

**Lecture 007** (February 7, 2007)

**Basic assumptions, ver. 2.0**

Recall the set of basic assumptions for a set  $S$  of cofibrations from Lecture 006:

$S$  is a set of cofibrations such that:

- 1)  $A$  is cofibrant for all  $i : A \rightarrow B$  in  $S$ .
- 2)  $S$  includes the set  $J$  of maps

$$\Sigma_T^\infty C[-n] \rightarrow \Sigma_T^\infty D[-n], \quad n \geq 0,$$

which are induced by the  $\alpha$ -bounded trivial cofibrations  $C \rightarrow D$  of pointed simplicial presheaves.

- 3)  $S$  includes all cofibrations

$$(A \wedge \Delta_+^m) \cup (B \wedge \partial \Delta_+^m) \rightarrow B \wedge \Delta_+^m, \quad m \geq 0,$$

for  $A \rightarrow B$  in  $S$ .

We shall now replace 3) by the following more inclusive assertion:

- 3') If  $i : A \rightarrow B$  is in  $S$  then all cofibrations

$$(A \wedge D) \cup (B \wedge C) \rightarrow B \wedge D$$

induced by  $i$  and all  $\alpha$ -bounded cofibrations  $C \rightarrow D$  of pointed simplicial presheaves are in  $S$ .

Note that the  $T$ -spectrum  $A \wedge C$  is cofibrant if  $A$  is cofibrant, so that the objects

$$(A \wedge D) \cup (B \wedge C)$$

appearing in 3') are all cofibrant  $T$ -spectra.

If  $A$  and  $X$  are  $T$ -spectra, then  $\mathbf{Hom}(A, X)$  is the pointed simplicial presheaf defined by

$$\mathbf{Hom}(A, X)(U) = \mathbf{hom}(A|_U, X|_U)$$

for  $U \in \mathcal{C}$ . The functor  $X \mapsto \mathbf{Hom}(A, X)$  is right adjoint to the functor

$$s\text{Pre}_*(\mathcal{C}) \rightarrow \text{Spt}_T(\mathcal{C})$$

which defined for pointed simplicial presheaves  $K$  by  $K \mapsto A \wedge K$

**Lemma 1:** Suppose that  $J \subset I \subset S$  and that  $I$  generates  $S$  in the sense that  $S$  is the smallest subset of itself which contains  $I$  and satisfies 3'). Then a map  $p : X \rightarrow Y$  is injective if and only if  $p$  is a strictl fibration and all maps

$$(i^*, p_*) : \mathbf{Hom}(B, X) \rightarrow \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A, Y)} \mathbf{Hom}(B, Y)$$

are trivial fibrations of simplicial presheaves for all  $i : A \rightarrow B$  in  $I$ .

**Proof:** Suppose that  $(i^*, p_*)$  is a trivial fibration for all  $i \in I$ . Then  $p$  has the right lifting property

with respect to all cofibrations

$$(A \wedge F) \cup (B \wedge E) \rightarrow B \wedge F$$

which are induced by cofibrations  $E \rightarrow F$  of pointed simplicial presheaves, for all  $i : A \rightarrow B$  in  $I$ , by 3').

In particular,  $p$  has the RLP with respect to all cofibrations induced by the map  $i$  and the inclusions

$$(D \wedge E) \cup (C \wedge F) \rightarrow D \wedge F$$

induced by all cofibrations  $E \rightarrow F$  and  $C \rightarrow D$  of pointed simplicial presheaves.  $\square$

## Fibrations

Now suppose that  $\mathcal{S}$  is generated over  $\mathcal{J}$  by cofibrant replacements of the maps

$$\Sigma_T^\infty T[-1 - n] \rightarrow S_T[-n].$$

The resulting model structure on  $\text{Spt}_T(\mathcal{C})$  will be called the stable model structure, and the  $L$ -equivalences and  $L$ -fibrations will be called stable equivalences and stable fibrations, respectively.

In this case, we know from formal nonsense that the stably fibrant objects are precisely the injective objects: it follows from [Lemma 1](#) that an object  $X$  is stably fibrant if and only if  $X$  is strictly fibrant and all maps  $\sigma_* : X^n \rightarrow \Omega_T X^{n+1}$  are weak equivalences of pointed simplicial presheaves.

There is a similar criterion for injective fibrations: a map  $p : X \rightarrow Y$  is an injective fibration if and only if  $p$  is a strict fibration and all diagrams of pointed simplicial presheaf maps

$$\begin{array}{ccc} X^n & \xrightarrow{\sigma_*} & \Omega_T X^{n+1} \\ \downarrow & & \downarrow \\ Y^n & \xrightarrow{\sigma_*} & \Omega_T Y^{n+1} \end{array}$$

are homotopy cartesian.

In effect, [Lemma 1](#) says that  $p$  is an injective fibration if and only if  $p$  induces trivial fibrations of simplicial presheaves

$$\mathbf{Hom}(D, X) \rightarrow \mathbf{Hom}(C, X) \times_{\mathbf{Hom}(C, Y)} \mathbf{Hom}(C, X)$$

for all generators  $C \rightarrow D$  of the set  $S$ . These generators include all cofibrations

$$\Sigma_T^\infty A[-n] \rightarrow \Sigma_T^\infty B[-n]$$

arising from all  $\alpha$ -bounded trivial cofibrations  $A \rightarrow B$  of pointed simplicial sets and  $n \geq 0$ , so that  $p : X \rightarrow Y$  must be a strict fibration. The other members of the generating set for  $S$  are cofibrant replacements for the maps

$$\Sigma_T^\infty T[-1 - n] \rightarrow S_T[-n]$$

for  $n \geq 0$ , which means that all maps

$$X^n \rightarrow Y^n \times_{\Omega_T Y^{n+1}} \Omega_T X^{n+1}$$

must be weak equivalences.

Every stable fibration is an injective fibration. The point of this section is to establish a converse assertion which holds in a wide variety of cases. The basic message is that we need some kind of compactness assumption on  $T$  for this to work out.

Starting with a  $T$ -spectrum  $X$ , define functors  $X \mapsto Q^k X$  for  $k \geq 0$  by specifying that

$$Q^0 X = F X$$

and

$$Q^{k+1} X = \Omega_T Q^k X[1]$$

(fake  $T$ -loops). The map  $Q^k X \rightarrow Q^{k+1} X$  is the canonical map

$$Q^k X \xrightarrow{\sigma_*} \Omega_T Q^k X[1].$$

Set

$$QX = F(\varinjlim_k Q^k X).$$

Say that  $T$  is *compact up to equivalence* if for any filtered diagram  $i \mapsto X_i$  of globally fibrant pointed simplicial presheaves the map

$$\varinjlim_i \Omega_T X_i \rightarrow \Omega_T F(\varinjlim_i X_i)$$

is a local weak equivalence.

Observe that if  $T$  and  $T'$  are compact up to equivalence, then  $T \wedge T'$  is compact up to equivalence. In effect, there is a natural isomorphism

$$\Omega_{T \wedge T'} Y \cong \Omega_T \Omega_{T'} Y,$$

and there is a commutative diagram

$$\begin{array}{ccc}
\varinjlim_i \Omega_T \Omega_{T'} X_i & \xrightarrow{\simeq} & \Omega_T F(\varinjlim_i \Omega_{T'} X_i) \\
\downarrow & & \downarrow \simeq \\
\Omega_T \Omega_{T'} F(\varinjlim_i X_i) & \xrightarrow[\Omega_T j]{\simeq} & \Omega_T F(\Omega_{T'} F(\varinjlim_i X_i))
\end{array}$$

The top horizontal map is a weak equivalence since  $T$  is compact up to equivalence, and vertical map on the right is a weak equivalence since  $T'$  is compact up to equivalence.

**Lemma 2:** Suppose that  $T$  is compact up to equivalence. Then  $QX$  is stably fibrant.

**Proof:** All objects  $Q^k X$  are strictly fibrant. In the diagram

$$\begin{array}{ccc}
\varinjlim_k Q^k X & \xrightarrow{j} & F(\varinjlim_k Q^k X) \\
\cong \downarrow & \searrow \sigma_* & \downarrow \sigma_* \\
\varinjlim_k \Omega_T Q^k X[1] & \longrightarrow & \Omega_T(\varinjlim_k Q^k X)[1] \longrightarrow \Omega_T F(\varinjlim_k Q^k X)[1]
\end{array}$$

the indicated vertical map is an isomorphism by cofinality, and the bottom horizontal composite is a strict equivalence since  $T$  is compact up to equivalence. It follows that

$$QX = F(\varinjlim_k Q^k X)$$

is a strictly fibrant  $T$ -spectrum for which the canonical map

$$\sigma_* : QX \rightarrow \Omega_T QX[1]$$

is a strict equivalence. This means that  $QX$  is stably fibrant.  $\square$

**Lemma 3:** Suppose that  $T$  is compact up to equivalence, and that  $i \mapsto Z_i$  is a filtered diagram of  $T$ -spectra such that every transition morphism  $Z_i \rightarrow Z_j$  induces a strict equivalence

$$QZ_i \rightarrow QZ_j.$$

Then each canonical map  $Z_i \rightarrow \varinjlim_i Z_i$  induces a strict equivalence

$$QZ_i \rightarrow Q(\varinjlim_i Z_i).$$

**Proof:** The map

$$\varinjlim_i Q^k Z_i \rightarrow Q^k(\varinjlim_i Z_i).$$

is a strict equivalence for each  $k \geq 0$ , since the  $k$ -fold wedge  $T^{\wedge k}$  is compact up to equivalence. Note that if  $k = 0$ , this is a property of globally fibrant models. It follows that the induced map

$$\varinjlim_i QZ_i \rightarrow Q \varinjlim_i Z_i$$

is a strict equivalence. Finally, all maps

$$QZ_i \rightarrow \varinjlim_i QZ_i$$

are strict equivalences by the assumption on the filtered diagram  $i \mapsto Z_i$ .  $\square$

**Lemma 4:** Suppose that  $T$  is compact up to equivalence. Then the natural stably fibrant model  $j : X \rightarrow LX$  induces a strict equivalence

$$QX \rightarrow QLX.$$

**Proof:** Say that a map  $f : X \rightarrow Y$  of  $T$ -spectra is an equivalence after level  $N$  if the component maps  $X^k \rightarrow Y^k$  are local weak equivalences for  $k > N$ . If  $f : X \rightarrow Y$  is an equivalence after level  $N$ , then the map  $Q^k X \rightarrow Q^k Y$  is a strict equivalence for  $k > N$ , and so the map  $QX \rightarrow QY$  is a strict equivalence. Each generating map for the set  $S$  is an equivalence after level  $N$  for some  $N$ , as are all of its pushouts. It follows that the canonical map  $j : X \rightarrow LX$  can be written as the composite

$$X = X_0 \rightarrow X_1 \rightarrow \dots$$

of maps between objects  $X_s, s \in \beta$  for some infinite cardinal  $\beta$  such that each map  $X_s \rightarrow X_{s+1}$

induces a strict equivalence  $QX_s \rightarrow QX_{s+1}$ , and such that  $X_t = \varinjlim_{s < t} X_s$  for all limit ordinals  $t < \beta$ .

It follows from the [Lemma 3](#) that each map  $X = X_0 \rightarrow X_s$  induces a strict equivalence

$$QX \rightarrow QX_s$$

for  $s < \beta$ , and hence that the map

$$X = X_0 \rightarrow \varinjlim_{s < \beta} X_s = LX$$

induces a strict equivalence

$$QX \rightarrow QLX. \quad \square$$

**Lemma 5:** Suppose that  $T$  is compact up to equivalence. Then the canonical map  $X \rightarrow QX$  is a stably fibrant model for each  $T$ -spectrum  $X$ .

**Proof:** [Lemma 2](#) says that  $QX$  is stably fibrant. In the diagram

$$\begin{array}{ccc} X & \longrightarrow & QX \\ j \downarrow & & \downarrow j_* \\ LX & \longrightarrow & QLX \end{array} \quad (1)$$

the map  $j_*$  is a strict equivalence by [Lemma 4](#).

All maps  $LX \rightarrow Q^k LX$  in the defining diagram for  $QLX$  are strict equivalences. In effect, if  $LX \rightarrow Q^k LX$  is a strict equivalence, then  $Q^k LX$  is stably fibrant and the map

$$Q^k LX \rightarrow \Omega_T Q^k LX[1] = Q^{k+1} LX$$

is a strict equivalence, so that

$$LX \rightarrow Q^{k+1} LX$$

is a strict equivalence. It follows that the map  $LX \rightarrow QX$  is a strict equivalence.

One concludes from the diagram (1) that  $X \rightarrow QX$  is a stable equivalence.  $\square$

**Corollary 5.5:** The canonical map  $X \rightarrow Q^k X$  is a stable equivalence for each  $k \geq 0$ .

**Proof:** The map  $FX \rightarrow \Omega_T FX[1]$  induces a level equivalence  $QFX \rightarrow Q(\Omega_T FX[1])$ , and is therefore a stable equivalence.  $\square$

**Lemma 6:** Suppose that  $T$  is compact up to equivalence. Suppose given a pullback diagram

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{f_*} & X \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{f} & Y \end{array}$$

of  $T$ -spectra such that  $p$  is a strict fibration and  $f$  is a stable equivalence. Then the map  $f_*$  is a stable equivalence.

**Proof:** The induced diagram

$$\begin{array}{ccc} Q(Z \times_Y X) & \longrightarrow & QX \\ \downarrow & & \downarrow \\ QZ & \longrightarrow & QY \end{array}$$

is strictly homotopy cartesian, and the map  $QZ \rightarrow QY$  is a strict equivalence. It follows that the map

$$Q(Z \times_Y X) \rightarrow QX$$

is a strict equivalence, and so  $f_*$  is a stable equivalence.  $\square$

**Corollary 7:** Suppose that  $T$  is compact up to equivalence. Then the stable model structure on the category of  $T$ -spectra is proper.

**Lemma 8:** Suppose that  $p : X \rightarrow Y$  is a strict fibration between stably fibrant  $T$ -spectra. Then  $p$  is a stable fibration.

**Proof:** Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array} \tag{2}$$

where  $i$  is a cofibration and a stable equivalence. Then the induced map  $i_* : LA \rightarrow LB$  is a strict equivalence, as are the stably fibrant model maps  $j : X \rightarrow LX$  and  $j : Y \rightarrow LY$ . The induced diagram

$$\begin{array}{ccc} LA & \longrightarrow & LX \\ i_* \downarrow & & \downarrow p_* \\ LB & \longrightarrow & LY \end{array}$$

has a factorization

$$\begin{array}{ccccc} LA & \xrightarrow{j_A} & V_X & \xrightarrow{p_X} & LX \\ i_* \downarrow & & \downarrow i' & & \downarrow p_* \\ LB & \xrightarrow{j_B} & V_Y & \xrightarrow{q_Y} & LY \end{array}$$

such that  $j_A$  and  $j_B$  are strict trivial cofibrations and  $p_X$  and  $p_Y$  are strict fibrations. In the pullback diagram

$$\begin{array}{ccc} V_X \times_{LX} X & \longrightarrow & X \\ j_{X*} \downarrow & & \downarrow j_X \\ V_X & \xrightarrow{p_X} & LX \end{array}$$

the map  $j_{X*}$  is a strict equivalence. The corresponding map  $j_{Y*}$  in the diagram

$$\begin{array}{ccccc} LA & \xrightarrow{j_A} & V_X & \xleftarrow{j_{X*}} & V_X \times_{LX} X \\ \downarrow & & \downarrow & & \downarrow \\ LB & \xrightarrow{j_B} & V_Y & \xleftarrow{j_{Y*}} & V_Y \times_{LY} Y \end{array}$$

is also a strict equivalence. It follows that the induced map

$$V_X \times_{LX} X \rightarrow V_Y \times_{LY} Y$$

is a strict equivalence, and that the diagram (2) has a factorization

$$\begin{array}{ccccc} A & \longrightarrow & V_X \times_{LX} X & \longrightarrow & X \\ i \downarrow & & \downarrow \simeq & & \downarrow p \\ B & \longrightarrow & V_Y \times_{LY} Y & \longrightarrow & Y \end{array}$$

in which the middle vertical map is a strict equivalence. The result follows by a standard argument: one factorizes the middle vertical map as a trivial strict cofibration followed by a trivial strict fibration.  $\square$

**Proposition 9:** Suppose that  $T$  is compact up to equivalence, and suppose that  $p : X \rightarrow Y$  is a strict fibration. Then  $p$  is a stable fibration if and only if the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & LX \\ p \downarrow & & \downarrow Lp \\ Y & \xrightarrow{i} & LY \end{array} \quad (3)$$

is strictly homotopy cartesian.

**Proof:** Suppose that the diagram (3) is strictly

homotopy cartesian. There is a factorization

$$\begin{array}{ccc} LX & \xrightarrow{j} & Z \\ & \searrow Lp & \downarrow q \\ & & LY \end{array}$$

of  $Lp$  such that  $j$  is a stable equivalence and  $q$  is an injective fibration. But then  $Z$  is injective, hence stably fibrant, so that  $j$  is a strict equivalence. It also follows from [Lemma 8](#) that  $q$  is a stable fibration. By pulling back  $q$  along  $i$ , we see from the hypothesis that the induced map

$$X \rightarrow Y \times_{LY} Z$$

is a strict equivalence. Every trivial strict fibration is an stable fibration, and it follows that  $p$  is a retract of an stable fibration, and hence is itself a stable fibration.

Suppose that  $p : X \rightarrow Y$  is a stable fibration. Find a factorization

$$\begin{array}{ccc} LX & \xrightarrow{j} & Z \\ & \searrow Lp & \downarrow q \\ & & LY \end{array}$$

such that  $j$  is a trivial stable cofibration and  $q$  is a stable fibration. Then  $j$  is a stable equivalence between stably fibrant objects, and is therefore a

strict equivalence. In the pullback diagram

$$\begin{array}{ccc} Y \times_{LY} Z & \xrightarrow{pr} & Z \\ q_* \downarrow & & \downarrow q \\ Y & \longrightarrow & LY \end{array}$$

the projection map  $pr$  is a stable equivalence since the stable model structure is proper by [Corollary 7](#). It follows that the comparison

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \times_{LY} Z \\ p \searrow & & \swarrow q_* \\ & Y & \end{array}$$

is a stable equivalence between stable fibrations, and  $\theta$  is therefore a strict equivalence by a standard argument.  $\square$

**Remark:** The assumption that  $T$  is compact up to equivalence in [Proposition 9](#) is only required for showing that the diagram (3) is strictly homotopy cartesian if  $p$  is a stable fibration.

**Corollary 9.5:** Suppose given a diagram

$$\begin{array}{ccc} X & \xrightarrow{j_X} & X' \\ p \downarrow & & \downarrow p' \\ Y & \xrightarrow{j_Y} & Y' \end{array}$$

such that  $p$  and  $p'$  are strict fibrations, and  $j_X$

and  $j_Y$  are strict equivalences. Then  $p$  is a stable fibration if and only if  $p'$  is a stable fibration.

**Proof:** The diagram

$$\begin{array}{ccc} X & \longrightarrow & LX \\ p \downarrow & & \downarrow \\ Y & \longrightarrow & LY \end{array}$$

is strictly equivalent to the diagram

$$\begin{array}{ccc} X' & \longrightarrow & LX' \\ p' \downarrow & & \downarrow \\ Y' & \longrightarrow & LY' \end{array}$$

Thus the first diagram is strictly homotopy cartesian if and only if the second diagram is strictly homotopy cartesian.  $\square$

**Theorem 10:** Suppose that  $T$  is compact up to equivalence. Then every injective fibration of  $T$ -spectra  $p : X \rightarrow Y$  between strictly fibrant objects is a stable fibration.

**Proof:** Forget that  $p$  is a strict fibration for the moment, and say that a map  $p : X \rightarrow Y$  between strictly fibrant objects is semi-injective if all diagrams

$$\begin{array}{ccc} X^n & \longrightarrow & \Omega_T X^{n+1} \\ \downarrow & & \downarrow \\ Y^n & \longrightarrow & \Omega_T Y^{n+1} \end{array}$$

Certainly all injective fibrations are semi-injective.

All induced maps  $p_k : Q^k X \rightarrow Q^k Y$  are semi-injective if  $p$  is semi-injective. In effect, if  $p$  is semi-injective, then the induced map  $p_* : \Omega_T X[1] \rightarrow \Omega_T Y[1]$  is semi-injective, since the functor  $Z \mapsto \Omega_T Z[1]$  preserves strict homotopy cartesian diagrams of strictly fibrant objects.

All diagrams

$$\begin{array}{ccc} X^n & \longrightarrow & Q^k X^n \\ p \downarrow & & \downarrow p_s \\ Y^n & \longrightarrow & Q^k Y^n \end{array} \quad (4)$$

are homotopy cartesian for  $k \geq 0$ . In effect, if the diagram (4), call it  $D_k$ , is homotopy cartesian, then  $p_k$  is semi-injective so that the composite diagram

$$\begin{array}{ccccc} X^n & \longrightarrow & Q^k X^n & \longrightarrow & Q^{k+1} X^n \\ \downarrow & & \downarrow & & \downarrow \\ Y^n & \longrightarrow & Q^k Y^n & \longrightarrow & Q^{k+1} Y^n \end{array}$$

is homotopy cartesian, so that  $D_{k+1}$  is homotopy cartesian.

It follows that the composite diagram

$$\begin{array}{ccccc}
X^n & \longrightarrow & \varinjlim_k Q^k X^n & \xrightarrow{\cong} & F(\varinjlim_k Q^k X^n) \\
\downarrow & & \downarrow & & \downarrow \\
Y^n & \longrightarrow & \varinjlim_k Q^k Y^n & \xrightarrow{\cong} & F(\varinjlim_k Q^k Y^n)
\end{array}$$

is homotopy cartesian if  $p$  is semi-injective.

Thus, if  $p$  is an injective fibration, then  $p$  is a strict fibration and the diagram

$$\begin{array}{ccc}
X & \longrightarrow & QX \\
\downarrow & & \downarrow \\
Y & \longrightarrow & QY
\end{array}$$

is strictly homotopy cartesian. The horizontal maps are stably fibrant models by [Lemma 5](#), so the map  $p$  is a stable fibration by [Proposition 9](#).  $\square$

**Corollary 10.5:** Suppose that  $T$  is compact up to equivalence. Suppose that  $p : X \rightarrow Y$  is a strict fibration. Then  $p$  is a stable fibration if and only if there is a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{j_X} & X' \\
p \downarrow & & \downarrow p' \\
Y & \xrightarrow{j_Y} & Y'
\end{array} \tag{5}$$

such that  $j_X$  and  $j_Y$  are strict equivalences,  $Y'$  is strictly fibrant and  $p'$  is an injective fibration.

**Proof:** Suppose that the diagram (5) exists meeting these conditions. Then  $p'$  is a stable fibration by [Theorem 10](#), and so  $p$  is a stable fibration by [Corollary 9.5](#).

Suppose that  $p$  is a stable fibration, and construct a diagram (5) such that  $j_X, j_Y$  are strict equivalences,  $Y'$  is strictly fibrant and  $p'$  is a strict fibration. Then  $p'$  is a stable fibration by [Corollary 9.5](#), and so  $p'$  is an injective fibration.  $\square$

## $(S^1 \wedge K)$ -spectra

Suppose that  $K$  is compact up to equivalence. The class of pointed simplicial presheaves which are compact up to equivalence is closed under finite smash products and includes all finite pointed simplicial sets. It follows that  $S^1 \wedge K$  is compact up to equivalence, so all results of the previous section apply to  $(S^1 \wedge K)$ -spectra.

An  $(S^1 \wedge K)$ -spectrum  $X$  determines a  $K$ -spectrum object  $X^{*,*}$  in spectra, which at  $K$ -level  $n$  is the spectrum

$$X^{n,*} : X^0 \wedge K^n, X^1 \wedge K^{n-1}, \dots, \\ X^{n-1} \wedge K, X^n, S^1 \wedge X^n, S^2 \wedge X^n, \dots$$

The bonding maps are the maps  $\sigma_*$  defined by the composites

$$S^1 \wedge X^{n-j} \wedge K^j \xrightarrow{S^1 \wedge \tau \wedge K^{n-1}} S^1 \wedge K \wedge X^{n-j} \wedge K^{j-1} \\ \xrightarrow{\sigma \wedge K^{n-1}} X^{n-(j-1)} \wedge K^{j-1}$$

up to level  $n - 1$  (ie. for  $j \geq 1$ ) and are identities beyond. The  $K$ -bonding maps

$$X^{n,*} \wedge K \rightarrow X^{n+1,*}$$

are identity maps up to level  $n - 1$ .

Each  $K$ -spectrum object  $Z$  has a functorially associated “diagonal”  $T$ -spectrum  $dZ$  with

$$(dZ)^n = Z^{n,n}$$

and with bonding maps given by the composites

$$S^1 \wedge K \wedge Z^{n,n} \xrightarrow{S^1 \wedge \tau} S^1 \wedge Z^{n,n} \wedge K \xrightarrow{S^1 \wedge \sigma^K} S^1 \wedge Z^{n+1,n} \xrightarrow{\sigma} Z^{n+1,n+1}.$$

There is a natural isomorphism

$$d(X^{*,*}) \cong X$$

for all  $(S^1 \wedge K)$ -spectra  $X$ .

Suppose that  $X$  is an  $(S^1 \wedge K)$ -spectrum, and write

$$Q_\ell X = \varinjlim_n \Omega_T^n X,$$

where the filtered colimit is taken over all finite  $n$ . Then, by definition,  $QX = FQ_\ell X$  is the natural strictly fibrant model for  $Q_\ell X$ .

**Lemma 11:** A map  $f : X \rightarrow Y$  of  $(S^1 \wedge K)$ -spectra is a stable equivalence if and only if it induces a strict equivalence  $Q_\ell X \rightarrow Q_\ell Y$ .

An  $(S^1 \wedge K)$ -spectrum  $X$  has bigraded presheaves of stable homotopy groups  $\pi_{s,t}X$ , defined by

$$\pi_{s,t}X(U) = \varinjlim_{n \geq 0} [S^{n+s} \wedge K^{n+t}|_U, X^n|_U]$$

where the homotopy classes of maps are computed for pointed simplicial presheaves on  $\mathcal{C}/U = \text{Sch} \downarrow U$ , and the transition maps are defined by suspension with  $T$  in the “obvious” way: a representing map

$$\alpha : S^{n+s} \wedge K^{n+t} \rightarrow X^n$$

is sent to the composite

$$\begin{aligned} S^{n+1+s} \wedge K^{n+1+t} &\xrightarrow{S^1 \wedge \tau \wedge K^{n+t}} S^1 \wedge K \wedge S^{n+s} \wedge K^{n+t} \\ &\xrightarrow{1 \wedge \alpha} S^1 \wedge K \wedge X^n \xrightarrow{\sigma} X^{n+1}. \end{aligned}$$

These stable homotopy group presheaves are specializations of bigraded stable homotopy groups  $\pi_{s,t}Y$  which are defined for  $K$ -spectrum objects  $Y$  in spectra by

$$\pi_{s,t}Y(U) = \varinjlim_{k,l} [S^{k+s} \wedge K^{l+t}, Y^{k,l}]_U,$$

where the notation indicates that the homotopy classes are computed over  $U \in \mathcal{C}$ . This means that there are natural isomorphisms

$$\pi_{s,t}Y \cong \pi_{s,t}dY$$

of presheaves for all  $K$ -spectrum objects  $Y$ . It follows that there are natural isomorphisms

$$\pi_{s,t}X^{*,*} \cong \pi_{s,t}X$$

for all  $(S^1 \wedge K)$ -spectra  $X$ .

The bonding maps  $Y^n \wedge K \rightarrow Y^{n+1}$  in a  $K$ -spectrum object  $Y$  induce homomorphisms of stable homotopy classes of maps

$$\begin{aligned} [S[s] \wedge K^{n+t}, Y^n]_U &\rightarrow [S[s] \wedge K^{n+t+1}, Y^{n+1}]_U \\ &\rightarrow [S[s] \wedge K^{n+t+2}, Y^{n+2}]_U \rightarrow \dots \end{aligned}$$

for all  $U \in \mathcal{C}$ , and the filtered colimit of the system is  $\pi_{s,t}Y(U)$ .

Note that there are isomorphisms of presheaves

$$\pi_k Q_\ell X^n(U) \cong \pi_{k-n, -n} X(U).$$

**Lemma 12:** Suppose that a map  $f : X \rightarrow Y$  of  $K$ -spectrum objects in spectra consists of stable equivalences  $X^n \rightarrow Y^n$ ,  $n \geq 0$ . Then the induced map  $dX \rightarrow dY$  is a stable equivalence of  $(S^1 \wedge K)$ -spectra.

**Proof:** If all maps  $X^n \rightarrow Y^n$  are stable equivalences, then all homomorphisms

$$[S[s] \wedge K^{n+t}, X^n]_U \rightarrow [S[s] \wedge K^{n+t}, Y^n]_U$$

are isomorphisms, so that all induced maps

$$\pi_{s,t}X \rightarrow \pi_{s,t}Y$$

are isomorphisms of presheaves of groups. It follows that all induced maps

$$\pi_k Q_\ell dX \rightarrow \pi_k Q_\ell dY$$

of presheaves of homotopy groups (pointed by the canonical base point) are isomorphisms. Finally,  $Q_\ell dX^n \cong \Omega \Omega_K Q_\ell dX^{n+1}$  so that all  $Q_\ell dX^n$  are presheaves of  $H$ -spaces. It follows that all maps  $Q_\ell dX^n \rightarrow Q_\ell dY^n$  are sectionwise weak equivalences, so that the morphism  $Q_\ell dX \rightarrow Q_\ell dY$  is a strict equivalence. The desired result then follows from the [Lemma 11](#).  $\square$

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a strict fibre sequence of  $(S^1 \wedge K)$ -spectra. Every map  $f : Z \rightarrow W$  of  $K$ -spectrum objects has a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & V \\ & \searrow f & \downarrow q \\ & & W \end{array}$$

where  $q$  is a strict fibration in each  $K$ -level and  $j$  is a cofibration and a strict weak equivalence in

each  $K$ -level. Take such a factorization

$$\begin{array}{ccc} X^{*,*} & \xrightarrow{j} & V \\ & \searrow p & \downarrow q \\ & & Y^{*,*} \end{array}$$

for the map induced by the  $T$ -spectrum map  $p : X \rightarrow Y$ , and let  $\overline{F}$  be the fibre of  $q$ . Then there are induced comparisons of fibre sequences of simplicial presheaves

$$\begin{array}{ccccc} F^n & \xrightarrow{i} & X^n & \xrightarrow{p} & Y^n \\ \downarrow & & \simeq \downarrow & & \downarrow \cong \\ \overline{F}^{n,n} & \longrightarrow & Z^{n,n} & \longrightarrow & Y^{n,n} \end{array}$$

for each  $n \geq 0$ , and it follows (by properness for pointed simplicial presheaves) that the induced map  $F \rightarrow d\overline{F}$  is a strict weak equivalence.

Strict fibre and cofibre sequences coincide up to natural stable equivalence in  $(S^1 \wedge K)$ -spectra. The proof comes in three parts:

**Lemma 13:** Suppose that  $p : X \rightarrow Y$  is a strict fibration of  $(S^1 \wedge K)$ -spectra, with fibre  $F$ . Then the canonical map

$$X/F \rightarrow Y$$

is a stable equivalence.

**Proof:** Take a factorization

$$\begin{array}{ccc} X^{*,*} & \xrightarrow{j} & V \\ & \searrow p_* & \downarrow q \\ & & Y^{*,*} \end{array}$$

of the corresponding map of bispectrum objects, where  $q$  is a strict fibration and  $j$  is a strict trivial cofibration in all  $K$ -levels. Let  $\overline{F}$  be the fibre of  $q$ . Then the map

$$V/\overline{F} \rightarrow Y^{*,*}$$

is a stable equivalence in all  $K$ -levels, and so the induced map

$$d(V/\overline{F}) \rightarrow d(Y^{*,*})$$

is a stable equivalence of  $(S^1 \wedge K)$ -spectra by [Lemma 12](#). The map

$$d(X^{*,*}/F^{*,*}) \rightarrow d(V/\overline{F})$$

is a strict equivalence of  $(S^1 \wedge K)$ -spectra because

$$F = d(F^{*,*}) \rightarrow d\overline{F}$$

is a strict equivalence. □

**Lemma 14:** Suppose that

$$\begin{array}{ccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\ f_1 \downarrow & & \downarrow f_2 & & \downarrow f_3 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \end{array}$$

is a comparison of level cofibre sequences of  $(S^1 \wedge K)$ -spectra. If any two of the maps  $f_1, f_2$  and  $f_3$  are stable equivalences, then so is the third.

**Proof:** It suffices to assume that all objects are cofibrant.

Suppose that  $A$  is cofibrant and that  $\Sigma_T A$  denotes the fake suspension of an  $(S^1 \wedge K)$ -spectrum  $A$ . Then the canonical map  $\Sigma_T A \rightarrow A[1]$  is a stable equivalence. In effect, if  $Z$  is stably fibrant, then the induced map

$$\mathbf{hom}(A[1], Z) \rightarrow \mathbf{hom}(\Sigma_T A, Z) \quad (6)$$

is isomorphic to the map

$$\mathbf{hom}(A, Z[-1]) \rightarrow \mathbf{hom}(A, \Omega_T Z)$$

which is induced by the shift  $Z[-1] \rightarrow \Omega_T Z$  (suitably defined) of the canonical strict equivalence  $Z \rightarrow \Omega_T Z[1]$ , where  $\Omega_T Z$  is the fake loop spectrum. It follows that the simplicial set map (6) is a weak equivalence for all stably fibrant  $Z$ , and so the map  $\Sigma_T A \rightarrow A[1]$  of cofibrant objects is a stable equivalence.

It follows that the natural map  $\Sigma_T B \rightarrow B[1]$  is a stable equivalence for all  $(S^1 \wedge K)$ -spectra  $B$ ,

because each  $B$  is strictly equivalent to a cofibrant object.

Shifting preserves stable equivalences, so the map  $\Sigma_T B[-1] \rightarrow B$  is also a stable equivalence.

The bispectrum object  $(\Sigma_T B)^{*,*}$  is naturally isomorphic to the fake  $S^1$ -suspension  $\Sigma_{S^1} \Sigma_K B^{*,*}$  of the fake  $K$ -suspension of  $B^{*,*}$ . The object  $\Sigma_K B^{*,*}$  is an  $S^1$ -spectrum object in  $K$ -spectra, and so one can use the telescope construction (cofibrant model) for partial spectrum objects to show that there are natural  $S^1$ -stable equivalences

$$\Sigma_K B^{*,*} \wedge S^1 \simeq \Sigma_{S^1} \Sigma_K B^{*,*}.$$

See Lecture 003. The “map” consists of  $S^1$ -stable equivalences in each  $K$ -level, and therefore induces a natural stable equivalence

$$d(\Sigma_K B^{*,*}) \wedge S^1 \simeq d(\Sigma_{S^1} \Sigma_K B^{*,*}) \cong \Sigma_T B$$

by [Lemma 12](#).

It follows that there are natural stable equivalences

$$d(\Sigma_K B^{*,*})[-1] \wedge S^1 \simeq \Sigma_T B[-1] \simeq B.$$

All functors in this picture preserve cofibrant objects and level cofibre sequences, so the process can be iterated. The original comparison of cofibre sequences therefore induces a comparison of

fibre sequences

$$\begin{array}{ccccc}
 \mathbf{hom}(B_3, Z) & \longrightarrow & \mathbf{hom}(B_2, Z) & \longrightarrow & \mathbf{hom}(B_1, Z) \\
 f_3^* \downarrow & & \downarrow f_2^* & & \downarrow f_1^* \\
 \mathbf{hom}(A_3, Z) & \longrightarrow & \mathbf{hom}(A_2, Z) & \longrightarrow & \mathbf{hom}(A_1, Z)
 \end{array}$$

of fibre sequences of infinite loop spaces. Thus if any two of the vertical maps are (stable) equivalences, then so is the third.  $\square$

**Lemma 15:** Suppose that  $i : A \rightarrow B$  is a level cofibration of  $(S^1 \wedge T)$ -spectra, and take a factorization

$$\begin{array}{ccc}
 B & \xrightarrow{j} & Z \\
 & \searrow \pi & \downarrow p \\
 & & B/A
 \end{array}$$

of the quotient map  $\pi : B \rightarrow B/A$ , where  $j$  is a strict trivial cofibration and  $p$  is a fibration. Let  $F$  be the fibre of  $p$ . Then the induced map  $A \rightarrow F$  is a stable equivalence.

**Proof:** The canonical map  $p_* : Z/F \rightarrow B/A$  associated to the fibration  $p : Z \rightarrow B/A$  is a stable equivalence by [Lemma 13](#). There is also a

commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & B/A \\ \downarrow & & \simeq \downarrow j & & \downarrow j_* \\ F & \longrightarrow & Z & \longrightarrow & Z/F \end{array}$$

But  $p_*j_* = 1$  so that  $j_*$  is a stable equivalence. It follows that the map  $A \rightarrow F$  in the diagram (which is the map of interest) is a stable equivalence, by [Lemma 14](#)  $\square$

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level strict fibre sequence of  $K$ -spectrum objects and that  $Y$  is strictly fibrant in all  $K$ -levels. Then all induced sequences

$$\Omega_K^{t+n} F^n \rightarrow \Omega_K^{t+n} X^n \rightarrow \Omega_K^{t+n} Y^n$$

are strict fibre sequences of presheaves of spectra, and all spectra  $\Omega_K^{t+n} Y^n$  are strictly fibrant. It follows that there is a long exact sequence in presheaves of stable homotopy groups of the form

$$\cdots \rightarrow \pi_s \Omega_K^{t+n} F^n \rightarrow \pi_s \Omega_K^{t+n} X^n \rightarrow \pi_s \Omega_K^{t+n} Y^n \xrightarrow{\partial} \pi_{s-1} \Omega_K^{t+n} F^n \rightarrow \cdots$$

There are, as well, comparisons of fibre sequences

$$\begin{array}{ccccc} \Omega_K^{t+n} F^n & \longrightarrow & \Omega_K^{t+n} X^n & \longrightarrow & \Omega_K^{t+n} Y^n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_K^{t+n+1} F^{n+1} & \longrightarrow & \Omega_K^{t+n+1} X^{n+1} & \longrightarrow & \Omega_K^{t+n+1} Y^{n+1} \end{array}$$

induced by the respective  $K$ -spectrum object structures. Thus taking a filtered colimit in  $n$  gives a long exact sequence

$$\cdots \rightarrow \pi_{s,t}F \xrightarrow{i_*} \pi_{s,t}X \xrightarrow{p_*} \pi_{s,t}Y \xrightarrow{\partial} \pi_{s-1,t}F \rightarrow \cdots \quad (7)$$

in presheaves of bigraded stable homotopy groups. Note that the “degree”  $s$  changes while the “weight”  $t$  does not.

Any strict fibration  $p : X \rightarrow Y$  has a strictly fibrant replacement

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ j_X \downarrow & & \downarrow j_Y \\ X' & \xrightarrow{p'} & Y' \end{array}$$

where  $Y'$  is strictly fibrant in all  $K$ -levels,  $p'$  is a strict fibration in all  $K$ -levels, and the vertical maps are strictly trivial cofibrations in all  $K$ -levels. Then the induced map  $F \rightarrow F'$  to the fibre  $F'$  of  $p'$  is a strict equivalence. Also, any two such fibrant replacements are strictly equivalent, and hence equivalent in all sections and all levels. It follows, by making choices that don't matter, that there is a long exact sequence of the form (7) for any strict fibre sequence

$$F \xrightarrow{i} X \xrightarrow{p} Y.$$

**Corollary 16:** Every level cofibre sequence

$$A \rightarrow B \rightarrow B/A$$

has a naturally associated long exact sequence

$$\dots \pi_{s,t}A \rightarrow \pi_{s,t}B \rightarrow \pi_{s,t}(B/A) \xrightarrow{\partial} \pi_{s-1,t}A \rightarrow \dots$$

of presheaves of stable homotopy groups.

**Corollary 17:** There are natural isomorphisms

$$\pi_{s+1,t}(Y \wedge S^1) \cong \pi_{s,t}Y$$

for all  $(S^1 \wedge K)$ -spectra  $Y$ .

**Corollary 18:** (additivity) Suppose that  $X$  and  $Y$  are  $(S^1 \wedge K)$ -spectra. Then the canonical map

$$c : X \vee Y \rightarrow X \times Y$$

is a stable equivalence.

**Proof:** The map  $c$  induces an isomorphism

$$\pi_{s,t}(X \vee Y) \cong \pi_{s,t}(X \times Y)$$

for all  $s, t$ . In effect,  $\pi_{s,t}(X \vee Y) \cong \pi_{s,t}X \oplus \pi_{s,t}Y$  from the long exact sequence for a cofibre sequence ([Corollary 16](#)), and  $\pi_{s,t}(X \times Y) \cong \pi_{s,t}X \oplus \pi_{s,t}Y$  on account of the long exact sequence for a fibre sequence ([7](#)). It follows from [Lemma 12](#) that the map  $c$  is a stable equivalence.  $\square$