Lecture 004 (October 12, 2005)

Simplicial modules

Suppose that R is a commutative ring with identity, and write $R - \mathbf{Mod}$ for the category of R-modules, and let $sR - \mathbf{Mod}$ be the category of simplicial R-modules. This section will describe the homotopy theory of simplicial R-modules, in the style of [3]. See also section III.2 of [1].

There are adjoint functors

$$R : s\mathbf{Set} \leftrightarrows sR - \mathbf{Mod} : u$$

where RX is the free simplicial R-module on a simplicial set X (explicitly $(RX)_n = R(X_n)$ is the free R-module on the set X_n), and u forgets the R-module structure.

Say that a map $f : A \to B$ of simplicial R-modules is a weak equivalence (resp. fibration) if the underlying map $uA \to uB$ is a weak equivalence (resp. fibration) of simplicial sets.

A cofibration in $sR\mathbf{Mod}$ is a map which has the left lifting property with respect to all trivial fibrations. Every cofibration $i : K \to L$ of simplicial sets induces a cofibration $i_* = Ri : RK \to RL$ of simplicial R-modules.

Plainly, a map $p : A \to B$ of simplicial R-modules is a fibration if and only if it has the RLP with respect to the induced cofibrations

$$R\Lambda^n_k \to R\Delta^n.$$

Similarly, p is a trivial fibration if and only if it has the RLP with respect to all induced maps

$$R\partial\Delta^n \to R\Delta^n.$$

Here are some other things to know (see III.2 of [1]):

- $u(A)$ is a Kan complex, for all simplicial R-modules A (really for all simplicial groups). Every surjective homomorphism $p : A \to B$ is a fibration.

- $$\pi_k(u(A), 0) \cong H_k(NA),$$

  where NA is the *normalized chain complex* of A:

  $$NA_n = \cap_{i=0}^{n-1} \ker d_i,$$

  and $\partial = (-1)^n d_n : NA_n \to NA_{n-1}$.

- The *Moore complex* MA of A has $MA_n = A_n$ and

  $$\partial = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

The inclusions $NA_n \to A_n$ induce a natural chain homotopy equivalence $NA \to MA$.

Observations: 1) $f : A \to B$ is a weak equivalence if and only if f induces a homology isomorphism $NA \to NB$ (equivalently $MA \to MB$). Note that $f_* : uA \to uB$ is an H-map of H-spaces, so it's enough to look at the maps

$$\pi_k(uA, 0) \to \pi_k(uB, 0).$$

2) $H_n(MR(X)) = H_n(X, R)$ for all simplicial sets X.

3) Any homotopy $h : X \times \Delta^1 \to Y$ induces a homotopy

$$uR(X) \times \Delta^1 \xrightarrow{\gamma} uR(X \times \Delta^1) \xrightarrow{uRh} uRY$$

where $\gamma : uA \times K \to u(A \otimes RK)$ is the natural map defined in degree n by the map

$$\bigsqcup_{\sigma \in K_n} uA_n \to u(\bigoplus_{\sigma \in K_n} A_n).$$

4) Consequence: All maps $\Delta^n \to *$ induce weak equivalences $R\Delta^n \to R*$, or are homology isomorphisms.

Lemma 1: All inclusions $\Lambda^n_k \subset \Delta^n$ induce weak equivalences $R\Lambda^n_k \to R\Delta^n$.

Proof: Suppose that the subcomplex

$$\Delta\langle \sigma_1, \dots, \sigma_r \rangle \subset \Lambda^n_k$$

is generated by a list of non-degenerate $(n-1)$-simplices $\{\sigma_i\}$ of Λ^n_k. We show that the map

$$\Delta\langle \sigma_0, \dots, \sigma_r \rangle \to *$$

is a homology isomorphism for all such subobjects, hence for Λ^n_k itself.

Note that $r \leq n$, and that there are pushouts

$$\begin{tikzcd}
\Delta\langle \sigma_r \cap \sigma_1, \dots, \sigma_r \cap \sigma_{r-1} \rangle \arrow[r] \arrow[d] & \Delta\langle \sigma_1, \dots, \sigma_{r-1} \rangle \arrow[d] \\
\Delta^{n-1} \arrow[r, "\sigma_r"'] & \Delta\langle \sigma_1, \dots, \sigma_r \rangle
\end{tikzcd}$$

Then $r - 1 \leq n - 1$ so that

$$\Delta\langle \sigma_r \cap \sigma_1, \dots, \sigma_r \cap \sigma_{r-1} \rangle \subset \Delta^{n-1}_s$$

for some s. Then by induction on dimension n and number of simplices r (and the fact that the free module functor preserves pushouts), the map

$$\Delta\langle \sigma_0, \dots, \sigma_r \rangle \to *$$

is a homology isomorphism. $\square$

Exercise: Any short exact sequence

$$0 \to A \to B \xrightarrow{p} C \to 0$$

induces a short exact sequence

$$0 \to NA \to NB \to NC \to 0$$

of normalized chain complexes, and hence induces a long exact sequence in homology groups.

Lemma 2: Every weak equivalence $f : X \to Y$ of simplicial sets induces a weak equivalence $f_* : RX \to RY$ of simplicial R-modules.

Proof: Every trivial cofibration $i : A \to B$ is a retract of an anodyne extension, and hence induces a weak equivalence $i_* : RA \to RB$, by Lemma 1. Every weak equivalence $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

where j is a trivial cofibration and π is left inverse to a trivial cofibration $Y \to Z$. The Lemma follows. $\square$

Here's the function complex object for simplicial R-modules:

$$\mathbf{hom}(A, B)_n = \hom(A \otimes R\Delta^n, B).$$

Write $A \otimes K = A \otimes RK$ for simplicial R-modules A and simplicial sets K. There are natural isomorphisms

$$\begin{aligned} \hom(A \otimes K, B) &\cong \hom(RK, B)) \\ &\cong \hom(RK, \mathbf{hom}(A, B)). \end{aligned}$$

Theorem: With the definitions given above, the category $sR-\mathbf{Mod}$ of simplicial R-modules satisfies the definitions for a proper closed simplicial model category.

Proof: A map p is a fibration if and only if it has the RLP with respect to all maps $R\Lambda^n_k \to R\Delta^n$. A small object argument then implies that any map $f : A \to B$ has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "f"'] & C \arrow[d, "p"] \\ & B \end{tikzcd}$$

where p is a fibration and j is a cofibration which has the LLP with respect to all fibrations. It follows from the construction, Lemma 2, and appropriate long exact sequence arguments that j is also a weak equivalence. The other factorization $f = q \cdot i$ with q a trivial fibration and i a cofibration is accomplished similarly, and more easily.

This proves **CM5**.

Suppose that $i : A \to B$ is a cofibration and a weak equivalence. There is a factorization

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[dr, "j"'] & C \arrow[d, "p"] \\
& B
\end{tikzcd}$$

as above, where p is a fibration and j is a weak equivalence which has the left lifting property with respect to all fibrations. Then p is a weak equivalence, so the lifting exists in the diagram

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "i"'] & C \arrow[d, "p"] \\
B \arrow[r, "1"'] \arrow[ur, dotted] & B
\end{tikzcd}$$

so that i is a retract of j and therefore has the same lifting property. This proves **CM4**. The rest of the closed model axioms are trivial to verify.

The simplicial model structure follows from the natural isomorphism

$$\mathbf{hom}(RK, A) \cong \mathbf{hom}(K, uA)$$

and the simplicial model axiom for simplicial sets. Right properness is a consequence of right properness for simplicial sets, while left properness can be proved by comparing long exact sequences. $\square$

Remark: Every cofibration $i : A \to B$ of $sR-\mathbf{Mod}$ is a monomorphism in all simplicial degrees. In effect, i has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & C \arrow[d, "p"] \\ & B \end{tikzcd}$$

where j is in the saturation of the set of morphisms $R(\partial\Delta^n) \to R(\Delta^n)$ and is therefore a monomorphism. Then i has the LLP with respect to the trivial fibration p, so that i is a retract of j in the usual way.

Write $\mathbf{Ch}_+(R)$ for the category of ordinary chain complexes

$$C : \ \dots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

of R-modules. The normalized chains functor

$$N : sR-\mathbf{Mod} \to \mathbf{Ch}_+(R)$$

is part of an equivalence of categories, which is often called the *Dold-Kan correspondence.*

The inverse functor

$$\Gamma : \mathbf{Ch}_+(R) \to sR-\mathbf{Mod}$$

is a bit fussy to define. It is is given in simplicial degree n by the assignment

$$(\Gamma C)_n = \bigoplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} C_k.$$

The simplicial structure map $\theta^* : (\Gamma C)_n \to (\Gamma C)_m$ arising from an ordinal number morphism $\theta : \mathbf{m} \to \mathbf{n}$ is defined by the diagram

$$\begin{tikzcd}
C_k \arrow[r, "in_\sigma"] \arrow[d, "d^*"'] & \oplus_{\mathbf{n} \twoheadrightarrow \mathbf{k}} C_k \arrow[d, "\theta^*"] \\
C_r \arrow[r, "in_\gamma"'] & \oplus_{\mathbf{m} \twoheadrightarrow \mathbf{r}} C_r
\end{tikzcd}$$

where $\sigma\theta = d\gamma$ is an epi-monic factorization of $\sigma\theta$, and finally $d^* = (-1)^n \partial$ if $d = d_n$ and is 0 otherwise.

The following result is due to Quillen:

Lemma (Fibration Lemma): A map $p : A \to B$ of simplicial R-modules is a fibration if and only if the induced maps $NA_k \to NB_k$ are surjective for $k \geq 1$.

Proof: It's easy to see that any fibration p must induce surjective maps $NA_k \to NB_k$ for $k \geq 1$, so we'll only prove the converse.

Note that any abelian group homomorphism $E \to F$ induces a fibration $K(E, 0) \to K(F, 0)$, where $K(E, 0)$ is the constant simplicial abelian group associated to E. Note that

$$NK(E,0) \cong E[0]$$

where the thing on the right is the chain complex which is E in degree 0 and is 0 in all other degrees.

It's also easy to see, from the Dold-Kan correspondence, that if a chain map $C \to D$ is surjective in all degrees, then the induced simplicial R-module map $\Gamma C \to \Gamma D$ is surjective in all degrees, and is therefore a fibration.

Form the diagram

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, "p"] \arrow[ddr] & & \\
& K(\pi_0 A, 0) \times_{K(\pi_0 B, 0)} B \arrow[r, "\tilde{p}"] \arrow[d] & B \arrow[d] \\
& K(\pi_0 A, 0) \arrow[r, "p_*"'] & K(\pi_0 B, 0)
\end{tikzcd}$$

The normalized chains functor preserves pullbacks, and the chain map

$$NA \to \pi_0 A[0] \times_{\pi_0 B[0]} NB$$

is surjective in all degrees by our assumptions. Thus the map

$$A \to K(\pi_0 A, 0) \times_{K(\pi_0 B, 0)} B$$

is a fibration. The map $\tilde{p}$ is the pullback of a fibration p_*, so that p is a fibration. $\square$

Remarks:

1) The Dold-Kan correspondence and the model structure for simplicial R-modules therefore induce a model structure for $\mathbf{Ch}_+(R)$ for which the weak equivalences are homology isomorphisms and the fibrations are chain maps $C \to D$ which are surjective in non-zero degrees. This model structure on chain complexes is one of the first examples of a model structure that one normally meets.

2) **Warning:** The Fibration Lemma fails in more exotic contexts, such as sheaves of simplicial abelian groups and sheaves of chain complexes. In such cases, there is a good model structure for simplicial modules and it forces a model structure on chain complexes through the Dold-Kan correspondence. The catch is that the fibrations may not be so easy to describe.

If A is a simplicial R-module and K is a pointed simplicial set, then $A \otimes K$ is defined by

$$A \otimes K = A \otimes_R \tilde{R}(K),$$

where $\tilde{R}(K) = RK/R*$.

Other things that we will need to know are the existence of natural isomorphisms

$$\tilde{R}(K \wedge L) \cong \tilde{R}(K) \otimes \tilde{R}(L) = K \otimes \tilde{R}(L),$$

and that there exists a natural map

$$\gamma : u(A) \wedge K \to u(A \otimes K).$$

Here's a final thing about simplicial modules that we're going to need:

Lemma 3: Suppose that A is a simplicial abelian group. Then the canonical map

$$\eta : A \to \mathbf{hom}(S^1, A \otimes S^1)$$

is a weak equivalence.

Proof: Write Δ^1_* for the simplicial set Δ^1, pointed by the vertex 0. Then there is a contracting homotopy $h : \Delta^1_* \wedge \Delta^1_* \to \Delta^1_*$ given by the picture

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 1 \end{tikzcd}$$

and this map h determines a natural contracting homotopy

$$h_* : \mathbf{hom}(\Delta^1_*, B) \otimes \Delta^1_* \to \mathbf{hom}(\Delta^1_*, B).$$

for all simplicial abelian groups B. Of course, $\mathbf{hom}(\Delta^1_*, B) = PB$. Also, $B \otimes \Delta^1_*$ is the simplicial model for the cone on B, and there is a natural short exact sequence

$$0 \to B \to B \otimes \Delta^1_* \to B \otimes S^1 \to 0.$$

The homotopy h_* induces a composite morphism

$$\begin{tikzcd}
A \otimes \Delta^1_* \arrow[r, "\eta\otimes 1"] \arrow[ddr, "\gamma"'] & \mathbf{hom}(S^1, A \otimes S^1) \otimes \Delta^1_* \arrow[d] \\
 & \mathbf{hom}(\Delta^1_*, A \otimes S^1) \otimes \Delta^1_* \arrow[d, "h_*"] \\
 & \mathbf{hom}(\Delta^1_*, A \otimes S^1)
\end{tikzcd}$$

and one can show that there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[d] & \mathbf{hom}(S^1, A \otimes S^1) \arrow[d] \\
A \otimes \Delta^1_* \arrow[r, "\simeq"] \arrow[d] & \mathbf{hom}(\Delta^1_*, A \otimes S^1) \arrow[d] \\
A \otimes S^1 \arrow[r, "1"'] & A \otimes S^1
\end{tikzcd}$$

This is a comparison of fibre sequences, and so the map η is a weak equivalence by properness. $\square$

Corollary 4: The natural map

$$\epsilon_A : \mathbf{hom}(S^1, A \otimes S^1) \otimes S^1 \to A$$

induces isomorphisms in π_k for $k \geq 1$.

Write $\Omega A = \mathbf{hom}(S^1, A)$.

Proof: There is a diagram

$$\begin{tikzcd}
\Omega A \arrow[r, "\eta", "\simeq"'] \arrow[dr, "1"'] & \Omega(\Omega A \otimes S^1) \arrow[d, "\Omega\epsilon"] \\
& \Omega A
\end{tikzcd}$$

Thus, $\Omega\epsilon$ is a weak equivalence, which means that ϵ has the claimed effect in homotopy groups. $\square$

Spectra in simplicial modules

A *spectrum in simplicial R-modules* (or *spectrum object in simplicial R-modules*) A consists of simplicial R-modules A^n, $n \geq 0$, together with bonding homomorphisms

$$\sigma : S^1 \otimes A^n \to A^{n+1},\ n \geq 0.$$

A morphism $f : A \to B$ of spectrum objects consists of simplicial R-module homomorphisms $A^n \to B^n$, $n \geq 0$, which respect the bonding homomorphisms in the obvious sense. Write $\mathbf{Spt}(R)$ for the corresponding category. This category is complete and cocomplete, because the category of simplicial R-modules has these properties.

The maps $\gamma : S^1 \wedge u(A^n) \to u(S^1 \otimes A^n$ give the pointed simplicial sets $u(A^n)$ the structure of a spectrum, and there is a corresponding *forgetful* functor

$$u : \mathbf{Spt}(R) \to \mathbf{Spt}.$$

The reduced free R-module functor $\tilde{R}$ determines a left adjoint to u. Explictly,

$$(\tilde{R}X)^n = \tilde{R}(X^n),$$

and the bonding morphisms are the composites

$$S^1 \otimes \tilde{R}(X^n) \cong \tilde{R}(S^1 \wedge X^n) \xrightarrow{\sigma_*} \tilde{R}(X^{n+1}).$$

Let's cut to the chase: a map $f : A \to B$ of spectrum objects is a *stable equivalence* (respectively *stable fibration*) if the underlying map $u(f) : uA \to uB$ of spectra is a stable equivalence (respectively stable fibration).

A *cofibration* in $\mathbf{Spt}(R)$ is a map which has the left lifting property with respect to all morphisms which are stable fibrations and stable equivalences. By adjointness, if $A \to B$ is a cofibration of spectra, then the induced map $\tilde{R}(A) \to \tilde{R}(B)$ is a cofibration of spectrum objects.

Lemma 4: The functor $\tilde{R} : \mathbf{Spt} \to \mathbf{Spt}(R)$ preserves stable equivalences.

Proof: The functor $\tilde{R}$ preserves level equivalences, so it suffices to show that if $A \to B$ is a stably trivial cofibration of spectra, then $\tilde{R}(A/B) \to 0$ is a stable equivalence. Thus, it suffices to show that $\tilde{R}(X) \to 0$ is a stable equivalence if $X \to *$ is a stable equivalence. We can assume that X is level fibrant.

Since X is level fibrant, the assumption that $X \to *$ is a stable equivalence implies that all spaces $\Omega^\infty X^n$ are contractible. Thus, if $K \subset X^n$ is a finite subcomplex of X^n, there is a $k \geq 0$ such that the composite

$$S^k \wedge K \to S^k \wedge X^n \xrightarrow{\sigma^k} X^{n+k}$$

is homotopically trivial. This means that the induced map

$$S^k \otimes \tilde{R}(K) \to S^k \otimes \tilde{R}(X^n) \to \tilde{R}(X^{n+k})$$

is also homotopically trivial, and so the morphism

$$\Sigma^\infty \tilde{R}(K)[-n] \to \tilde{R}(X)$$

induces 0 in all stable homotopy groups. Every element in $\pi_k^s(\tilde{R}(X))$ is in the image of such a map. All stable homotopy groups of $\tilde{R}(X)$ are therefore 0. $\square$

Once again, there is a function complex construction:

$$\mathbf{hom}(A, B)_n = \{A \otimes \Delta^n \to B\}.$$

Suppose that $i : A \to B$ is a level monomorphism in $\mathbf{Spt}(R)$ (as are all level cofibrations). Then there is a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} B/A \to 0$$

and the map π is a level surjection, hence a level fibration. In particular, the sequence is a level fibre sequence, and so there is a long exact sequence

$$\dots \pi_{k+1}^s(B/A) \xrightarrow{\partial} \pi_k^s A \xrightarrow{i_*} \pi_k^s(B) \xrightarrow{\pi_*} \pi_k^s(B/A) \to \dots$$

More generally, the long exact sequence exists for any short exact sequence in $\mathbf{Spt}(R)$, and is natural.

Theorem: With these definitions, the category $\mathbf{Spt}(R)$ of spectrum objects in simplicial R-modules has the structure of a proper closed simplicial model category.

Proof: The category $\mathbf{Spt}$ is cofibrantly generated. Thus, a map $p : A \to B$ is a stable fibration if and only if it has the right lifting property with respect to the maps

$$\tilde{R}(U) \to \tilde{R}(V)$$

induced by a set J of stably trivial cofibrations $U \to V$. All induced maps $\tilde{R}(U) \to \tilde{R}(V)$ are stable equivalences by Lemma 4. Note that level inclusions which are stable equivalences are closed under pushout, by a long exact sequence argument. It follows from a (possibly transfinite) small object argument that every map $f : A \to B$ in $\mathbf{Spt}(R)$ has a factorization

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "f"'] & C \arrow[d, "p"] \\ & D \end{tikzcd}$$

where j is a stably trivial cofibration which has the

LLP with respect to all fibrations and p is a fibration. The proof of the other statement involved in the factorization axiom **CM5** uses the fact that a map $p : A \to B$ is a stable fibration and a stable equivalence if and only if it has the right lifting property with respect to all morphisms

$$\tilde{R}(\Sigma^{\infty}\partial\Delta^n_+[k]) \to \tilde{R}(\Sigma^{\infty}\Delta^n_+[k])$$

This proof is similar, and easier.

If $i : A \to B$ is a stably trivial cofibration, then i is a retract of a map which has the LLP with respect to all fibrations, on account of a factorization for i in the style displayed above. Thus, every stably trivial cofibration has the LLP with respect to all fibrations, proving **CM4**.

The other closed model axioms are trivial to verify.

There is a natural isomorphism

$$\mathbf{hom}(\tilde{R}(K), A) \cong \mathbf{hom}(K, u(A)),$$

so that Quillen's axiom **SM7** follows from the corresponding statement for spectra, and so $\mathbf{Spt}(R)$ has a simplicial model structure.

Right properness follows from right properness for **Spt**, and left properness is proved by comparing long exact sequences. $\square$

Here are some things to notice:

0) Every spectrum object in simplicial R-modules is level fibrant.

1) The forgetful functor u and its left adjoint $\tilde{R}$ determine a Quillen adjunction

$$\tilde{R} : \mathbf{Spt} \leftrightarrows \mathbf{Spt}(R) : u$$

If $R = \mathbb{Z}$ the canonical map $X \to u(\mathbb{Z}(X))$ is called the Hurewicz homomorphism.

2) There is also a Quillen adjunction

$$\Sigma^\infty : sR - \mathbf{Mod} \leftrightarrows \mathbf{Spt}(R) : \text{0-level}$$

where $(\Sigma^\infty A)^n = S^n \otimes A$ (suspension spectrum) and the "0-level" functor is defined by $B \mapsto B^0$.

3) Suppose that A is a simplicial R-module, and consider the spectrum object $\Sigma^\infty A$. The bonding maps $S^1 \otimes S^n \otimes A \to S^{n+1} \otimes A$ are canonical isomorphisms, with adjoints

$$S^n \otimes A \to \mathbf{hom}(S^1, S^1 \otimes S^n \otimes A)$$

given by adjunction maps η. All of these maps η are weak equivalences by Lemma 3, and so $\Sigma^\infty A$ is stably fibrant, ie. $u(\Sigma^\infty A)$is an Ω-spectrum. It also follows that there are isomorphisms

$$\pi_n^s = \begin{cases} \pi_n(A) & \text{if } n \geq 0 \text{, and} \\ 0 & \text{if } n < 0. \end{cases}$$

In particular,

$$\pi_n^s(\tilde{R}(\Sigma^\infty(X))) \cong \pi_n^s(\Sigma^\infty \tilde{R}(X))$$

coincides with the reduced homology group $\tilde{H}_n(X, R)$ for $n \geq 0$ and is 0 otherwise.

Recall that there is a natural map

$$\gamma : u(A) \wedge K \to u(A \otimes K)$$

for pointed simplicial sets K and simplicial R-modules R. There are various ways to describe this map, but in simplicial degree n it is the obvious map

$$\bigvee_{K_n - *} A_n \to \bigoplus_{K_n - *} A_n.$$

The construction can be iterated, meaning that there are commutative diagrams

$$\begin{tikzcd} L \wedge u(A) \wedge K \arrow[r, "1\wedge\gamma"] \arrow[d, "\gamma\wedge 1"'] & L \wedge u(A \otimes K) \arrow[d, "\gamma"] \\ u(L \otimes A) \wedge K \arrow[r, "\gamma"'] & u(L \otimes A \otimes K) \end{tikzcd}$$

The map γ may therefore promoted to the spectrum level, so there is a natural map

$$\gamma : u(B) \wedge K \to u(B \otimes K)$$

for spectrum objects B and pointed simplicial sets K.

Theorem: The map

$$\gamma : u(B) \wedge K \to u(B \otimes K)$$

is a stable equivalence for all spectrum objects B and pointed simplicial sets K.

Proof: The simplicial set K has a (pointed) skeletal decomposition $\mathrm{sk}_n K \subset K$, and there are pushout diagrams

$$\begin{tikzcd}
\bigvee_{x \in NK_n} \partial\Delta^n_+ \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} K \arrow[d] \\
\bigvee_{x \in NK_n} \Delta^n_+ \arrow[r] & \mathrm{sk}_n K
\end{tikzcd}$$

of pointed simplicial sets.

Smashing with $u(B)$ gives a homotopy cocartesian diagram, which can be compared to the diagram of spectra underlying the pushout diagram

$$\begin{tikzcd}
\oplus_{x \in NK_n}(B \otimes \partial\Delta^n_+) \arrow[r] \arrow[d] & B \otimes \mathrm{sk}_{n-1} B \arrow[d] \\
\oplus_{x \in NK_n}(B \otimes \Delta^n_+) \arrow[r] & B \otimes \mathrm{sk}_n K
\end{tikzcd}$$

in $\mathbf{Spt}(R)$ via the map γ. The underlying diagram of spectra is homotopy cocartesian since both vertical maps have the same cofibres.

Inductively, one can assume that

$$u(B) \wedge \operatorname{sk}_{n-1} K \to u(B \otimes \operatorname{sk}_{n-1} K)$$

is a stable equivalence for all K. The statement for 0-skeleta is a consequence of additivity. It follows that it suffices to show that the map

$$\gamma : u(B) \wedge (\bigvee_{NK_n} \Delta^n_+) \to u(B \otimes (\bigvee_{NK_n} \Delta^n_+)).$$

By additivity, this reduces to the statement that

$$\gamma : u(B) \wedge \Delta^n_+ \to u(B \otimes \Delta^n_+)$$

is a stable equivalence. But both displayed functors preserve homotopy equivalences, so this particular instance of γ is equivalent to

$$\gamma : u(B) \wedge S^0 \to u(B \otimes S^0),$$

which is an isomorphism. □

Example: $H_n(X, R) \cong \pi^s_n(H(R) \wedge X)$. Here $H(R)$ is the Eilenberg-Mac Lane spectrum $\tilde{R}(\mathbf{S})$; it's also the sphere spectrum for $\mathbf{Spt}(R)$. Generally speaking, this is why one calls the groups $E_*(X) = \pi^s_*(E \wedge X)$ the E-homology of the space X.

Chain complexes

Given a chain complex D, identify D with a $\mathbb{Z}$-graded chain complex by putting in 0 in negative degrees, and define a new chain complex $D[k]$ by

$$D[k]_p = \begin{cases} D_{k+p} & \text{if } p > 0, \\ \ker(\partial : D_k \to D_{k-1}) & \text{if } p = 0. \end{cases}$$

In other words, for $n \geq 0$, $D[-n]$ shifts up n times while $D[n]$ is the good truncation of a shift down.

There are two suspension constructions for simplicial R-modules [2], [3]:

- the standard suspension $S^1 \otimes A = \tilde{R}(S^1 \otimes A)$,
- the Eilenberg-Mac Lane (or Kan) suspension $\overline{W}A = \Gamma(NA[-1])$.

There is an alternative construction for $\overline{W}A$.

Every simplicial abelian group can be written as a coequalizer

$$\bigoplus_{\theta:\mathbf{m}\to\mathbf{n}} A_n \otimes \Delta^m_+ \rightrightarrows \bigoplus_{n\geq 0} A_n \otimes \Delta^n_+ \to A$$

There is a pointed cosimplicial set $\mathbf{n} \mapsto \Delta^{n+1}_*$, where Δ^{n+1}_* is Δ^{n+1} pointed by 0, and $\theta : \mathbf{m} \to \mathbf{n}$

induces $\theta_* : \mathbf{m}+\mathbf{1} \to \mathbf{n}+\mathbf{1}$ which is defined by

$$\theta_*(j) = \begin{cases} 0 & j = 0, \\ \theta(j-1)+1 & j > 0. \end{cases}$$

The simplicial set maps $d^0 : \Delta^n \to \Delta^{n+1}$ determine a map of cosimplicial spaces, and a pointwise monomorphism of cosimplicial simplicial abelian groups

$$\tilde{R}\Delta^n_+ \to \tilde{R}\Delta^{n+1}_*$$

One checks that there is an isomorphism of cosimplicial chain complexes

$$N(\tilde{R}\Delta^{n+1}_* / N\tilde{R}\Delta^n_+) \cong N\tilde{R}\Delta^n_+[-1]$$

and so $\Gamma NA[-1]$ is defined by the coequalizer

$$\bigoplus_{\theta:\mathbf{m}\to\mathbf{n}} A_n \otimes N\tilde{R}\Delta^m_+[-1] \rightrightarrows \bigoplus_{n\geq 0} A_n \otimes N\tilde{R}\Delta^n_+[-1] \to \Gamma NA[-1]$$

It follows that there is a natural short exact sequence

$$0 \to A \xrightarrow{d^0} CA \to \overline{W}A \to 0$$

where the "cone" CA is defined by the coequalizer

$$\bigoplus_{\theta:\mathbf{m}\to\mathbf{n}} A_n \otimes \Delta^{m+1}_* \rightrightarrows \bigoplus_{n\geq 0} A_n \otimes \Delta^{n+1}_* \to CA$$

The inclusion $d^0 : \Delta^n \to \Delta^{n+1}$ contracts to the vertex $0 \in \mathbf{n}+\mathbf{1}$, via the homotopy

$$h : \Delta^n_+ \wedge \Delta^1_* \to \Delta^{n+1}_*$$

(Δ^1 is pointed by 0) which is given by the picture

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[d] \\
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n+1
\end{tikzcd}$$

The homotopies h form a map of cosimplicial spaces, and hence determine a natural map

$$A \otimes \Delta^1_* \to CA,$$

which in turn induces a natural map

$$h : S^1 \otimes A \to \overline{W}A$$

This map is a natural equivalence (because $A \otimes \Delta^1_*$ and CA are both contractible). It is even a natural homotopy equivalence, since the comparison map

$$h : S^1 \otimes \tilde{\mathbb{Z}}\Delta^n_+ \to \overline{W}\tilde{\mathbb{Z}}\Delta^n_+$$

is a homotopy equivalence of cosimplicial simplicial abelian groups.

Write g for the natural homotopy inverse for f.

Every spectrum object $\sigma : S^1 \otimes A^n \to A^{n+1}$ in simplicial R-modules determines a "Kan" spectrum object

$$\overline{W}A^n \xrightarrow{g} S^1 \otimes A^n \xrightarrow{\sigma} A^{n+1}$$

and hence a spectrum object

$$\tilde{\sigma} : NA^n[-1] \cong N(\overline{W}A^n) \to NA^{n+1}$$

in chain complexes.

Let $\sigma_* : A^n \to \Omega A^{n+1}$ be the adjoint of σ. Corollary 4 says that the evaluation map

$$ev : \Omega A^{n+1} \otimes S^1 \to A^{n+1}$$

is a homology isomorphism above degree 0, and further that there is an induced equivalence

$$ev_*[1] : N\Omega A^{n+1} \to NA^{n+1}[1]$$

(as $\mathbb{Z}$-graded chain complexes), on account of the diagram

$$\begin{tikzcd}
N(S^1 \otimes \Omega A^{n+1}) \arrow[r, "Nev"] & NA^{n+1} \\
N(\overline{W}\Omega A^{n+1}) \arrow[u, "Ng"] \arrow[r, "\cong"'] & N(\Omega A^{n+1})[-1] \arrow[u, "ev_*"']
\end{tikzcd}$$

There is, finally, a natural commutative diagram of chain complex maps

$$\begin{tikzcd}
NA^n \arrow[r, "N\sigma_*"] \arrow[dr, "\sigma\tilde{[}1]"'] & N\Omega A^{n+1} \arrow[d, "\simeq"', "ev_*[1]"] \\
& NA^{n+1}[1]
\end{tikzcd}$$

Identify all chain complexes NA^n with $\mathbb{A}$-graded chain complexes, and let QNA be the colimit of the diagram

$$NA^0 \to NA^1[1] \to NA^2[2] \to \ldots$$

Then one can show

Proposition: The map $f : A \to B$ is a stable equivalence of spectrum objects in simplicial R-modules if and only if the induced map $f_* : QNA \to QNB$ is a quasi-isomorphism of $\mathbb{Z}$-graded chain complexes.

One can go further [3], to show that the stable model structure on $\mathbf{Spt}(R)$, together with the Dold-Kan correspondence N, Γ induces a model structure on the category $\mathbf{Ch}(R)$ of $\mathbb{Z}$-graded chain complexes of R-modules. The weak equivalences in $\mathbf{Ch}(R)$ are the quasi-isomorphisms, and the fibrations are the surjective homomorphisms of chain complexes. This correspondence further induces an equivalence of the stable homotopy category for $\mathbf{Spt}(R)$ with the full derived category $\mathrm{Ho}(\mathbf{Ch}(R))$ for chain complexes of R-modules.

References

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