Lecture 005 (October 27, 2005)

Grothendieck sites

What I shall do in this section is define the concept of Grothendieck site, and list some of my favourite examples.

A *small Grothendieck site* $\mathcal{C}$ is a small category is a small category, equipped with a Grothendieck topology which consists of a family of "covering sieves" aka, subfunctors

$$R \subset \hom(U), U \in \mathcal{C},$$

which satisfy a short list of axioms:

- If $R \subset \hom(\ ,U)$ is covering and $\phi : V \to U$ is a morphism of $\mathcal{C}$, then $\phi^{-1}(R) \subset \hom(\ ,V)$ is covering.
- Suppose that R is a covering sieve for U and that $S \subset \hom(\ ,U)$ is some other subfunctor. If $\pi^{-1}(S)$ is covering for all $\phi : V \to U$ in R, then S is covering.
- $\hom(\ ,U)$ is covering for all $U \in \mathcal{C}$.

Here is a very small group of examples:

1) Suppose that X is a topological space, and let $op|_X$ be the category of open subsets $U \subset X$ with their inclusions. A subfunctor $R \subset \hom(\ ,U)$ is covering if the list of inclusions

$$V \subset U$$

in R defines an open covering of U. Conversely, any open covering $U_\alpha \subset U$ defines a covering sieve, which consists of all open subsets $V \subset U$ which factor some U_α.

2) The defining Zariski topology for a scheme S defines a Grothendieck site $Zar|_S$ whose objects are the Zariski open subschemes of S.

3) Suppose that S is a scheme which is locally of finite type. The étale maps $\phi : U \to S$ (smooth of relative dimension 0, or flat and unramified) are the objects of a category $et|_S$, called the étale site for S (see [7], for example). The morphisms are commutative diagrams of scheme homomorphisms

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & {}_{\phi}\searrow \quad \swarrow_{\psi} & \\ & S & \end{array}$$

Here are some little things to know: the family of étale maps is closed under composition and base change, and includes all open immersions. Also, in any picture like the above, f is forced to be étale since ϕ and ψ are étale.

A family of étale maps $\phi_\alpha : V_\alpha \to U$ is covering if

$$\cup_\alpha \, \phi_\alpha(V_\alpha) = U.$$

The étale covering families generate the sieves for the étale topology on $et|_S$.

4) The Nisnevich topology [8], [3] on the category $et|_S$ is generated by those étale covering families $\phi_\alpha : V_\alpha \to U$ such that all (residue) fields $\mathrm{Sp}(K) \to U$ lift in the sense that there is a commutative diagram

$$\begin{array}{ccc} & \nearrow & V_\alpha \\ & & \downarrow{\scriptstyle \phi_\alpha} \\ \mathrm{Sp}(K) & \longrightarrow & U \end{array}$$

for some α.

5) All of the standard geometric topologies have big site versions. The big étale site $(\mathrm{Sch}\,|_S)_{et}$ for S is the category of morphisms $Y \to S$ which are locally of finite type, with a topology generated by all étale covering families of Y for S-schemes $Y \to S$. There are similarly defined big Zariski and Nisnevich and flat sites, denoted by $(\mathrm{Sch}\,|_S)_{Zar}$ and

$(\mathrm{Sch}\,|_S)_{Nis}$ respectively. These are not small sites, but they can be treated as such by assuming that there is some infinite cardinal α such that $|Y| < \alpha$ for all $Y \to S$ in $Sch|_S$. This is an approximation technique, in the sense that homotopy theoretic calculations do not vary with changes in α so long as α is big enough [3].

6) There are various flat topologies. Write $(\mathrm{Sch}\,|_S)_{fl}$ for the category of all S-schemes. The covering sieves for the flat topology are generated by sets $f_\alpha : Z_\alpha \to Y$ of flat morphisms which are faithfully flat in the sense that

$$\cup_\alpha f_\alpha(Z_\alpha) = Y.$$

Note that $Z \to Y$ flat means that it is determined locally (Zariski topology) by flat ring homomorphisms $A \to B$, ie. $N \mapsto B \otimes_A N$ is exact.

7) One often cuts the étale site $et|_S$ to the finite étale site, whose morphisms consist of the finite étale maps $U \to S$. If one does this for $S = \mathrm{Sp}(K)$ for some field K then the finite étale maps $U \to \mathrm{Sp}(K)$ have the form

$$U = \mathrm{Sp}(L_1) \sqcup \cdots \sqcup \mathrm{Sp}(L_k) \to \mathrm{Sp}(K)$$

where each L_i/K is a finite separable field extension. The functor $U \mapsto \hom(\mathrm{Sp}(K_{sep}), U)$ defines an isomorphism of the finite étale site for K with the category of finite discrete "modules" for the absolute Galois group $Gal(K_{sep}/K)$ of K.

8) Suppose that $G = \{G_i\}$ is a profinite group. A discrete G-module is a G-set X with G-action $G \times X \to X$ which is induced from an action $G_i \times X \to X$ by one of the finite quotients. The category $G-\mathbf{Set}_{df}$ of finite discrete G-sets with G-equivariant morphisms is a Grothendieck site: the covering families $X_\alpha \to X$ are the set-theoretic coverings [3].

9) The h-topology [10] on $Sch|_S$ is generated by finite families $p_i : U_i \to Y$ such that each p_i is of finite type and the sums

$$\sqcup_i : U_i \to Y$$

are universal topological epimorphisms, ie.such maps must be epimorphisms on the level of underlying topological spaces, and they must be stable under base change. Any surjective proper morphism of finite type is covering for the h-topology. Blowups are covering maps for this topology.

10) The qfh-topology [10] on $Sch|_S$ is generated by all finite families $p_i : U_i \to Y$ which are h-coverings, and such that all p_i are quasi-finite (ie.

of finite type with finite fibres). Every finite surjective morphism $U \to Y$ generates a covering sieve for the qfh-topology. The abelian sheaves for this topology are presheaves with transfers, as defined by Voevodsky.

11) Suppose that I is an infinite set, and consider the poset $\mathcal{P}(I)$ of subsets of I. There is a topology on $\mathcal{P}(I)$ which is generated by the finite covers $U_i \subset V$ of a subset V. It is equivalent in a rather strong sense (ie. has the same sheaves as) the Zariski topology on $\mathrm{Sp}(\Pi_{j\in I}\ F_j)$, where F_j is a list of fields indexed by the set I. This is the setting for the theory of ultrafilters and ultraproducts [4].

12) Here is an important technical example. Suppose that $\mathcal{C}$ is a site and take $U \in \mathcal{C}$. Then the slice category $\mathcal{C}/U$ of morphisms $V \to U$ in $\mathcal{C}$ has a Grothendieck topology, for which a family of maps

$$\begin{array}{ccc} W & \xrightarrow{f} & V \\ & \searrow \quad \swarrow_{\phi} & \\ & U & \end{array}$$

in $\mathcal{C}/U$ defines a covering sieve for ϕ if and only if the list of maps $f : W \to V$ defines a covering sieve for V in $\mathcal{C}$. An obvious example is

$op|_X/U \cong op|_U$ for $U \subset X$ open in a topological space X. Also, if $S' \to S$ is an S-scheme, then $(\mathrm{Sch}\,|)_S/S' \cong (\mathrm{Sch}\,|_{S'})$, and if $(\mathrm{Sch}\,|_S)$ carries one of the geometric topologies, then the induced topology on $(\mathrm{Sch}\,|_S)/S'$ matches the corresponding topology on $(\mathrm{Sch}\,|_{S'})$ through this isomorphism.

Presheaves and sheaves

This will be a very quick tour. My favourite reference for basic sheaf theory is still [9].

A *presheaf* (of sets) on a site $\mathcal{C}$ is a set-valued contravariant functor $F : \mathcal{C}^{op} \to \mathbf{Set}$. A presheaf on $\mathcal{C}$ in anything else is also a contravariant functor defined on $\mathcal{C}$ and taking values in the something else.

Example: a *simplicial presheaf* X is a contravariant functor $X : \mathcal{C}^{op} \to s\mathbf{Set}$, or alternatively a simplicial object in presheaves of sets. The morphisms in all presheaf categories are natural transformations of functors. In particular, $\mathrm{Pre}(\mathcal{C})$ will denote the category of presheaves defined on $\mathcal{C}$ and $s\,\mathrm{Pre}(\mathcal{C})$ is the category of simplicial presheaves.

Any (covering) sieve $R \subset \hom(\ ,U)$ determines a map

$$F(U) \to \varinjlim_{\phi:V\to U\in R} F(V)$$

We say that F is a *sheaf* if this map is an isomorphism (aka. bijection) for all covering sieves of all objects $U \in \mathcal{C}$.

Equivalently, F is a sheaf if any R-compatible family $x_\phi \in F(V)$, $\phi : V \to U \in R$ determines a unique element $x \in F(U)$ such that $\phi^*(x) = x_\phi$ for all $\phi \in R$.

Any inclusion $S \subset R$ of covering sieves determines a restriction function on the level of compatible families, and the covering sieves of $U \in C$ form a filtered system. Write $G(U)_R$ for the set of R-compatible families relative to U for a presheaf G. Then there is a new presheaf LG which is defined by the filtered colimit

$$LG(U) = \varinjlim_{R} G(U)_R$$

where the colimit is indexed over the covering sieves $R \subset \hom(\ ,U)$. There is a canonical presheaf map

$$G(U) \to LG(U)$$

which arises from the fact that $\hom(\ ,U)$ is covering. Then L^2G is a sheaf and the map $\eta : G \to$

L^2G defined by the composite

$$G \to LG \to L^2G$$

is the associated sheaf map.

One often writes $\tilde{G} = L^2G$. The associated sheaf functor $G \mapsto \tilde{G}$ is left adjoint to the inclusion of the full subcategory $\mathrm{Shv}(\mathcal{C})$ of sheaves on $\mathcal{C}$ in the presheaf category $\mathrm{Pre}(\mathcal{C})$.

The associated sheaf functor is defined by filtered colimits, so it preserves finite limits, and therefore preserves algebraic structures. For example, if A is a presheaf of abelian groups, then $\tilde{A}$ is a sheaf of abelian groups.

The associated sheaf functor also respects diagrams in presheaves, so that there is an associated sheaf functor $X \mapsto \tilde{X}$ from simplicial presheaves to simplicial sheaves. Write $s\,\mathrm{Shv}(\mathcal{C})$ for the category of simplicial sheaves on $\mathcal{C}$: it is the full subcategory of $s\,\mathrm{Pre}(\mathcal{C})$ of objects X for which all presheaves X_n of simplices are sheaves. Of course, the associated sheaf functor is left adjoint to the inclusion of simplicial sheaves in simplicial presheaves: write $\eta : X \to \tilde{X}$ for the canonical map.

Simplicial presheaves, simplicial sheaves

Every simplicial set X has naturally defined homotopy groups $\pi_n(X,x)$, $n \geq 1$ and $x \in X_0$. One either defines $\pi_n(X,x) = \pi_n(F(X), j(x))$ where $j : X \to FX$ is a natural fibrant model, or sets $\pi_n(X,x) = \pi_n(|X|,x)$ to be the homotopy groups of the topological realization — the two approaches produce the same list of groups up to natural isomorphism ("Milnor theorem").

The path components $\pi_0 X$ of a simplicial set are defined more directly by the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X$$

arising from identifying the two faces $d_0(\omega), d_1(\omega)$ of a 1-simplex ω.

A map of simplicial sets $f : X \to Y$ is a weak equivalence if

- $\pi_0 X \to \pi_0 Y$ is a bijection, and
- all maps $\pi_n(X,x) \to \pi_n(Y, f(x))$ are isomorphisms.

Collecting together base points determines a function

$$\pi_n X = \bigsqcup_{x \in X_0} \pi_n(X,x) \to \bigsqcup_{x \in X_0} * = X_0$$

which gives $\pi_n X$ the structure of a group object over X_0. Further, any map $f : X \to Y$ determines commutative diagrams

$$\begin{array}{ccc} \pi_n X & \xrightarrow{f_*} & \pi_n Y \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow[f]{} & Y_0 \end{array}$$

A weak equivalence $f : X \to Y$ of simplicial sets is a map for which the induced map on path components is a bijection and all diagrams above are pullbacks.

Simplicial presheaves X have presheaves $\pi_n X \to X_0$ of homotopy group objects and presheaves $\pi_0 X$ of path components. Write $\tilde{\pi}_n X$ for the respective associated sheaves. A map $f : X \to Y$ is a (local) weak equivalence if

- the induced map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ of sheaves of path components is an isomorphism, and
- all diagrams of sheaves

$$\begin{array}{ccc} \tilde{\pi}_n X & \xrightarrow{f_*} & \tilde{\pi}_n Y \\ \downarrow & & \downarrow \\ \tilde{X}_0 & \xrightarrow[f_*]{} & \tilde{Y}_0 \end{array}$$

  are pullbacks.

The second condition is equivalent to the assertion that all induced maps

$$\tilde{\pi}_n(X|_U, x) \to \tilde{\pi}_n(Y|_U, f(x))$$

are isomorphisms of sheaves, for all $x \in X(U)_0, U \in \mathcal{C}$. Here $X|_U$ is the composite

$$(\mathcal{C}/U)^{op} \to \mathcal{C}^{op} \xrightarrow{X} s\mathbf{Set}.$$

In the presence of an adequate family of stalks, $f : X \to Y$ is a weak equivalence if and only if it induces a weak equivalence $f_x : X_x \to Y_x$ of simplicial sets in all stalks.

Example: The associated sheaf map $\eta : X \to \tilde{X}$ is a weak equivalence. In effect, it has the local right lifting property with respect to all inclusions $\partial\Delta^n \to \Delta^n$, and one can show that any map $p : X \to Y$ which has such a local lifting property must be a weak equivalence. For the latter, if p has the local RLP, then $p_* : \operatorname{Ex}^\infty X \to \operatorname{Ex}^\infty Y$ has the same local RLP, and so all induced maps of presheaves

$$\pi_n(\operatorname{Ex}^\infty X|_U, x) \to \pi_n(\operatorname{Ex}^\infty Y|_U, f(x))$$

and

$$\pi_0 \operatorname{Ex}^\infty X \to \pi_0 \operatorname{Ex}^\infty Y$$

are local epimorphisms and local monomorphisms.

A *cofibration* of simplicial presheaves is a monomorphism, and a *global fibration* (or an injective fibration) is a map which has the RLP with respect to all maps which are cofibrations and local weak equivalences.

The function complex $\mathbf{hom}(X,Y)$ for simplicial presheaves X,Y is the simplicial set with

$$\hom(X,Y)_n = \{X \times \Delta^n \to Y\}.$$

Here Δ^n is the constant simplicial presheaf associated to the simplicial set Δ^n: one sometimes sees the constant simplicial presheaf associated to a simplicial set K denoted by $\Gamma^* K$.

Theorem 1: [1], [5]. With these definitions, the category $s\,\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves on an arbitrary small Grothendieck site has a proper closed simplicial model structure. This model structure is cofibrantly generated.

Suppose that α is an infinite cardinal such that $\mathrm{Mor}(\mathcal{C})| < \alpha$. Then the set of α-bounded trivial cofibrations forms a generating family for the class of trivial cofibrations. One proves this by establishing a bounded cofibration condition with respect to the cardinal α.

The U-sections functor $X \mapsto X(U)$ has a left adjoint

$$Y \mapsto Y \otimes U = Y \times \hom(\ ,U).$$

The objects $\Delta^n \otimes U$ and their quotients are α-bounded, so that map $p : Z \to W$ of simplicial presheaves is a global fibration and a local weak equivalence if and only if it has the RLP with respect to all α-bounded cofibrations $A \subset B$. In particular, if p is a trivial global fibration then all maps $p : X(U) \to Y(U)$ in sections are trivial fibrations of simplicial sets. The α-bounded cofibrations therefore form a generating set for the class of all cofibrations.

The same definitions work for simplicial sheaves: a map of simplicial sheaves is a weak equivalence if it is a weak equivalence of simplicial presheaves, and a cofibration of simplcial presheaves is just a monomorphism. A map $p : X \to Y$ of simplicial presheaves is said to be a global fibration if it has the RLP with respect to all trivial cofibrations. Note that the associated sheaf functor preserves cofibrations and trivial cofibrations, while the inclusion of sheaves in presheaves preserves fibrations and trivial fibrations. The definition of the function space objects is the same for simplicial sheaves as for simplicial presheaves.

The model structure for simplicial sheaves was first displayed by Joyal in a letter to Grothendieck [6]. This is the model structure of the next result, but it is an easy consequence of the theorem for simplicial presheaves.

Theorem 2:

1) With these definitions, the category $s\,\mathrm{Shv}(\mathcal{C})$ of simplicial presheaves on an arbitrary small Grothendieck site has a proper closed simplicial model structure. This model structure is cofibrantly generated.

2) The model structures for simplicial sheaves and simplicial sheaves on $\mathcal{C}$ are Quillen equivalent. In particular, they generate equivalent homotopy categories.

Presheaves of spectra

Write $s\,\mathrm{Pre}(\mathcal{C})_*$ for the category of pointed simplicial presheaves $* \to X$, with base point preserving maps

$$\begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

Recall that this category has a proper closed simplicial model structre for which a map f as above is a local weak equivalence (resp. global fibration, cofibration) if and only if the underlying map $f : X \to Y$ of simplicial presheaves is a local weak equivalence (resp. global fibration, cofibration). The function complex $\mathbf{hom}(X, Y)$ is defined in simplicial degree n by

$$\mathbf{hom}(X, Y)_n = \{X \wedge \Delta^n_+ \to Y\}.$$

Warning: A map f of pointed simplicial presheaves is a local weak equivalence if and only if it induces an isomorphism on sheaves of path components, and induces isomorphisms

$$\tilde{\pi}_n(X|_U, x) \to \tilde{\pi}_n(Y|_U, f(x))$$

for all $U \in \mathcal{C}$ and all (local) choices of base points $x \in X(U)$. It is **not** sufficient to check that the obvious maps $\tilde{\pi}_n(X,*) \to \tilde{\pi}_n(Y,*)$ (based at the canonical point $* \to X$) are isomorphisms.

A *presheaf of spectra* X consists of pointed simplicial presheaves X^n, $n \geq 0$ together with bonding maps

$$\sigma : S^1 \wedge X^n \to X^{n+1},\ n \geq 0.$$

Here, S^1 is identified with the constant pointed simplicial presheaf $U \mapsto S^1$. A map $f : X \to Y$ of presheaves of spectra consists of pointed simplicial presheaf maps $f : X^n \to Y^n$, $n \geq 0$, which respect structure in the obvious sense. Write $\mathrm{Spt}(\mathcal{C})$ for the category of presheaves of spectra on $\mathcal{C}$.

Generally speaking, any functorial construction for spectra also applies to presheaves of spectra, and we will see that, in general outline, the stable homotopy theory of presheaves of spectra is an analogue of ordinary stable homotopy theory.

Note that the ordinary category of spectra is the category of presheaves of spectra on the one-object, one-morphism category, so that general results about presheaves of spectra apply to spectra. Also, if I is a small category, then the category of I-diagrams $X : I \to \mathbf{Spt}$ is a category of presheaves of spectra on I^{op}, where I^{op} has the trivial topology: this means that results about presheaves of spectra apply to all categories of small diagrams of spectra.

Some examples:

1) Any spectrum A (of pointed simplicial sets) determines an associated constant presheaf of spectra Γ^*A on $\mathcal{C}$, where

$$\Gamma^*A(U) = A,$$

and every morphism $\phi : V \to U$ induces the identity morphism $A \to A$. We shall often write $A = \Gamma^*A$ when there is no possibility of confusion. The sphere spectrum $\mathbf{S}$ in $\mathrm{Spt}(\mathcal{C})$ is the constant object $\Gamma^*\mathbf{S}$ associated to the ordinary sphere spectrum.

2) The functor $A \mapsto \Gamma^*A$ is left adjoint to the global sections functor $\Gamma_* : \mathrm{Spt}(\mathcal{C}) \to \mathbf{Spt}$, where

$$\Gamma_*X = \varprojlim_{U \in \mathcal{C}} X(U).$$

3) If A is a sheaf (or presheaf) of abelian groups, the Eilenberg-Mac Lane presheaf of spectra $H(A)$ is the presheaf of spectra underlying the suspension object

$$A, S^1 \otimes A, S^2 \otimes A, \dots$$

in the category of presheaves of spectra in simplicial abelian groups. Note that , as a simplicial presheaf, $S^n \otimes A = K(A,n)$, and if $j : K(A,n) \to FK(A,n)$ is a globally fibrant model of $K(A,n)$ then there are natural isomorphisms

$$\pi_j \Gamma_* FK(A,n) = \begin{cases} H^{n-j}(\mathcal{C}, A) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

We'll see later that these isomorphisms assemble to give an identification of the stable homotopy groups of global sections of a (stably) fibrant model for $H(A)$ with the cohomology of $\mathcal{C}$ with coefficients in A. In other words all sheaf cohomology groups are stable homotopy groups.

4) Every chain complex (bounded or unbounded) D determines a presheaf of spectra $H(D)$, which computes the hypercohomology of $\mathcal{C}$ with coefficients in D, via computing stable homotopy groups of global sections of a stably fibrant model. In fact, we shall see that spectrum objects in presheaves of simplicial R-modules give a model for the full derived category.

5) There is a presheaf of spectra K on $\mathrm{Sch}|_S$, called the algebraic K-theory spectrum, such that $\pi_j K(U)$ is the j^{th} algebraic K-group $K_j(U)$. The

construction of this object is still rather unsatisfactory, after all these years: in its most general form, it starts with a pseudo-functor on S-schemes taking values in pseudo-simplicial symmetric monoidal categories [2], [3].

Say that a map $f : X \to Y$ of presheaves of spectra is a strict weak equivalence (respectively strict fibration) if all maps $f : X^n \to Y^n$ are local equivalences (respectively global fibrations).

A cofibration $i : A \to B$ of $\mathrm{Spt}(\mathcal{C})$ is a map for which

- $i : A^0 \to B^0$ is a cofibration, and
- all maps

  $$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \to B^{n+1}$$

  are cofibrations.

The function complex $\mathbf{hom}(X, Y)$ for presheaves of spectra X, Y is defined in simplicial degree n in the usual way:

$$\mathbf{hom}(X, Y)_n = \{X \wedge \Delta^n_+ \to Y\}.$$

Proposition: With these definitions, the category $s\,\mathrm{Pre}(\mathcal{C})$ satisfies the axioms for a proper closed simplicial model category.

Proof: Exercise.

Of course, the Proposition is just an opening act.

A presheaf of spectra X has presheaves $\pi_n^s X$ of stable homotopy groups, defined by

$$U \mapsto \pi_n^s X(U).$$

Write $\tilde{\pi}_n^s X$ for the sheaf associated to the presheaf $\pi_n^s X$. The sheaves $\tilde{\pi}_n^s X$, $n \in \mathbb{Z}$, are the sheaves of stable homotopy groups of X.

Say that a map $f : X \to Y$ of presheaves of spectra is a *stable equivalence* if it induces isomorphisms

$$\tilde{\pi}_n^s X \xrightarrow{\cong} \tilde{\pi}_n^s Y$$

for all $n \in \mathbb{Z}$.

Observe that every strict equivalence is a stable equivalence.

Say that $p : Z \to W$ is a *stable fibration* if it has the RLP with respect to all maps which are cofibrations and stable equivalences.

Theorem 3: With the definitions of cofibration, stable equivalence and stable cofibration given above, the category $\mathrm{Spt}(\mathcal{C})$ satisfies the axioms for a proper closed simplicial model category.

Lemma 1: A map $p : X \to Y$ is a stable fibration and a stable equivalence if and only if all maps $p : X^n \to Y^n$ are trivial global fibrations of simplicial presheaves.

Proof: If all $p : X^n \to Y^n$ are trivial global fibrations, then p has the RLP with respect to all cofibrations, and is therefore a stable fibration. p is also a stable equivalence because it is a strict equivalence.

Suppose that p is a stable fibration and a stable equivalence. Then p has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & {}_{p}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

where j is a cofibration and q is a trivial strict fibration. But then j is stable equivalence as well as a cofibration, so that the lifting exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ W & \xrightarrow[q]{} & Y \end{array}$$

so that p is a retract of q and is therefore a trivial strict fibration. $\square$

Choose an infinite cardinal α such that $|\operatorname{Mor}(\mathcal{C})| <$

α. Say that a presheaf of spectra A is α-bounded if all pointed simplicial sets $A^n(U)$, $n \geq 0$, $U \in \mathcal{C}$ are α-bounded. Observe that every presheaf of spectra X is a union of its α-bounded subobjects.

Lemma 2: Suppose given a cofibration $i : X \to Y$ which is a stable equivalence, and suppose that $A \subset Y$ is an α-bounded subobject. Then there is an α-bounded subobject $B \subset Y$ such that $A \subset B$ and the cofibration $B \cap X \to B$ is a stable equivalence.

Proof: Note that $\tilde{\pi}_n^s Z = 0$ if and only if for all $x \in \pi_n^s Z(U)$ there is a covering sieve $\phi : V \to U$ such that $\phi^*(x) = 0$ for all ϕ in the covering.

The sheaves $\tilde{\pi}_n^s(Y/X)$ are trivial (sheafify the natural long exact sequence for a cofibration), and

$$\tilde{\pi}_n^s(Y/X) = \varinjlim_{C} \tilde{\pi}_n^s(C/C \cap X)$$

where C varies over all α-bounded subobjects of Y. The list of elements of all $x \in \pi_n^s(A/A \cap X)(U)$ is α-bounded. For each such x there is an α-bounded subobject $B_x \subset X$ such that

$$x \mapsto 0 \in \tilde{\pi}_n(B_x/B_x \cap X).$$

It follows that there is an α-bounded subobject

$$B_1 = A \cup (\cup_x B_x)$$

such that all $x \mapsto 0 \in \tilde{\pi}_n(B_1/B_1 \cap X)$.

Write $A = B_0$. Then inductively, we can produce an ascending sequence

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of α-bounded subobjects of Y such that all presheaf homomorphisms

$$\pi_n^s(B_i/B_i \cap X) \to \tilde{\pi}_n^s(B_{i+1}/B_{i+1} \cap X)$$

are trivial. Set $B = \cup_i B_i$. Then B is α-bounded and all sheaves $\tilde{\pi}_n(B/B \cap X)$ are trivial. $\square$

Lemma 3: The class of stably trivial cofibrations has a generating set, namely the set I of all α-bounded stably trivial cofibrations.

Proof: The class of cofibrations of $\mathrm{Spt}(\mathcal{C})$ is generated by the set J of cofibrations

$$\Sigma^\infty A[-n] \to \Sigma^\infty B[-n]$$

which are induced by α-bounded cofibrations $A \to B$ of pointed simplicial presheaves.

Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

where j is a cofibration, B is α-bounded, and f is a stable equivalence. Then f has a factorization $f = q \cdot i$ where i is a cofibration and q is a strictly trivial fibration, hence a stable equivalence, and the lifting exists in the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \Big| & & \big\downarrow{\scriptstyle i} \\ {\scriptstyle j}\Big| & & Z \\ \Big\downarrow & \nearrow & \big\downarrow{\scriptstyle q} \\ B & \longrightarrow & Y \end{array}$$

The cofibration $i : X \to Z$ is a stable equivalence, and the image $\theta(B) \subset Z$ is α-bounded, so there is an α-bounded subobject $D \subset Z$ with $\theta(B) \subset D$ such that $D \cap X \to D$ is a stable equivalence. It follows that there is a factorization

$$\begin{array}{ccccc} A & \longrightarrow & D \cap X & \longrightarrow & X \\ {\scriptstyle j}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

of the original diagram through an α-bounded stably trivial cofibration.

Now suppose that $i : C \to D$ is a stably trivial cofibration. Then i has a factorization

$$\begin{array}{ccc} C & \xrightarrow{j} & E \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle p} \\ & & D \end{array}$$

where j is a cofibration in the saturation of the set of α-bounded stably trivial cofibrations and p has the RLP with respect to all α-bounded stably trivial cofibrations. The map j is a stable equivalence since the class of stably trivial cofibrations is closed under pushout (by a long exact sequence argument) and composition. It follows that p is a stable equivalence, and therefore has the RLP with respect to all α-bounded cofibrations and hence with respect to all cofibrations. It follows that i is a retract of the map j. $\square$

Proof of Theorem 3:

According to Lemma 3, a map is a stable fibration if and only if it has the RLP with respect to all α-bounded stably trivial cofibrations. A (possibly transfinite) small object argument therefore implies that every map $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where j is a stably trivial cofibration and p is a stable fibration.

Similarly, Lemma 1 implies that a map is a stable fibration and a stable weak equivalence if and only if it is a strict fibration and a strict weak equivalence. There is a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

for any map $f : X \to Y$ where i is a cofibration and p is a strict fibration and a strict equivalence — this gives the corresponding factorization for the stable structure.

We have therefore proved **CM5**. The non-trivial part of **CM4** is a consequence of Lemma 1. The remaining closed model axioms are trivial.

The closed simplicial model structure is proved with the same argument as was used for ordinary spectra in Lecture 002: one shows by induction on n that if $i : A \to B$ is a stably trivial cofibration then all maps

$$i \wedge \partial\Delta^n_+ : A \wedge \partial\Delta^n_+ \to B \wedge \partial\Delta^n_+$$

are stable equivalences.

Left and right properness are consequences of long exact sequences in stable homotopy groups. □

Since the stable model structure on $\mathrm{Spt}(\mathcal{C})$ is cofibrantly generated there is a functorial stably fibrant model construction.

$$j : X \to LX$$

Note that if X and Y are stably fibrant, any stable equivalence $f : X \to Y$ must be a strict equivalence. This is a consequence of Lemma 1. It follows that a map $f : X \to Y$ of arbitrary presheaves of spectra is a stable equivalence if and only if the induced map $LX \to LY$ is a strict equivalence. Thus a map $f : X \to Y$ is an L-equivalence if and only if it is a stable equivalence.

We also have the following:

A4 The functor L preserves strict equivalence.

A5 The maps $j_{LX}, Lj_X : LX \to LLX$ are strict weak equivalences.

A6′ Stable equivalences are preserved by pullback along stable fibrations.

We therefore have the following formal consequence:

Corollary: A map $p : X \to Y$ of $\mathrm{Spt}(\mathcal{C})$ is a stable fibration if and only it is a strict fibration and the diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & LX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Lp} \\ Y & \xrightarrow[j]{} & LY \end{array}$$

is strictly homotopy cartesian.

Lemma: Suppose that $p : X \to Y$ is a stable fibration. Then the diagrams

$$\begin{array}{ccc} X^n & \xrightarrow{\sigma_*} & \Omega X^{n+1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Omega p} \\ Y^n & \xrightarrow[\sigma_*]{} & \Omega Y^{n+1} \end{array}$$

are strictly homotopy cartesian.

Proof: Since p is a stable fibration, any stably trivial cofibration $\theta : A \to B$ induces a homotopy cartesian diagram

$$\begin{array}{ccc} \mathbf{hom}(B, X) & \xrightarrow{p_*} & \mathbf{hom}(B, Y) \\ {\scriptstyle \theta^*}\downarrow & & \downarrow{\scriptstyle \theta^*} \\ \mathbf{hom}(A, X) & \xrightarrow[p_*]{} & \mathbf{hom}(A, Y) \end{array}$$

If $\theta : A \to B$ is a stable equivalence between cofibrant objects, then the diagram above is still homotopy cartesian. In effect, θ has a factorization $\theta = \pi \cdot j$ where j is a trivial cofibration and $\pi \cdot i = 1$ for some trivial cofibration i. It follows that the diagram above is a retract of a homotopy cartesian diagram, and is therefore homotopy cartesian.

The diagrams of the statement of the Lemma arise from the stable equivalences

$$\Sigma^\infty S^1[-1-n] \to \mathbf{S}[-n]. \quad \Box$$

Corollary: If X is stably fibrant, then all X^n are globally fibrant and all adjoint bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$ are local weak equivalences.

Proposition: A presheaf of spectra X is stably fibrant if and only if all X^n are globally fibrant and all adjoint bonding maps $\sigma_* : X^n \to \Omega X^{n+1}$ are local weak equivalences.

Proof: Suppose that all X^n are globally fibrant and all $\sigma_* : X^n \to \Omega X^{n+1}$ are local weak equivalences. Then the simplicial presheaves X^n and ΩX^{n+1} are globally fibrant and cofibrant, so that all σ_* are homotopy equivalences. It follows that all spaces $X^n(U)$ are fibrant and that all maps $\sigma_* : X^n(U) \to \Omega X^{n+1}(U)$ are weak equivalences of pointed simplicial sets. It follows that all maps

$$\pi_k X^n(U) \to \pi^s_{k-n} X(U)$$

are isomorphisms.

Suppose that $j : X \to LX$ is a stably fibrant model for X. Then all spaces $LX^n(U)$ are fibrant and all maps $LX^n(U) \to \Omega LX^{n+1}(U)$ are weak equivalences, and so all maps

$$\pi_k LX^n(U) \to \pi^s_{k-n} LX(U)$$

are isomorphisms. The map j induces an isomorphism in all sheaves of stable homotopy groups, and so the maps $j : X^n \to LX^n$ induce isomorphisms

$$\tilde{\pi}_k X^n \to \tilde{\pi}_k LX^n$$

of sheaves of homotopy groups for $k \geq 0$. X and LX are presheaves of infinite loop spaces, and so the maps $X^n \to LX^n$ are local weak equivalences of simplicial presheaves. In particular, $j : X \to LX$ is a strict weak equivalence.

Now consider the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & \nearrow & \\ B & & \end{array}$$

where i is a stably trivial cofibration. Then the induced map $i_* : LA \to LB$ is a strict equivalence of stably fibrant objects, by assumption.

We've seen the following argument before, in the development of the stable model structure for spectra in Lemma 2 of Lecture 002:

Take a factorization

$$\begin{array}{ccc} LA & \xrightarrow{L\alpha} & LX \\ & {\scriptstyle j'}\searrow \quad \nearrow{\scriptstyle p} & \\ & Z & \end{array}$$

where j' is a cofibration and a strict weak equivalence and p is a strict fibration. Then LB is strictly fibrant, so there is a map $\zeta : Z \to LB$ such that the diagram

$$\begin{array}{ccc} LA & \xrightarrow{j'} & Z \\ {\scriptstyle Li}\downarrow & \swarrow{\scriptstyle \zeta} & \\ LB & & \end{array}$$

commutes. The map ζ is therefore a strict equivalence. Form the pullback

$$\begin{array}{ccc} Z \times_{LX} X & \xrightarrow{p_*} & X \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j} \\ Z & \xrightarrow[p]{} & LX \end{array}$$

and observe that the map j_* is a strict weak equivalence since j is a strict weak equivalence and p is a strict fibration. Then there is a diagram

$$\begin{array}{ccccc} A & \longrightarrow & Z \times_{LX} X & \xrightarrow{p_*} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \zeta j_*} & & \\ B & \xrightarrow[j]{} & FB & & \end{array}$$

in which the top composite is the map $\alpha : A \to X$. The vertical map ζj_* is a strict weak equivalence and X is strictly fibrant, so the lifting problem can be solved. □

Corollary: The presheaf of spectra

$$QX = F\Omega^\infty FX$$

is stably fibrant, for any presheaf of spectra X.

The natural map $\eta : X \to QX$ defined by the composite

$$X \to FX \to \Omega^\infty FX \to F\Omega^\infty FX$$

is plainly a stable equivalence, so that $\eta : X \to QX$ is a natural stably fibrant model.

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