

  $$X \xrightarrow{\eta} \mathbb{Z}(X) \to \tilde{\mathbb{Z}}(X),$$

  denoted by h for "Hurewicz". If A is a simplicial abelian group, there is a natural simplicial map

$$\gamma : S^1 \wedge A \to \tilde{\mathbb{Z}}(S^1) \otimes A =: S^1 \otimes A.$$

The *Eilenberg-Mac Lane* spectrum $H(A)$ associated to A consists of the spaces

$$A,\ S^1 \otimes A,\ S^2 \otimes A,\ \ldots$$

with bonding maps

$$S^1 \wedge (S^n \otimes A) \xrightarrow{\gamma} S^1 \otimes (S^n \otimes A) \cong S^{n+1} \otimes A.$$

- X = spectrum, K = pointed simplicial set. There is a spectrum $\mathbf{hom}_*(K, X)$ with

$$\mathbf{hom}_*(K, X)^n = \mathbf{hom}_*(K, X^n),$$

and with bonding map

$$S^1 \wedge \mathbf{hom}_*(K, X^n) \to \mathbf{hom}_*(K, X^{n+1})$$

adjoint to the composite

$$S^1 \wedge \mathbf{hom}_*(K, X^n) \wedge K \xrightarrow{S^1 \wedge ev} S^1 \wedge X^n \xrightarrow{\sigma} X^{n+1}.$$

There is a natural bijection

$$\hom(X \wedge K, Y) \cong \hom(X, \mathbf{hom}_*(K, Y)).$$

X = spectrum and $n \in \mathbb{Z}$: the *shifted spectrum* $X[n]$ is the spectrum with

$$X[n]^m = \begin{cases} * & m+n < 0 \\ X^{m+n} & m+n \geq 0 \end{cases}$$

Examples: $X[-1]^0 = *$ and $X[-1]^n = X^{n-1}$ for $n \geq 1$. $X[1]^n = X^{n+1}$ for all $n \geq 0$.

Remarks: 1) There is a natural map $\Sigma X \to X[1]$ defined by the bonding maps. We'll see later that this is a stable equivalence, and that there is a stable equivalence $\Sigma X \simeq X \wedge S^1$.

2) There is a natural bijection

$$\hom(X[n], Y) \cong \hom(X, Y[-n])$$

and of course $X[n][-n] \cong X$, so that all shift operators are invertible.

3) There is a natural bijection

$$\hom(\Sigma^\infty K[-n], Y) \cong \hom(K, Y^n),$$

so that the n^{th} level functor $Y \to Y^n$ has a left adjoint.

4) Given a spectrum X, the n^{th} *layer* $L_n X$ is the spectrum

$$X^0, \ \ldots \ , X^n, \ S^1 \wedge X^n, \ S^2 \wedge X^n, \ \ldots$$

There are obvious maps $L_nX \to L_{n+1}X \to X$ and a natural isomorphism

$$\varinjlim_n L_nX \cong X.$$

The functor $X \mapsto L_nX$ is left adjoint to truncation up to level n. The system of maps

$$\Sigma^\infty X^0 = L_0X \to L_1X \to \dots$$

is called the *layer filtration* of X. Here's an exercise: show that there are pushout diagrams

$$\begin{array}{ccc} \Sigma^\infty(S^1 \wedge X^n)[-n-1] & \longrightarrow & L_nX \\ {\scriptstyle \sigma_*}\downarrow & & \downarrow \\ \Sigma^\infty X^{n+1}[-n-1] & \longrightarrow & L_{n+1}X \end{array}$$

The category of spectra is "the" (or rather "a") home for stable homotopy theory, and stable homotopy theory itself is the outcome of formally inverting the suspension. It will take a few definitions and results to put the theory in place.

Strict structure

Say that a map $f : X \to Y$ is a *strict (levelwise) weak equivalence* (resp. *strict (levelwise) fibration*) if all maps $f : X^n \to Y^n$ are weak equivalences (resp. fibrations) of (pointed) simplicial sets.

A *cofibration* is a map $iA \to B$ of spectra such that

- $i : A^0 \to B^0$ is a cofibration of (pointed) simplicial sets, and
- all maps
$$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \to B^{n+1}$$
are cofibrations.

NB: All cofibrations are levelwise cofibrations.

Given spectra X, Y, the function complex $\mathbf{hom}(X, Y)$ is a simplicial set with

$$\mathbf{hom}(X, Y)_n = \hom(X \wedge \Delta^n_+, Y).$$

Proposition: With these definitions the category **Spt** of spectra satisfies the axioms for a proper closed simplicial model category. This model structure is also cofibrantly generated.

Proof: Suppose given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

where i is a cofibration and p is a strict fibration and strict weak equivalence. Then the lifting θ_0 exists in the diagram

$$\begin{array}{ccc} A^0 & \xrightarrow{\alpha} & X^0 \\ {\scriptstyle i}\downarrow & \overset{\theta^0}{\nearrow} & \downarrow{\scriptstyle p} \\ B^0 & \xrightarrow[\beta]{} & Y^0 \end{array}$$

and then θ_1 exists in the diagram

$$\begin{array}{ccc} (S^1 \wedge B^0) \cup_{(S^1 \wedge A^0)} A^1 & \xrightarrow{(\theta^0_*,\alpha)} & X^1 \\ {\scriptstyle cof}\downarrow & \overset{\theta^1}{\nearrow} & \downarrow{\scriptstyle p} \\ B^1 & \xrightarrow[\beta]{} & Y^1 \end{array}$$

Proceed inductively to show that the lifting problem can be solved. The lifting problem is solved in a similar way if i is a trivial cofibration and p is a strict fibration. We have proved the lifting axiom **CM4**.

Suppose that $f : X \to Y$ is a map of spectra, and find a factorization

$$\begin{array}{ccc} X^0 & \xrightarrow{i^0} & Z^0 \\ & \searrow^{f} & \downarrow{\scriptstyle p^0} \\ & & Y^0 \end{array}$$

in level 0, where i^0 is a cofibration and p^0 is a fibration. Form the diagram

$$\begin{array}{ccccc} S^1 \wedge X^0 & \longrightarrow & & & X^1 \\ \downarrow{\scriptstyle S^1 \wedge i^0} & & & \swarrow^{i_*} & \\ S^1 \wedge Z^0 & \longrightarrow & (S^1 \wedge Z^0) \cup X^1 & & \downarrow{\scriptstyle f} \\ \downarrow{\scriptstyle S^1 \wedge p^0} & & & \searrow^{f_*} & \\ S^1 \wedge Y^0 & \longrightarrow & & & Y^1 \end{array}$$

and find a factorization

$$\begin{array}{ccc} (S^1 \wedge Z^0) \cup X^1 & \xrightarrow{j} & Z^1 \\ & \searrow^{f_*} & \downarrow{\scriptstyle p^1} \\ & & Y^1 \end{array}$$

where j is a cofibration and p^1 is a trivial fibration. Write $i^1 = j \cdot i_*$. Then we have factorized f as a cofibration followed by a trivial fibration up to level 1. Proceed inductively to show that $f = p \cdot i$ where p is a trivial strict fibration and i is a cofibration. The other factorization statment has the same proof, giving **CM5**.

The simplicial model structure is inherited from pointed simplicial sets, as is properness. The generating sets for the cofibrations and trivial cofibrations, respectively are the maps

$$\Sigma^\infty(\Lambda^n_k)_+[m] \to \Sigma^\infty\Delta^n_+[m]$$

and

$$\Sigma^\infty(\partial\Delta^n)_+[m] \to \Sigma^\infty\Delta^n_+[m]$$

respectively. $\square$