Lecture 003 (November 10, 2005)

Suspensions and shift

Here's a fine old result:

Theorem: [Freudenthal Suspension Theorem] Suppose that a pointed space X is n-connected where $n \geq 0$. Then the homotopy fibre F of the canonical map $\eta : X \to \Omega(X \wedge S^1)$ is $2n$-connected.

In particular the suspension homomorphism

$$\pi_i X \to \pi_i(\Omega(X \wedge S^1)) \cong \pi_{i+1}(X \wedge S^1)$$

is an isomorphism for $i \leq 2n$ and is an epimorphism for $i = 2n+1$. This is the original meaning of the word stable: the suspensions $X \wedge S^n$ are at least n-connected, and so the map

$$\pi_i(S^n \wedge X) \to \pi^s_{i-n}(\Sigma^\infty X)$$

are isomorphisms in the *stable range* $n \leq i \leq 2n$ (really $0 \leq i \leq 2n$).

The proof of the Theorem (see [2], III.3.10, p.175) involves using the transgression (aka. differential for the Serre spectral sequence) for the path-loop fibration for $X \wedge S^1$ to show that the induced maps

$$\tilde{H}_i(X) \to \tilde{H}_i(\Omega(X \wedge S^1))$$

are isomorphisms for $i \leq 2n+1$. Then it follows from a Hurewicz theorem argument that F has the advertised connectivity.

Here's an easy observation:

Lemma 1: The natural map $\sigma_* : X \to \Omega X[1]$ is a stable equivalence if X is strictly fibrant.

Proof: This is an easy cofinality argument, using the fact that $\Omega^\infty X$ is the filtered colimit of the system

$$X \to \Omega X[1] \to \Omega^2 X[2] \to \dots \quad \square$$

Lemma 2: Suppose that X is a pointed space. Then the canonical map

$$\eta : \Sigma^\infty X \to \Omega\Sigma(\Sigma^\infty X)$$

is a stable homotopy equivalence.

Proof: The map

$$\pi_{n+k}(S^n \wedge X) \to \pi_k^s(\Sigma^\infty X)$$

is an isomorphism for $0 \leq k \leq n-2$ by the Freudenthal Suspension Theorem (NB: X might not be connected, but $S^1 \wedge X$ is). In particular, the map

$$\pi_{2k}(S^{k+2} \wedge X) \to \pi_k^s(\Sigma^\infty X)$$

is an isomorphism, as is the map

$$\pi_{2k}(S^{k+2} \wedge X) \longrightarrow \pi_{2k}(\Omega(S^{k+3} \wedge X)).$$

Similarly, the map

$$\pi_{2k}(\Omega(S^{k+3} \wedge X)) \longrightarrow \pi_k^s(\Omega\Sigma(\Sigma^\infty X))$$

is an isomorphism (via Lemma 1). Now use the commutative diagram

$$\begin{array}{ccc} \pi_{2k}(S^{k+2} \wedge X) & \xrightarrow{\cong} & \pi_k^s(\Sigma^\infty X) \\ \cong \downarrow & & \downarrow \\ \pi_{2k}(\Omega(S^{k+3} \wedge X)) & \xrightarrow[\cong]{} & \pi_k^s(\Omega\Sigma(\Sigma^\infty X)) \end{array}$$

Remark: What we've really shown in Lemma 2 is that the composite

$$\Sigma^\infty X \xrightarrow{\eta} \Omega\Sigma(\Sigma^\infty X) \xrightarrow{\Omega j} \Omega F(\Sigma(\Sigma^\infty X))$$

is a natural stable equivalence.

Lemma 3: Suppose that Y is a spectrum. Then the composite

$$Y \xrightarrow{\eta} \Omega\Sigma Y \xrightarrow{\Omega j} \Omega F(\Sigma Y)$$

is a stable equivalence.

Proof: We show that the maps

$$L_n Y \xrightarrow{\eta} \Omega\Sigma L_n Y \xrightarrow{\Omega j} \Omega F(\Sigma L_n Y)$$

arising from the layer filtration for Y are stable equivalences.

In the layer filtration

$$L_nY: \ Y^0, \ldots, , Y^n, S^1 \wedge Y^n, S^2 \wedge Y^n, \ldots$$

the canonical map $\Sigma^\infty Y^n[-n] \to L_nY$ is a stable equivalence. In fact, it's better than that: the maps

$$(\Sigma^\infty Y^n[-n])^r \to L_nY^r$$

are isomorphisms for $r \geq n$. Thus, the maps

$$(\Omega F(\Sigma(\Sigma^\infty Y^n[-n])))^r \to \Omega F(\Sigma(L_nY))^r$$

are weak equivalences for $r \geq n$, so that

$$\Omega F(\Sigma(\Sigma^\infty Y^n[-n])) \to \Omega F(\Sigma(L_nY))$$

is a stable equivalence. The map $\eta : X \to \Omega\Sigma X$ respects shift. Now use Lemma 2 to show that the composite

$$\Sigma^\infty Y[-n] \to \Omega\Sigma(\Sigma^\infty Y[-n]) \to \Omega F(\Sigma(\Sigma^\infty Y[-n]))$$

is a stable equivalence. □

Theorem A: Suppose that X is a spectrum. Then the canonical map

$$\sigma : \Sigma X \to X[1]$$

is a stable equivalence.

Proof: The map σ is adjoint to the map $\sigma_* : X \to \Omega X[1]$, so that there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & \Omega\Sigma X & \xrightarrow{\Omega j} & \Omega F(\Sigma X) \\ & \searrow^{\sigma_*} & \downarrow{\scriptstyle \Omega\sigma} & & \downarrow{\scriptstyle \Omega F\sigma} \\ & & \Omega X[1] & \xrightarrow[\Omega j]{} & \Omega F(X[1]) \end{array}$$

The composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega(FX)[1]$$

is a stable equivalence by Lemma 1, and the shifted cofibration $j[1] : X[1] \to (FX)[1]$ is a strictly fibrant model of $X[1]$. There is a commutative triangle

$$\begin{array}{ccc} X[1] & \xrightarrow{j} & F(X[1]) \\ & \searrow^{j[1]} & \downarrow{\scriptstyle \simeq} \\ & & (FX)[1] \end{array}$$

in which the indicated map is a strict equivalence. It follows that the composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j} \Omega F(X[1])$$

is a stable equivalence. The composite

$$X \xrightarrow{\eta} \Omega\Sigma X \xrightarrow{\Omega j} \Omega F(\Sigma X)$$

is a stable equivalence by Lemma 3. The map $\Omega F\sigma$ is therefore a stable equivalence, so that Lemma 1 implies that $F\sigma : F(\Sigma X) \to F(X[1])$ is a stable equivalence: in effect, $\Omega F\sigma$ is a stable equivalence, so that $\Omega F\sigma[1]$ is a stable equivalence, so $F\sigma$ is a stable equivalence. $\square$

Here's another, still elementary but much fussier, result:

Theorem B: The functors $X \mapsto X \wedge S^1$ and $X \mapsto \Sigma X$ are naturally stably equivalent.

Sketch Proof: ([3], Lemma 1.9, p.7) The isomorphisms $\tau : S^1 \wedge X^n \to X^n \wedge S^1$ and the bonding maps $\sigma \wedge S^1$ together define a spectrum with the space $S^1 \wedge X^n$ in level n, and with bonding maps $\tilde{\sigma}$ defined by the diagrams

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X^n & \xrightarrow{\tilde{\sigma}} & S^1 \wedge X^{n+1} \\ {\scriptstyle S^1 \wedge \tau}\downarrow \cong & & \cong \downarrow {\scriptstyle \tau} \\ S^1 \wedge X^n \wedge S^1 & \xrightarrow[\sigma \wedge S^1]{} & X^{n+1} \wedge S^1 \end{array}$$

and it's not hard to see that there are commutative diagrams

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X^n & \searrow^{S^1 \wedge \sigma} & \\ {\scriptstyle \tau \wedge X^n}\downarrow & & S^1 \wedge X^{n+1} \\ S^1 \wedge S^1 \wedge X^n & \nearrow_{\tilde{\sigma}} & \end{array}$$

It's also not hard to see that there is a diagram

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge S^1 \wedge X^n & \xrightarrow{(S^1\wedge\sigma)(S^1\wedge S^1\wedge\sigma)} & \\ \downarrow{\scriptstyle (3,2,1)\wedge X^n} & & S^1 \wedge X^{n+2} \\ S^1 \wedge S^1 \wedge S^1 \wedge X^n & \xrightarrow{\tilde{\sigma}\cdot(S^1\wedge\tilde{\sigma})} & \end{array}$$

where $(3,2,1)$ is induced on the smash factors making up S^3 by the corresponding cyclic permutation of order 3.

The spaces $S^1 \wedge X^0, S^1 \wedge X^2, \ldots$ and the respective composite bonding maps $(S^1 \wedge \sigma)(S^1 \wedge S^1 \wedge \sigma)$ and $\tilde{\sigma} \cdot (S^1 \wedge \tilde{\sigma})$ define "partial" spectrum structures from which the stable homotopy types of the original spectra can be recovered. The self map $(3,2,1)$ of the 3-sphere S^3 has degree 1 and is therefore homotopic to the identity. This homotopy can be used to describe a telescope construction (see [3], p.11-15, and the next section) which is stably equivalent to both of these partial spectra. $\square$

Corollary:

1) The functors $X \mapsto X[1]$, $X \mapsto \Sigma X$ and $X \mapsto X \wedge S^1$ are naturally stably equivalent.

2) The functors $X \mapsto X[-1]$, $X \mapsto \Omega X$ and $X \mapsto \mathbf{hom}_*(S^1, X)$ are naturally stably equivalent.

In other words, the suspension and loop functors (real or fake) are equivalent to shift functors, and therefore define equivalences $\mathrm{Ho}(\mathbf{Spt}) \to \mathrm{Ho}(\mathbf{Spt})$ of the stable category.

The telescope construction

Observe that a spectrum Y is cofibrant if and only if all bonding maps $\sigma : S^1 \wedge Y^n \to Y^{n+1}$ are cofibrations.

The telescope TX for a spectrum X is a natural cofibrant replacement, equipped with a canonical strict equivalence $TX \to X$. It is constructed inductively as follows: we find natural trivial cofibrations

$$X^k \xrightarrow{j_k} CX^k \xrightarrow{\alpha_k} TX^k \; k \leq n,$$

such that

- $X^0 = CX^0 = TX^0$ and j_0 and α_0 are identities,
- CX^n is the mapping cylinder for $\sigma : S^1 \wedge X^n \to X^{n+1}$, meaning that there is a pushout diagram

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle d^0}\downarrow & & \downarrow{\scriptstyle j_{n+1}} \\ (S^1 \wedge X^n) \wedge \Delta^1_+ & \xrightarrow[\zeta_{n+1}]{} & CX^{n+1} \end{array}$$

for each n. Write σ_* for the composite

$$S^1 \wedge X^n \xrightarrow{d^1} (S^1 \wedge X^n) \wedge \Delta^1_+ \xrightarrow{\zeta_{n+1}} CX^{n+1}$$

and observe that σ_* is a cofibration.

- Form the pushout diagram

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma_*} & CX^{n+1} \\ {\scriptstyle S^1\wedge j_n}\downarrow & & \\ S^1 \wedge CX^n & & \downarrow{\scriptstyle \alpha_{n+1}} \\ {\scriptstyle S^1\wedge \alpha_n}\downarrow & & \\ S^1 \wedge TX^n & \xrightarrow[\tilde{\sigma}]{} & TX^{n+1} \end{array}$$

Then $\tilde{\sigma}$ is a cofibration, and the maps j_{n+1}, α_{n+1} are trivial cofibrations.

Lemma: Suppose that X is a spectrum with bonding maps $\sigma : S^1 \wedge X^n \to X^{n+1}$. Suppose that X' is a spectrum with the same objects as X and with bonding maps $\sigma' : S^1 \wedge X^n \to X^{n+1}$. Suppose that $j : X' \to Z$ is a map of spectra such that there are fixed homotopies

$$\begin{array}{ccc} S^1 \wedge X^n & & \\ {\scriptstyle d^1}\downarrow & \searrow^{j\sigma'} & \\ (S^1 \wedge X^n) \wedge \Delta^1_+ & \xrightarrow{h} & Z^{n+1} \\ {\scriptstyle d^0}\uparrow & \nearrow_{j\sigma} & \\ S^1 \wedge X^n & & \end{array}$$

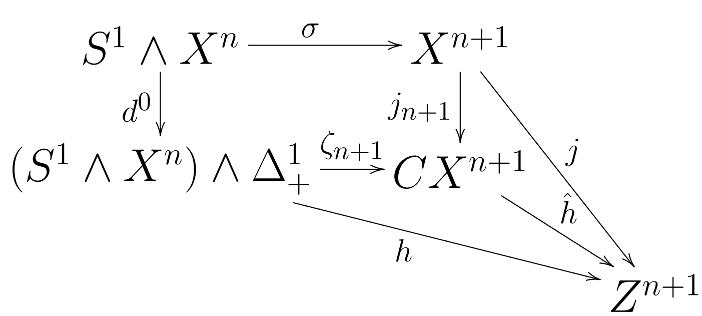
from $j\sigma'$ to $j\sigma$ in all levels. Then there is a canonically defined map $h_* : TX \to Z$. If $j : X' \to Z$ is a strict weak equivalence then the map h_* is a strict weak equivalence.

Corollary: If the map $j : X' \to Z$ of the Lemma is a strict weak equivalence, then there are strict weak equivalences

$$X \leftarrow TX \xrightarrow{h_*} Z \xleftarrow{j} X'$$

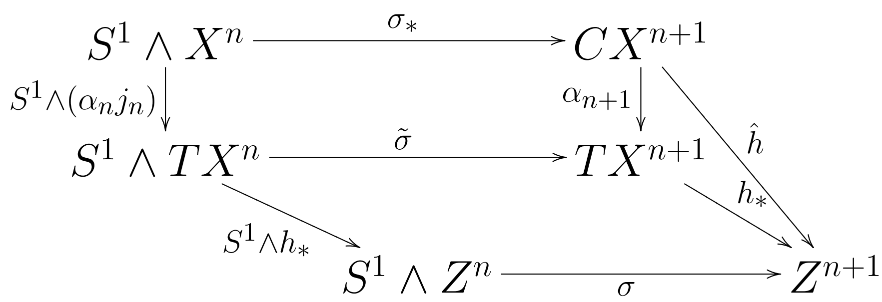
Proof: The identity homotopies on the bonding maps σ induce a strict weak equivalence $TX \to X$. $\square$

Proof of Lemma: Suppose that $h_* : TX^k \to Z^k$ has been defined for $k \leq n$ such that h_* is a map of spectra $TX \to Z$ up to level n. There is a unique map $\hat{h} : CX^{n+1} \to Z^{n+1}$ such that the diagram



commutes. Suppose that $h_* \alpha_k = \hat{h}$ up to level n. Then there is a unique map $h_* : TX^{n+1} \to Z^{n+1}$

such that the diagram



commutes.

To see the commutativity, note that

$$\hat{h}\sigma_* = \hat{h}(\zeta_{n+1}d^1) = j\sigma' = \sigma((S^1 \wedge j)$$

and $h * \alpha_n \jmath_n = \hat{h} j_n = j$ by the inductive assumption $h_* \alpha_n = \hat{h}$.

If j is a strict equivalence, then all maps $\hat{h}$ are weak equivalences, so that the maps h_* are weak equivalences. □

Remark: The construction of the Lemma is canonical and therefore natural in important special cases.

1) Suppose that $i \mapsto X_i$ and $i \mapsto X_i'$ are spectrum valued functors defined on an index category I such that $X_i^n = X_i'^n$ for all $i \in I$. Let $j : X' \to Z$ be a natural choice of strict fibrant model for the diagram X' and suppose finally that

there are natural homotopies

$$\begin{array}{ccc} S^1 \wedge X_i^n & & \\ \downarrow{\scriptstyle d^1} & \searrow^{j\sigma'} & \\ (S^1 \wedge X_i^n) \wedge \Delta^1_+ & \xrightarrow{h} & Z_i^n \\ \uparrow{\scriptstyle d^0} & \nearrow_{j\sigma} & \\ S^1 \wedge X_i^n & & \end{array}$$

where σ and σ' are the bonding maps for X and X' respectively. Then the homotopies h canonically determine a natural strict equivalence $h_* : TX \to Z$. This means that there are natural strict equivalences

$$X \leftarrow TX \xrightarrow{h_*} Z \xleftarrow{j} X'.$$

2) Suppose given partial spectra $X(1)$ and $X(2)$ having objects $S^1 \wedge X^{2n}$ and bonding maps $\sigma_1, \sigma_2 : S^3 \wedge X^{2n} \to S^1 \wedge X^{2n+2}$ respectively, such that the diagram

$$\begin{array}{ccc} S^3 \wedge X^{2n} & & \\ {\scriptstyle c}\downarrow & \searrow^{\sigma_1} & \\ & & S^1 \wedge X^{2n+2} \\ & \nearrow_{\sigma_2} & \\ S^3 \wedge X^{2n} & & \end{array}$$

commutes, where c is induced by the cyclic permutation $(3, 2, 1)$. Choose a natural fibrant model

$j : Y \to FY$ for simplicial sets which preserves products (like the Ex^∞ functor), so that it can be used to define a strict fibrant model $FX(2)$ for $X(2)$, with spaces $F(S^1 \wedge X^{2n})$ and bonding maps

$$S^2 \wedge F(S^1 \wedge X^{2n}) \to F(S^3 \wedge X^{2n}) \xrightarrow{F(\sigma_2)} F(S^1 \wedge X^{2n+2})$$

Then the map $j\sigma_1$ factors through the composite

$$\begin{array}{ccc} S^3 \wedge X^{2n} \xrightarrow{(j\cdot c)\wedge j} F(S^3) \wedge F(X^{2n}) & \longrightarrow & F(S^3 \wedge X^{2n}) \\ & & \downarrow F(\sigma_2) \\ & & F(S^1 \wedge X^{2n+2}) \end{array}$$

The map $j \cdot c : S^3 \to F(S^3)$ is homotopic to the identity via some choice of homotopy h, so that $j\sigma_2$ is homotopic to $j\sigma_1$. It follows that there are strict equivalences

$$X(1) \leftarrow TX(1) \xrightarrow{h_*} FX(2) \leftarrow X(2).$$

If $X(1)$ and $X(2)$ are the outputs of functors defined on spectra (such as those arising from the comparison of fake with real suspension), then these equivalences are natural.

Fibrations and cofibrations

Suppose that $i : A \to X$ is a levelwise cofibration of spectra with cofibre $\pi : X \to X/A$. Suppose that $\alpha : S^r \to X^n$ represents a homotopy element such that the composite

$$S^r \xrightarrow{\alpha} X^n \xrightarrow{\pi} X^n/A^n$$

represents $0 \in \pi_r(X/A)^n$. Then, by comparing cofibre sequences, there is a commutative diagram

$$\begin{array}{ccccccc}
S^r & \longrightarrow & CS^r & \longrightarrow & S^1 \wedge S^r & \xrightarrow{\simeq} & S^1 \wedge S^r \\
\downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle S^1\wedge\alpha} \\
X^n & \xrightarrow[\pi]{} & (X/A)^n & \longrightarrow & S^1 \wedge A^n & \xrightarrow{S^1\wedge i} & S^1 \wedge X^n \\
 & & & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\
 & & & & A^{n+1} & \xrightarrow[i]{} & X^{n+1}
\end{array}$$

where $CS^r \simeq *$ is the cone on S^r. It follows that the image of $[\alpha]$ under the suspension map

$$\pi_r X^n \to \pi_{r+1} X^{n+1}$$

is in the image of the map $\pi_{r+1}A^{n+1} \to \pi_{r+1}X^{n+1}$. We have proved the following:

Lemma: Suppose that $A \to X \to X/A$ is a levelwise cofibre sequence of spectra. Then the sequence

$$\pi_k^s A \to \pi_k^s X \to \pi_k^s(X/A)$$

is exact.

Corollary: Any levelwise cofibre sequence

$$A \to X \to X/A$$

induces a long exact sequence

$$\dots \xrightarrow{\partial} \pi_k^s A \to \pi_k^s X \to \pi_k^s (X/A) \xrightarrow{\partial} \pi_{k-1}^s A \to \dots$$

This sequence is the long exact sequence in stable homotopy groups for a level cofibre sequence of spectra.

Proof: The map $X/A \to A \wedge S^1$ in the Puppe sequence induces the boundary map

$$\pi_k^s(X/A) \to \pi_k^s(A \wedge S^1) \cong \pi_k^s(A[1]) \cong \pi_{k-1}^s A.$$

since $A \wedge S^1$ is naturally stably equivalent to the shifted spectrum $A[1]$. □

Corollary: Suppose that X and Y are spectra. Then the inclusion $X \vee Y \to X \times Y$ is a natural stable equivalence.

Proof: The sequence

$$0 \to \pi_k^s X \to \pi_k^s (X \vee Y) \to \pi_k^s Y \to 0$$

arising from the level cofibration $X \subset X \vee Y$ is split exact, as is the sequence

$$0 \to \pi_k^s X \to \pi_k^s (X \times Y) \to \pi_k^s Y \to 0$$

arising from the fibre sequence $X \to X \times Y \to Y$. It follows that the map $X \vee Y \to X \times Y$ induces an isomorphism in all stable homotopy groups. $\Box$

Corollary: The stable homotopy category Ho(**Spt**) is *additive*: the sum of two maps $f, g : X \to Y$ is represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xleftarrow{\simeq} Y \vee Y \xrightarrow{\nabla} Y.$$

Corollary: Suppose that

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C & \xrightarrow[j]{} & D \end{array}$$

is a pushout in **Spt** where i is a levelwise cofibration. Then there is a long exact sequence in stable homotopy groups

$$\dots \xrightarrow{\partial} \pi_k^s A \xrightarrow{(i,\alpha)} \pi_k^s C \oplus \pi_k^s B \xrightarrow{j-\beta} \pi_k^s D \xrightarrow{\partial} \pi_{k-1}^s A \to \dots$$

This is the *Mayer-Vietoris sequence* for the cofibre square. The boundary map $\partial : \pi_k^s D \to \pi_{k-1}^s A$ is the composite

$$\pi_k^s D \to \pi_k^s D/C = \pi_k^s B/A \xrightarrow{\partial} \pi_{k-1}^s A.$$

Lemma A: Suppose that $A \xrightarrow{i} X \xrightarrow{\pi} X/A$ is a pointwise cofibre sequence in **Spt**, and let F be the strict homotopy fibre of the map $\pi : X \to X/A$. Then the induced map $i_* : A \to F$ is a stable equivalence.

Proof: Choose a strict fibration $p : Z \to X/A$ such that $Z \to *$ is a strict weak equivalence. Form the pullback

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi_*} & Z \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\pi]{} & X/A \end{array}$$

Then $\tilde{X}$ is the homotopy fibre of π and the maps $i : A \to X$ and $* : A \to Z$ together determine a map $i_* : A \to \tilde{X}$. We show that i_* is a stable equivalence.

Pull back the cofibre square

$$\begin{array}{ccc} A & \longrightarrow & * \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xrightarrow[\pi]{} & X/A \end{array}$$

along the fibration p to find a (levelwise) cofibre square

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & U \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & Z \end{array}$$

The spectrum Z is contractible, so a Mayer-Vietoris sequence argument implies that the map $\tilde{A} \to \tilde{X} \times U$ is a stable equivalence.
Also, from the fibre square

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & U \\ \downarrow & & \downarrow \\ A & \longrightarrow & * \end{array}$$

we see that the map $\tilde{A} \to A \times U$ is a stable equivalence. The map $i_* : A \to \tilde{X}$ induces a section $\theta : A \to \tilde{A}$ of the map $\tilde{A} \to A$ which composes with the projection $\tilde{A} \to U$ to give the trivial map $* : A \to U$. It follows that there is a commutative diagram

$$\begin{array}{ccccc} & \swarrow^{(1_A,*)} & A & \xrightarrow{i_*} & \tilde{X} \\ & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle (1_{\tilde{X}},*)} \\ A \times U & \xleftarrow{\simeq} & \tilde{A} & \xrightarrow{\simeq} & \tilde{X} \times U \\ & \searrow_{pr} & \downarrow & \swarrow & \\ & & U & & \end{array}$$

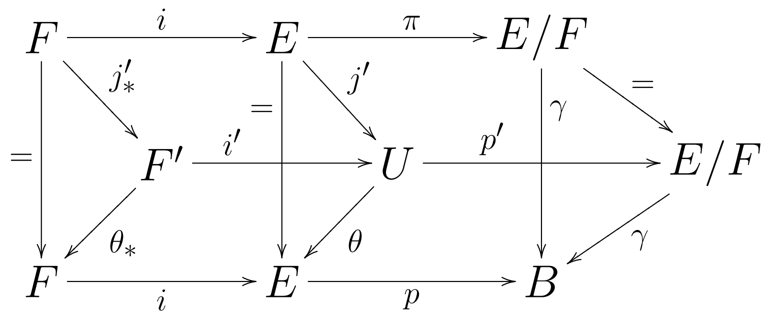
It follows that A is the stable homotopy fibre of the map $\tilde{A} \to U$, and so i_* is a stable equivalence. $\square$

Lemma B: Suppose that

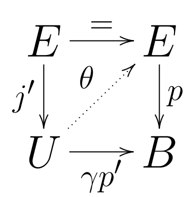
$$F \xrightarrow{i} E \xrightarrow{p} B$$

is a strict fibre sequence, where i is a levelwise cofibration. Then the induced map $\gamma : E/F \to B$ is a stable equivalence.

Proof: There is a diagram



where p' is a strict fibration, j' is a cofibration and a strict equivalence, and θ exists by a lifting property:



Then the map j'_* is a stable equivalence by Lemma A, so that θ_* is a stable equivalence. The map θ is a strict equivalence, so it follows from a comparison of long exact sequences in stable homotopy groups that γ is a stable equivalence. □

Remark: Lemma A and Lemma B together say that fibre and cofibre sequences coincide in the stable category. This is a famous slogan in this business.

Cofibrant generation

We will show in this section that the stable model structure on **Spt** is cofibrantly generated. This means that there are sets I and J of stably trivial cofibrations and cofibrations, such that $p : X \to Y$ is a stable fibration (respectively stably trivial fibration) if and only if it has the RLP with respect to all members of the set I (respectively all members of J).

Recall that a map $p : X \to Y$ is a stably trivial fibration if and only if it is a strict fibration and a strict weak equivalence. Thus p is a stably trivial fibration if and only if it has the RLP with respect to all maps

$$\Sigma^\infty \partial\Delta^n_+[m] \to \Sigma^\infty \Delta^n_+[m]$$

We have therefore found our set of maps J.

It remains to find a set of stably trivial cofibrations I which generates the full class of stably trivial cofibrations. We do this in a sequence of lemmas.

Lemma 1: Suppose given the diagram

$$\begin{array}{ccc} A \cap X & \longrightarrow & X \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & Y \end{array}$$

in spectra, where j is a cofibration and i is a levelwise cofibration. Then the induced map $j_* : A \cap X \to A$ is a cofibration.

Proof: From the diagram

$$\begin{array}{ccc} S^1 \wedge A^n \cup_{S^1 \wedge (A \cap X)^n} (A \cap X)^{n+1} & \longrightarrow & A^{n+1} \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ S^1 \wedge Y^n \cup_{S^1 \wedge X^n} X^{n+1} & \longrightarrow & Y^{n+1} \end{array}$$

one sees that it suffices to show that the comparison i_* is a cofibration of pointed simplicial sets. This follows from the fact that the diagram of pointed simplicial sets

$$\begin{array}{ccccc} S^1 \wedge A^n & \longleftarrow & S^1 \wedge (A \cap X)^n & \longrightarrow & (A \cap X)^{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ S^1 \wedge Y^n & \longleftarrow & S^1 \wedge X^n & \longrightarrow & X^{n+1} \end{array}$$

has the left lifting property with respect to all constant diagrams arising from trivial fibrations $p : Z \to W$: all vertical maps in the diagram are cofibrations, and the square on the left is cofibrant since the map $S^1 \wedge (A \cup X)^n \to S^1 \wedge Y^n$ is a cofibration. $\square$

Corollary: Suppose given level cofibrations $A \subset X \subset Y$ such that the map $A \subset Y$ is a cofibration. Then the map $A \subset X$ is a cofibration.

Lemma 2: Suppose that α is an infinite cardinal, and suppose given maps

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & Y \end{array}$$

such that A is α-bounded in the sense that α is an upper bound for all $|A^n_m|$. Suppose also that i is a levelwise cofibration and that j is a levelwise cofibration and a stable equivalence. Then there is an α-bounded subobject $B \subset Y$ such that $A \subset B \subset Y$ and the map $B \cap X \to B$ is a stable equivalence.

Proof: The cofibration $B \cap X \to B$ is a stable equivalence if and only if all stable homotopy groups $\pi^s_n(B/(B \cap X))$ vanish.

One needs to know that if A is α-bounded then all stable homotopy groups $\pi^s_n A$ of A are α-bounded. There is a strictly fibrant model $\mathrm{Ex}^\infty A$ of A which is α-bounded. For this, see III.4.8 of [2]:

$$\operatorname{Ex} X_n = \hom(\operatorname{sd} \Delta^n, X),$$

the "last vertex maps" $\operatorname{sd} \Delta^n \to \Delta^n$ induce a weak equivalence $X \to \operatorname{Ex} X$, and $\operatorname{Ex}^\infty X = \varinjlim \operatorname{Ex}^n X$ is a Kan complex. Finally, if Y is an α-bounded Kan complex then all of its simplicial homotopy groups are α-bounded.

Write $A_0 = A$. Y is a filtered colimit of its α-bounded subobjects, and the α-bounded set of elements of the homotopy groups $\pi_n^s(A_0/(A_0 \cap X))$ vanish in $\pi_n^s(A_1/(A_1 \cap X))$ for some α-bounded subobject $A_1 \subset X$ with $A_0 \subset A_1$.

Repeat the construction inductively to find α-bounded subcomplexes

$$A = A_0 \subset A_1 \subset A_2 \subset \dots$$

of Y such that all induced maps

$$\pi_n^s(A_i/(A_i \cap X)) \to \pi_n^s(A_{i+1}/(A_{i+1} \cap X))$$

are 0. Set $B = \cup_i A_i$. Then B is α-bounded and all groups $\pi_n^s(B/(B \cap X))$ vanish. □

Lemma 2 is an example of a "bounded cofibration condition", and such things abound in nature. Here's another example:

Lemma: Suppose that α is an infinite cardinal, and suppose given cofibrations

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & Y \end{array}$$

of (pointed) simplicial sets such that A is α-bounded. Suppose also that j is a weak equivalence. Then there is an α-bounded subobject $B \subset Y$ with $A \subset B$ such that the map $B \cap X \to B$ is a weak equivalence.

The proof of this result is slightly less trivial — see X.2.8 of [2].

Corollary: Suppose given level cofibrations

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & Y \end{array}$$

of spectra such that A is α-bounded. Suppose also that j is a level weak equivalence. Then there is an α-bounded subobject $B \subset Y$ with $A \subset B$ such that the map $B \cap X \to B$ is a level weak equivalence.

Proof: Exercise.

Corollary: Suppose given a map $X \to Y$ which is a level cofibration and a level (resp. stable) weak equivalence, and suppose that $X \subset A \subset X$ where A is α-bounded. Then there is an α-bounded subobject $B \subset X$ with $A \subset B$ such that $B \to X$ is a level (resp. stable) weak equivalence.

Corollary: Suppose given level cofibrations

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & Y \end{array}$$

of spectra such that A is α-bounded. Suppose also that j and i are level weak equivalences. Then there is an α-bounded subobject $B \subset Y$ with $A \subset B$ such that the maps $B \cap X \to B$ and $B \subset X$ are level weak equivalences.

Proof: Set $A = A_0$. There is an α-bounded $B_0 \subset Y$ with $A_0 \subset B_0$ such that $B_0 \cap X \to B_0$ is a level weak equivalence. There is an α-bounded subobject $A_1 \subset Y$ with $B_1 \subset A_0$ such that $A_0 \to X$ is a level weak equivalence. Continue inductively, and let B be the union of the subobjects

$$A = A_0 \subset B_0 \subset A_1 \subset B_1 \subset \dots \quad \square$$

Lemma 3: The class of stably trivial cofibrations has a generating set I.

Proof: Recall that the class of cofibrations is generated (ie. is the "saturation") of the set J of maps

$$\Sigma^\infty \partial\Delta^n_+[-k] \to \Sigma^\infty \Delta^n_+[-k],\ k \geq 0.$$

Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

where j is a cofibration and f is a stable equivalence. Then f has a factorization $f = q \cdot i$ where i is a stably trivial cofibration and q is a stably trivial fibration. Then there is a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \Big\downarrow & & \downarrow{\scriptstyle i} \\ {\scriptstyle j} & & Z \\ \Big\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle q} \\ B & \longrightarrow & Y \end{array}$$

where the lift θ exists since j is a cofibration and q is a stably trivial fibration.

Choose an infinite cardinal λ_B such that $\lambda_B > |B|$. Then $|\theta(B)| < \lambda_B$ and Lemma 2 says that there is a subobject $D \subset Z$ such that D is λ_B-bounded and such that the cofibration $D \cap X \to D$ (Lemma 1) is a stable equivalence. What we have, then, is a factorization

$$\begin{CD} A @>>> D \cap X @>>> X \\ @VjVV @VVV @VVfV \\ B @>>> D @>>> Y \end{CD}$$

of the original diagram through a set T_j of stably trivial cofibrations, namely the λ_B-bounded stably trivial cofibrations, which depends only on j.

Let $I = \cup_{j \in J} T_j$ and let $C(I)$ denote its saturation. Recall that J is the set of generators for the class of cofibrations.

Now suppose that $i : U \to V$ is a stably trivial cofibration. Then i has a factorization

$$\begin{array}{ccc} U & \xrightarrow{j} & W \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle q} \\ & & V \end{array}$$

where j is a member of $C(I)$ and q has the RLP with respect to all members of $C(I)$. But then q has the RLP with respect to all generators of the class of cofibrations by the construction above, so that q has the RLP with respect to all cofibrations.

In particular, there is a diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & W \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle q} \\ V & \xrightarrow[1]{} & V \end{array}$$

so that i is a retract of j. In particular, i is a member of $C(I)$ so that I is a generating set for the class of stably trivial cofibrations. $\square$

Suppose that α is an arbitrary choice of infinite cardinal. An α*-bounded* cofibration is a cofibration $A \to B$ such that B (hence A) is α-bounded in the sense that $|B^n_m| < \alpha$ for all n, m. .

Corollary: Suppose that α is an infinite cardinal. Then a map $p : X \to Y$ is a stable fibration if and only if it has the RLP with respect to all α-bounded stably trivial cofibrations.

Proof: Each member of the set I is α-bounded. $\square$

Remarks:

- The proof of Lemma 3b involves the verification of a "solution set condition" — see [4], but the argument given there was reverse engineered from an earlier argument of Beke. The overall idea is apparently due to J. Smith.

- The Corollary above is claimed in [1], essentially without proof. The proof envisaged there is supposed to proceed by analogy with a Zorn's Lemma argument that is given for the corresponding statement for simplicial sets (actually, for simplicial presheaves), but it's not clear that this actually works for spectra. The problem, ultimately, is that not every level cofibration is a cofibration in the **Spt**.

References

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