Lecture 002 (October 24, 2005)

Stable equivalences

X = pointed simplicial set: write

$$\Omega X = \mathbf{hom}_*(S^1, X).$$

$X \mapsto \Omega X$ is the simplicial loop space construction for pointed simplicial sets X.

The construction only makes homotopy theoretic sense (ie. preserves weak equivalences) if X is fibrant — in that case there are isomorphisms

$$\pi_{i+1}(X, *) \cong \pi_i(\Omega X, *),\ i \geq 0,$$

of simplicial homotopy groups ($*$ is the base point for X), by a standard long exact sequence argument. If X is not fibrant, then ΩX is most properly a derived functor:

$$\Omega X := \Omega X_f$$

where $j : X \to X_f$ is a fibrant model for X in the sense that j is a weak equivalence and X_f is fibrant. This construction can be made functorial, since $\mathbf{sSet}_*$ has functorial fibrant replacements.

In general, there is a natural bijection

$$\hom(Z \wedge S^1, X) \cong \hom(Z, \Omega X).$$

so that every morphism $f : Z \wedge S^1 \to X$ has a uniquely determined adjoint $f_* Z \to \Omega X$.

One can therefore say that a spectrum X consists of pointed simplicial sets $X^n, n \geq 0$, and (adjoint) bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$

Here are two constructions::

1) There is a natural (levelwise) fibrant model $j : Y \to FY$ is a natural (levelwise) fibrant model in the strict model structure for **Spt**.

2) Suppose that X is a spectrum. Set

$$\Omega^\infty X^n = \varinjlim \left(X^n \xrightarrow{\sigma_*} \Omega X^{n+1} \xrightarrow{\Omega\sigma_*} \Omega^2 X^{n+2} \to \dots \right).$$

The comparison diagram

$$\begin{array}{ccccccc}
X^n & \xrightarrow{\sigma_*} & \Omega X^{n+1} & \xrightarrow{\Omega\sigma_*} & \Omega^2 X^{n+2} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \Omega\sigma_*} & & \downarrow{\scriptstyle \Omega^2\sigma_*} & & \\
\Omega X^{n+1} & \xrightarrow[\Omega\sigma_*]{} & \Omega^2 X^{n+2} & \xrightarrow[\Omega^2\sigma_*]{} & \Omega^3 X^{n+3} & \longrightarrow & \cdots
\end{array}$$

determines a spectrum structure $\Omega^\infty X$ and a natural map $\omega : X \to \Omega^\infty X$.

Write $QY = \Omega^\infty FY$ and let $\eta : Y \to QY$ be the composite

$$Y \xrightarrow{j} FY \xrightarrow{\omega} \Omega^\infty FY = QY.$$

We often say that QY is the *stabilization* of Y.

Say that a map $f : X \to Y$ is a *stable equivalence or Q-equivalence* if and only if the induced map $f_* : QX \to QY$ is a strict equivalence.

Remarks:

1) All spaces QY^n are fibrant (NB: this is a special property of ordinary spectra), and the map $\sigma_* : QY^n \to \Omega QY^{n+1}$ is an isomorphism. In particular, all QY^n are H-spaces with groups $\pi_0 QY^n$ of path components. All induced maps $f_* : QX^n \to QY^n$ are H-maps. It follows that the maps $f_* : QX^n \to QY^n$ are weak equivalences if and only if all maps

$$\pi_i(QX^n, *) \to \pi_i(QY^n, *)$$

based at the distinguished base point are isomorphisms.

2) Define the *stable homotopy groups* $\pi_k^s Y$, $k \in \mathbb{Z}$ by

$$\pi_k^s Y = \varinjlim_{n+k \geq 0} (\cdots \to \pi_{n+k} FY^n \to \pi_{n+k+1} FY^{n+1} \to \dots)$$

where the maps of homotopy groups are induced by the maps $\sigma_* : FY^n \to \Omega FY^{n+1}$. There are isomorphisms

$$\pi_k(QY^n, *) \cong \pi_{k-n}^s Y,$$

so $f : X \to Y$ is a stable equivalence if and only if f induces an isomorphism in all stable homotopy groups.

The category of spectra **Spt** with the strict model structure and the stablization functor Q with natural maps $\eta : X \to QX$ fit into a general framework: suppose that $\mathbf{M}$ is a right proper closed model category with a functor $Q : \mathbf{M} \to \mathbf{M}$, and suppose further that there is a natural map $\eta_X : X \to QX$.

Say that a map $f : X \to Y$ is a Q-equivalence if the induced map $Qf : QX \to QY$ is a weak equivalence of $\mathbf{M}$. Q-cofibrations are cofibrations of $\mathbf{M}$. A Q-fibration is a map which has the right lifting property with respect to all maps which are cofibrations and Q-equivalences.

Here are some statements:

A4 The functor Q preserves weak equivalences of $\mathbf{M}$.

A5 The maps $\eta_{QX}, Q(\eta_X) : QX \to QQX$ are weak equivalences of $\mathbf{M}$.

A6′ Q-equivalences are stable under pullback along Q-fibrations.

Theorem: (Bousfield-Friedlander [2]) Suppose given a right proper closed model category $\mathbf{M}$ with functor $Q : \mathbf{M} \to \mathbf{M}$ and natural map $\eta : X \to QX$ as above. Suppose the Q-equivalences, cofibrations and Q-fibrations satisfy the axioms **A4**, **A5** and **A6′**. Then, with these definitions, $\mathbf{M}$ together with these three classes of maps satisfies the conditions for a right proper closed model category.

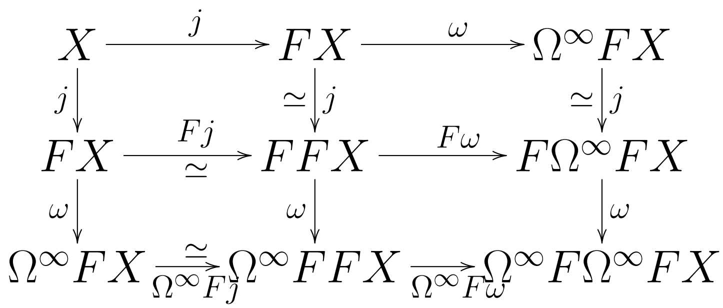
Proposition: The category **Spt** of spectra and the stabilization functor Q satisfy the axioms **A4**, **A5** and **A6′**.

Proof of Proposition:

A4 follows from

Lemma: Suppose given a natural transformation $f : X \to Y$ of functors $X, Y : I \to \mathbf{sSet}$ such that each component map $f_i : X_i \to Y_i$ is a weak equivalence. Then the map $f_* : \varinjlim_i X_i \to \varinjlim_i Y_i$ is a weak equivalence.

A5: Here's a picture:

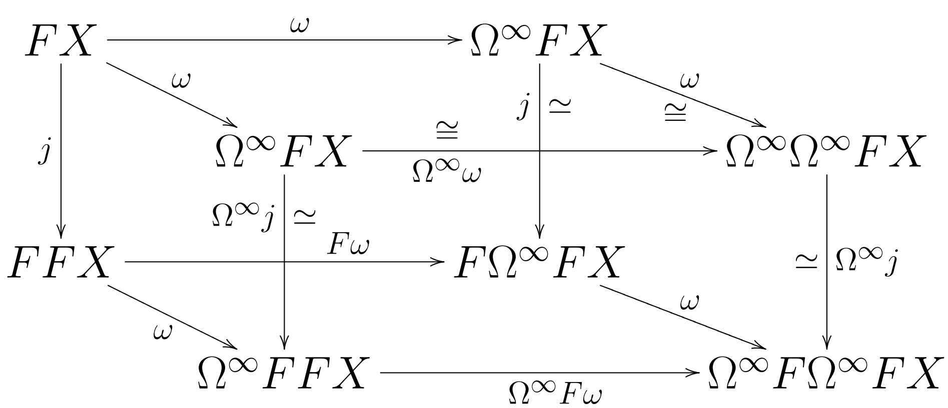


The indicated maps are strict weak equivalences, so it suffices to show that $\Omega^\infty F\omega$ and

$$\omega : F\Omega^\infty FX \to \Omega^\infty F\Omega^\infty FX$$

are strict weak equivalences.

Here's another picture:



It's an exercise to show that Ω^∞ is an isomorphism: actually

$$\omega = \Omega^\infty \omega : \Omega^\infty FX \to \Omega^\infty \Omega^\infty FX.$$

But then the required maps are strict equivalences.

A6: Every strict fibre sequence $F \to X \to Y$ induces a long exact sequence

$$\cdots \to \pi_k^s F \to \pi_k^s X \to \pi_k^s Y \xrightarrow{\partial} \pi_{k-1}^s F \to \cdots$$

Right properness follows from an exact sequence comparison. □

Remark: The model structure on **Spt** arising from the Bousfield-Friedlander Theorem via this Proposition is called the *stable model structure* for the category of spectra. The corresponding homotopy category Ho(**Spt**) is usually called "the" stable category. This is a misnomer: it is a basic aim of this course to demonstrate that there are **many** stable categories.

The proof of Bousfield-Friedlander Theorem is accomplished with a series of Lemmas:

Lemma 1: A map $p : X \to Y$ is a Q-fibration and a Q-weak equivalence if and only if it is a trivial fibration of **M**.

Proof: Every trivial fibration p has the RLP wrt all cofibrations, and is therefore a Q-fibration. p is also a Q-equivalence, by **A4**.
Suppose that $p : X \to Y$ is a Q-fibration and a

Q-equivalence. There is a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & \searrow^{p} & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

where j is a cofibration and π is a trivial fibration of $\mathbf{M}$. π is a Q-equivalence, so the cofibration j is a Q-equivalence. There is a diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[\pi]{} & Y \end{array}$$

Then p is a retract of π and is therefore a trivial fibration. $\square$

Lemma 2: Suppose that $p : X \to Y$ is a fibration of $\mathbf{M}$ and that the maps $\eta : X \to QX$, $\eta : Y \to QY$ are weak equivalences of $\mathbf{M}$. Then p is a Q-fibration.

Proof: Consider the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

There is a diagram

$$\begin{array}{ccccc} QA & & \xrightarrow{\quad Q\alpha \quad} & & QX \\ & \searrow^{j_\alpha} & & \nearrow_{p_\alpha} & \\ {\scriptstyle Qi}\downarrow & & Z & & \downarrow{\scriptstyle Qp} \\ & & \downarrow{\scriptstyle \pi} & & \\ QB & & \longrightarrow & & QY \\ & \searrow_{j_\beta} & & \nearrow_{p_\beta} & \\ & & W & & \end{array}$$

where j_α, j_β are trivial cofibrations of $\mathbf{M}$ and p_α, p_β are fibrations. There is an induced diagram

$$\begin{array}{ccccc} A & \longrightarrow & Z \times_{QX} X & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & W \times_{QY} Y & \longrightarrow & Y \end{array}$$

and the lifting problem is solved if we can show that π_* is a weak equivalence. But there is finally a diagram

$$\begin{array}{ccccc} QA & \xrightarrow{j_\alpha} & Z & \xleftarrow{pr} & Z \times_{QX} X \\ {\scriptstyle Qi}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_*} \\ QB & \xrightarrow[j_\beta]{} & W & \xleftarrow[pr]{} & W \times_{QY} Y \end{array}$$

The maps Qi, j_α and j_β are weak equivalences of $\mathbf{M}$ so that π is a weak equivalence. The maps pr are weak equivalences by right properness of $\mathbf{M}$ and the assumption on p. It follows that π_* is a weak equivalence of $\mathbf{M}$. $\square$

Lemma 3: Every map $f : QX \to QY$ has a factorization $f = q \cdot j$, where j is a cofibration and Q-equivalence and q is a Q-fibration.

Proof: f has a factorization $f = q \cdot j$ where j is a trivial cofibration and q is a fibration of $\mathbf{M}$. j is a Q-equivalence by **A4**, and q is a Q-fibration by Lemma 2. In effect, there is a diagram

$$\begin{array}{ccccc} QX & \xrightarrow[\simeq]{j} & Z & \xrightarrow{p} & QY \\ {\scriptstyle \eta}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \eta}\downarrow & & {\scriptstyle \simeq}\downarrow{\scriptstyle \eta} \\ QQX & \xrightarrow[Qj]{\simeq} & QZ & \xrightarrow[Qp]{} & QQY \end{array}$$

so that $\eta : Z \to QZ$ is a weak equivalence of $\mathbf{M}$. □

Lemma 4: Every map $f : X \to Y$ has a factorization $f = q \cdot j$, where j is a cofibration and Q-equivalence and q is a Q-fibration.

Proof: We know that the induced map $f_* : QX \to QY$ has a factorization

$$\begin{array}{ccccc} QX & & \xrightarrow{f_*} & & QY \\ & {\scriptstyle i}\searrow & & \nearrow{\scriptstyle p} & \\ & & X & & \end{array}$$

where p is a Q-fibration and i is a cofibration and a Q-equivalence, by Lemma 3. Form the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_*} & Z \times_{QY} Y & \xrightarrow{p_*} & Y \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta_*} & & \downarrow{\scriptstyle\eta} \\ QX & \xrightarrow[i]{} & Z & \xrightarrow[p]{} & QY \end{array}$$

Then η is a Q-equivalence by **A5**, wherever it occurs, and so η_* is a Q-equivalence by **A6′**. It follows that i_* is a Q-equivalence. The map i_* has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i_*} & Z \times_{QY} Y \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle\pi} & \\ & W & \end{array}$$

where j is a cofibration and π is a trivial strict fibration. Then π is a Q-equivalence and a Q-fibration by Lemma 1, so that j is a Q-equivalence, and the composite $p_* \cdot \pi$ is a Q-fibration. □

Proof of Bousfield-Friedlander Theorem:

The non-trivial closed model statements are the lifting axiom **CM4** and the factorization axiom **CM5**. **CM5** is a consequence of Lemma 1 and Lemma 4. **CM4** follows from Lemma 1.

The right properness of the model structure is the statement **A6′**. □

Lemma: Suppose that, in addition to the assumption of the Theorem, that $\mathbf{M}$ is left proper. Then the Q-structure on $\mathbf{M}$ is left proper.

Proof: Suppose given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[f_*]{} & B \cup_A C \end{array}$$

where f is a Q-equivalence and i is a cofibration. We must show that f_* is a Q-equivalence.

Find a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & D \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} \\ & & C \end{array}$$

where j is a cofibration and π is a trivial fibration of $\mathbf{M}$. Then the induced map $\pi_* : B \cup_A D \to B \cup_A C$ is a weak equivalence of $\mathbf{M}$ by a patching lemma argument (left properness). Also j is a stable equivalence as well as a cofibration, so that $j_* : B \to B \cup_A D$ is a cofibration and a stable equivalence. Then $f_* = \pi_* \cdot j_*$ is a stable equivalence. $\square$

Here's the other major abstract result in this game, again from [2]:

Theorem: Suppose that the model category $\mathbf{M}$ and the functor Q satisfy the conditions for the Bousfield-Friedlander Theorem. Then a map $p : X \to Y$ of $\mathbf{M}$ is a stable fibration if and only if it is a fibration of $\mathbf{M}$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Qp} \\ Y & \xrightarrow[\eta]{} & QY \end{array}$$

is homotopy cartesian in $\mathbf{M}$.

Let's call this the Stable Fibration Theorem.

Corollary A:

1) An object X of $\mathbf{M}$ is Q-fibrant if and only if it is fibrant and the map $\eta : X \to QX$ is a weak equivalence of $\mathbf{M}$.

2) A spectrum X is stably fibrant if and only if it is strictly fibrant and all adjoint bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$ are weak equivalences of pointed simplicial sets.

In other words the fibrant spectra are the "Ω-spectra" from days of yore.

Corollary B: Suppose given a diagram

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow[\simeq]{} & Y' \end{array}$$

in which p, p' are fibrations and the horizontal maps are weak equivalences of $\mathbf{M}$. Then p is a Q-fibration if and only if p' is a Q-fibration.

Proof of Stable Fibration Theorem:

Suppose that $p : X \to Y$ is a fibration and that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Qp} \\ Y & \xrightarrow[\eta]{} & QY \end{array}$$

is homotopy cartesian in $\mathbf{M}$. Then Qp has a factorization

$$\begin{array}{ccc} QX & \xrightarrow{i} & Z \\ & {\scriptstyle Qp}\searrow & \downarrow{\scriptstyle q} \\ & & QY \end{array}$$

where i is a trivial cofibration and q is a fibration. Then (see the proof of Lemma 3), q is a Q-fibration. Factorize the weak equivalence $\theta : X \to Y \times_{QY} Z$ (the square is homotopy cartesian) as

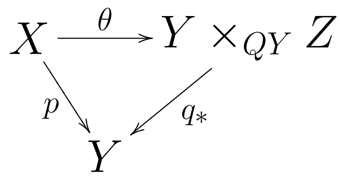
$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & {\scriptstyle \theta}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \times_{QY} Z \end{array}$$

where π is a trivial fibration of $\mathbf{M}$ and i is a trivial cofibration. Then $q_* \cdot \pi$ is a Q-fibration, and the lifting exists in the diagram

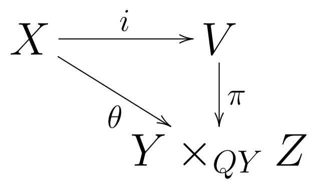
$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ W & \xrightarrow[q_*\pi]{} & Y \end{array}$$

Thus, p is a retract of a Q-fibration, and is therefore a Q-fibration itself.

Suppose that $p : X \to Y$ is a Q-fibration, and factorize $Qp = q \cdot i$ as above. Then the induced map $\eta_* : Y \times_{QY} Z \to Z$ is a Q-equivalence, so that θ is a Q-equivalence. In other words the picture



is a weak equivalence of fibrant objects in the category $\mathbf{M}/Y$ of objects fibred over Y, for the Q-structure on $\mathbf{M}$. The usual category of fibrant objects trick means that θ has a factorization



in $\mathbf{Spt}/Y$, where π is a Q-fibration and a Q-equivalence and i is a section of a map $V \to X$ which is a Q-fibration and a Q-equivalence. In particular, π and i are weak equivalences of $\mathbf{M}$ by Lemma 1, so that θ is a weak equivalence of $\mathbf{M}$. □

Closed simplicial model structure

Write $A \otimes K = A \wedge K_+$ for a spectrum A and a simplicial set K.

Lemma: Suppose that $i : A \to B$ is a stably trivial cofibration. Then all induced maps

$$(B \otimes \partial\Delta^n) \cup (A \otimes \Delta^n) \to B \otimes \Delta^n$$

are stably trivial cofibrations.

Quillen's simplicial model axiom follows easily: if $j : K \to L$ is a cofibration of simplicial sets and $i : A \to B$ is a cofibration of spectra, then the induced map

$$(B \otimes K) \cup (A \otimes L) \subset B \otimes L$$

is a cofibration which is a stable equivalence if either i is a stable equivalence (from the Lemma) or j is a weak equivalence of simplicial sets (simplicial model axiom for the strict structure).

Proof of Lemma: It suffices to show that

$$i \otimes \partial\Delta^n : A \otimes \partial\Delta^n \to B \otimes \partial\Delta^n$$

is a stable equivalence. There is a pushout diagram

$$\begin{array}{ccc} A \otimes \partial\Delta^{n-1} & \longrightarrow & A \otimes \Lambda^n_k \\ \downarrow & & \downarrow \\ A \otimes \Delta^{n-1} & \longrightarrow & A \otimes \partial\Delta^n \end{array}$$

There is also a corresponding diagram for B and an obvious comparison. The simplicial sets Λ^n_k and Δ^{n-1} are both weakly equivalent to a point, so it suffices to show that the comparison

$$i \otimes \partial\Delta^{n-1} : A \otimes \partial\Delta^{n-1} \to B \otimes \partial\Delta^{n-1}$$

is a stable equivalence. But this is the inductive step in an argument that starts with the case

$$i \otimes \partial\Delta^1 : A \otimes \partial\Delta^1 \to B \otimes \partial\Delta^1$$

and this map is isomorphic to the map

$$i \wedge i : A \wedge A \to B \wedge B.$$

Finally, a wedge (coproduct) of stably trivial cofibrations is stably trivial. □

Note: Bousfield gives a different proof of the Lemma in [1]. The result is also mentioned in Remark X.4.7 (on p.496) of [3], essentially without proof.

References

[1] A. K. Bousfield. On the telescopic homotopy theory of spaces. *Trans. Amer. Math. Soc.*, 353(6):2391–2426 (electronic), 2001.

[2] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 80–130. Springer, Berlin, 1978.

[3] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.