Lecture 008 (November 30, 2005)

Example: Presheaves of spectra

The simplicial circle S^1 is a finite pointed simplicial set, and is therefore compact up to equivalence.

As in the previous Lecture, for a presheaf of spectra X, write

$$QX = F(\varinjlim_n Q^n X) = F(\varinjlim_n \Omega^n FX[n]).$$

Write $\eta : X \to QX$ for the natural composite

$$X \to FX \to \varinjlim_n \Omega^n FX[n] \to F(\varinjlim_n \Omega^n FX[n]) = QX.$$

The simplicial circle S^1 is compact up to equivalence, so we have the following from the results of Lecture 007:

Lemma 1: The presheaf of spectra QX is stably fibrant, and the natural map $\eta : X \xrightarrow{\eta} QX$ is a stable equivalence.

It follows that a map $f : X \to Y$ is a stable equivalence if and only if it induces an isomorphism $\tilde{\pi}_s X \to \tilde{\pi}_s Y$ in all sheaves of stable homotopy groups. In effect f is a stable equivalence if an only if the map $QX \to QY$ is a level equivalence, and this is equivalent, since all QX^n are presheaves of H-spaces, to the requirement that all induced maps $\tilde{\pi}_k QX^n \to \tilde{\pi}_k QY^n$ of sheaves of homotopy groups (pointed at the distinguished base point) are isomorphisms. These sheaves of homotopy groups coincide with the sheaves of stable homotopy groups.

Cofibrations in the stable structure are the cofibrations of the underlying strict structure. It follows that the stable model structure on the category of presheaves of spectra in standard use [2], [3] is the localization of the strict model structure at the maps

$$\Sigma^\infty S^1[-1-n] \to S[-n],\ n \geq 0.$$

Note that the properness of the stable model structure is a consequence of the fact that S^1 is compact up to equivalence — see Corollary 7 of Lecture 007.

The standard recognition principle for a stable fibration of presheaves of spectra says that $p : X \to Y$ is a stable fibration if and only if it is a strict fibration and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ Y & \xrightarrow[\eta]{} & QY \end{array}$$

is strictly homotopy cartesian. This is a consequence of Proposition 9 of Lecture 007.

Example: T a finite simplicial set

In this case, S is generated over J by cofibrant replacements of the set of maps

$$\Sigma_T^\infty T[-1-n] \to S_T[-n]$$

where S_T is the sphere spectrum for presheaves of T-spectra, consisting of the list of pointed simplicial sets

$$S^0, T, T \wedge T, \dots$$

For a pointed simplicial presheaf, write $\Omega_T X$ for the pointed simplicial presheaf $\mathbf{Hom}(T, X)$. In this case, since T is a constant object,

$$\Omega_T X(U) = \mathbf{Hom}(T, X)(U) = \mathbf{hom}_*(T, X(U))$$

for all $U \in \mathcal{C}$.

The object T is compact up to equivalence; in fact the functor $X \mapsto \Omega_T X$ preserves filtered colimits on the nose since T is finite.

It follows, as before, that the object

$$QX = F(\varinjlim_k Q^k X) = F(\varinjlim_k \Omega_T^k F X[k])$$

is stably fibrant and the natural map

$$\eta : X \to QX$$

is a stable equivalence for all T-spectra X. A map $f : X \to Y$ of T-spectra is therefore a stable equivalence if and only if it induces a level equivalence $QX \to QY$.

It is rather hard to say more than that without knowing that the level objects QX^n have something like an H-space structure.

Suppose now that $T = S^1 \wedge K$ where K is a finite simplicial set. Then of course $S^1 \wedge K$ is still a finite simplicial set, and a map $f : X \to Y$ of $(S^1 \wedge K)$-spectra is a stable equivalence if and only if the map $QX \to QY$ is a level equivalence.

Write

$$Q_\ell X = \varinjlim_k \Omega_T^k FX[k],$$

so that $QX = FQ_\ell X$. Recall that there is an isomorphism of presheaves

$$\pi_k Q_\ell X^n \cong \pi_{k-n,-n} X.$$

All QX^n are presheaves of H-spaces, so that the maps $QX^n \to QY^n$ are weak equivalences if and only if all induced maps

$$\tilde{\pi}_k Q X^n \to \tilde{\pi}_k Q Y^n$$

are isomorphisms. It follows that $f : X \to Y$ is a stable equivalence if and only if it induces isomorphisms

$$\tilde{\pi}_{s,t} X \to \tilde{\pi}_{s,t} Y$$

in all sheaves associated to the presheaves $\pi_{s,t}$. In this case, the long exact sequences of bigraded presheaves of stable homotopy groups associated to strict fibre sequences and cofibre sequences given in Lecture 007 translate into long exact sequences of bigraded sheaves of stable homotopy groups, by applying the associated sheaf functor.

Example: T compact

A) Say that the site $\mathcal{C}$ has the *colimit descent property* if for any filtered system $i \mapsto Z_i$ of globally fibrant simplicial presheaves, any globally fibrant model

$$\varinjlim_i Z_i \xrightarrow{j} F(\varinjlim_i Z_i)$$

is a sectionwise weak equivalence.

Here are some examples:

1) Suppose that $\mathcal{C}$ is the Zariski site $Zar|_S$ of open subsets of a Noetherian scheme S. Every diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & U \cup V \end{array}$$

of open subsets of S is homotopy co-cartesian and hence induces a homotopy cartesian diagram

$$\begin{array}{ccc} Z(U \cup V) & \longrightarrow & Z(U) \\ \downarrow & & \downarrow \\ Z(V) & \longrightarrow & Z(U \cap V) \end{array}$$

if Z is globally fibrant.

The Brown-Gersten Descent Theorem [1] is the following:

Theorem: If S is a Noetherian scheme (or topological space) and if X is a non-empty presheaf of Kan complexes such that the diagram

$$\begin{array}{ccc} X(U \cup V) & \longrightarrow & X(U) \\ \downarrow & & \downarrow \\ X(V) & \longrightarrow & X(U \cap V) \end{array}$$

is homotopy cartesian for all pairs of open subsets U, V of S, then any globally fibrant model $j : X \to FX$ is a sectionwise weak equivalence.

Any filtered colimit of fibrant simplicial presheaves on $op|_S$ satisfies the conditions of the Theorem, so that $op|_S$ has the colimit descent property if S is Noetherian.

2) An *elementary Cartesian square* is a pullback diagram

$$\begin{array}{ccc} p^{-1}(U) & \longrightarrow & V \\ \downarrow & & \downarrow \phi \\ U & \xrightarrow[i]{} & X \end{array} \tag{1}$$

of scheme homomorphisms such that ϕ is étale, i is an open immersion and the induced map

$$\phi^{-1}(X - U) \to X - U$$

is an isomorphism of schemes. Here, $X - U$ and $\phi^{-1}(X - U)$ have the reduced closed subscheme structure. One shows that all residue fields of X lift to either U or V, so that the pair of maps i, ϕ forms a Nisnevich cover of X.

Diagrams defined by coverings involving two maps are homotopy cocartesian in general (we saw another example above), so all elementary Cartesian squares are homotopy cocartesian diagrams of simplicial presheaves. Thus, if Z is a globally fibrant simplicial presheaf on the Nisnevich site $(Sch\,|_S)_{Nis}$ (S of finite dimension) and X is an S-scheme then the induced square

$$\begin{array}{ccc} \mathbf{hom}(X,Z) & \longrightarrow & \mathbf{hom}(U,Z) \\ \downarrow & & \downarrow \\ \mathbf{hom}(V,Z) & \longrightarrow & \mathbf{hom}(\phi^{-1}(U),Z) \end{array}$$

is homotopy cartesian, but this is just the diagram of simplicial sets

$$\begin{array}{ccc} Z(X) & \longrightarrow & Z(U) \\ \downarrow & & \downarrow \\ Z(V) & \longrightarrow & Z(\phi^{-1}(U)) \end{array} \tag{2}$$

In other words, all globally fibrant simplicial presheaves and all elementary cartesian diagrams determine homotopy cartesian diagrams of simplicial sets like (2). The Nisnevich descent theorem is essentially a converse assertion:

Theorem: Suppose that a simplicial presheaf Z on the Nisnevich site $(\mathrm{Sch}\,|_S)_{Nis}$ satisfies the *cd*-excision property in the sense that all elementary cartesian squares (1) induce homotopy cartesian diagrams (2), and let $j : Z \to FZ$ be a choice of globally fibrant model for Z (for the Nisnevich topology). Then j is a sectionwise equivalence in the sense that all maps $j : Z(U) \to FZ(U)$ are equivalences of simplicial sets.

This result was first proved by Nisnevich in the stable case [6], hence the name — the main outcome is that the mod ℓ K-theory presheaf $K(\ ,\mathbb{Z}/\ell)$ (ℓ is a prime distinct from all residue characteristics of S) satisfies Nisnevich descent. The unstable version, for simplicial sheaves, was proved in [5]. The simplicial presheaf version appears in [4].

Again, if $i \mapsto Z_i$ is a filtered diagram of globally fibrant simplicial presheaves on $(\mathrm{Sch}\,|_S)_{Nis}$ then the filtered colimit $\varinjlim_i Z_i$ satisfies the conditions for the Nisnevich descent theorem, so that any globally fibrant model

$$j : \varinjlim_i Z_i \to F(\varinjlim_i Z_i)$$

is a sectionwise equivalence. Thus the Nisnevich site $(Sch|_S)_{Nis}$ has the colimit descent property.

Note that globally fibrant simplicial presheaves Z for any of the geometric topologies on $(Sch|_S)$ have an additional nice property: they are *flasque* in the sense that any (open) immersion $i : U \to V$ of S-schemes induces a fibration $i^* : Z(V) \to Z(U)$. This follows from the fact that $i : U \to V$ represents a cofibration of simplicial presheaves. The class of flasque presheaves of Kan complexes is closed under filtered colimits.

B) Say that a pointed simplicial presheaf T is *compact* if

1) The canonical map

$$\varinjlim_i \mathbf{Hom}(T, X_i) \to \mathbf{Hom}(T, \varinjlim_i X_i)$$

associated to any filtered system $i \mapsto X_i$ of pointed simplicial presheaves is an isomorphism.

2) If X is a flasque presheaf of pointed Kan complexes, then $\mathbf{hom}(T, X)$ is a flasque presheaf of Kan complexes.

3) Any sectionwise equivalence $X \to Y$ of presheaves of pointed Kan complexes induces a sectionwise equivalence

$$\mathbf{Hom}(T, X) \to \mathbf{Hom}(T, Y).$$

NB: The compactness property defined here is very much stronger than compactness up to equivalence. The latter is a much weakened version of condition 1).

Here are some statements which generate the main examples:

1) Any pointed S-scheme U represents a compact (simplicially discrete) simplicial presheaf

$$U = \hom(\ ,U).$$

2) Any finite pointed simplicial set K (identified with a constant simplicial presheaf) is compact.

3) If T_1 and T_2 are compact, then $T_1 \wedge T_2$ is compact.

4) If $i : U \to V$ is an immersion, then the (simplicial) presheaf V/U is compact.

Note that

$$\mathbf{Hom}(U, X)(V) = X(U \times V).$$

Here are some real examples of compact objects: $\mathbb{P}^1$, $\mathbb{G}_m$, S^1, $S^1 \wedge \mathbb{G}_m$, $\mathbb{A}^1/(\mathbb{A}^1 - \{0\})$.

Now suppose that $\mathcal{C}$ is defined by a geometric topology on some category of S-schemes which has the colimit descent property. Suppose that T is a compact pointed simplicial presheaf on $\mathcal{C}$.

The localized theory for T-spectra is generated by all maps

$$\Sigma_T^\infty A[n] \to \Sigma_T^\infty B[n]$$

induced by α-bounded trivial cofibrations $A \to B$ of pointed simplicial presheaves, and cofibrant replacements of all maps

$$\Sigma_T^\infty T[-1-n] \to S_T[-n].$$

The object T is compact, hence compact up to equivalence, so the results of Lecture 007 apply.

Form the objects QX and the natural maps

$$X \xrightarrow{\eta} QX$$

for all T-spectra X just as before. We know from formal nonsense (Corollary 5.5 of Lecture 007) that η is a stable equivalence, and that QX is stably fibrant.

Now suppose that $T = S^1 \wedge K$ where K is compact.

We have seen that every $(S^1 \wedge K)$-spectrum X has bigraded presheaves of stable homotopy groups $\pi_{s,t}X$ and that there is an isomorphism of presheaves

$$\pi_k Q_\ell X^n \cong \pi_{k-n,-n} X$$

for all k, n. Recall that

$$Q_\ell X = \varinjlim_k \Omega_T^k X[k],$$

so that $QX = FQ_\ell X$.

A map $f : X \to Y$ of $(S^1 \wedge K)$-spectra is a stable equivalence if and only if the induced map $QX^n \to QY^n$ is a level equivalence. Since the strict fibrant model $j : Q_\ell X \to FQ_\ell X$ is a sectionwise equivalence in all levels, we can conclude that $f : X \to Y$ is a stable equivalence if and only if the induced maps

$$\pi_{s,t} X \to \pi_{s,t} Y$$

are isomorphisms of presheaves for all s, t (the full argument for this claim requires the compactness of K). Note that this is equivalent to requiring that the induced maps

$$\tilde{\pi}_{s,t} X \to \tilde{\pi}_{s,t} Y$$

of associated sheaves are isomorphisms for all s, t. In effect, if the maps of associated sheaves are isomorphisms, then the maps $QX^n \to QY^n$ and all $\Omega^k_K QX^n \to \Omega^k_K QX^n$ are sectionwise equivalences, and so all maps of presheaves $\pi_{s,t}X \to \pi_{s,t}Y$ are isomorphisms.

Observe that the stable homotopy theory of $(S^1 \wedge K)$-spectra has all good properties, including properness, the coincidence of fibre and cofibre sequences, existence of long exact sequences for strict fibre sequences and cofibre sequences, and additivity, by the results of Lecture 007.

References

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