Lecture 009 (November 30, 2005)

Example: f-local T-spectra

Suppose that $\mathcal{C}$ is an arbitrary small site, let T be a pointed simplicial presheaf on $\mathcal{C}$, and suppose that $f : A \to B$ is a cofibration of pointed simplicial presheaves.

Let S_f be the set of cofibrations which is generated over J by all shifted suspensions of the cofibrations

$$(D \wedge A) \cup (C \wedge B) \to D \wedge B$$

arising from all α-bounded cofibrations $C \to D$, together with cofibrant replacements for all maps

$$\Sigma^\infty S^1[-1-n] \to S[-n]. \tag{1}$$

Recall that J is the set of maps

$$\Sigma_T^\infty C[-n] \to \Sigma_T^\infty D[-n]$$

arising from all α-bounded trivial cofibrations $C \to D$ of pointed simplicial presheaves. The resulting model structure is called the f-stable structure for T-spectra. Say that the L-equivalences are f-stable equivalences and that the L-fibrations are the f-stable fibrations.

Every stable equivalence of T-spectra is an f-local stable equivalence, since all the generators for the class of stably trivial cofibrations S are in the set S_f.

There is a corresponding closed simplicial model structure for pointed simplicial presheaves [1] in which

- An object Z is injective if and only if Z is globally fibrant and $Z \to *$ has the RLP with respect to all maps

$$(B \wedge C) \cup (A \wedge D) \to B \wedge D$$

- A map $g : V \to W$ is an f-equivalence if and only if the function

$$[W, Z] \xrightarrow{g^*} [V, Z]$$

  is a bijection for all injective Z.

- The cofibrations are the cofibrations of pointed simplicial presheaves.

- A map $p : X \to Y$ is an f-fibration if it has the RLP with respect to all maps which are cofibrations and f-equivalences.

- An object Z is f-fibrant (or injective) if and only if Z is globally fibrant and the morphism

$$f^* : \mathbf{Hom}(B, Z) \to \mathbf{Hom}(A, Z)$$

is a trivial global fibration of simplicial presheaves.

- The function complex $\mathbf{hom}(X, Y)$ is the ordinary pointed function complex.

Theorem A: The f-local model structure for pointed simplicial presheaves is proper if f is a global section $* \to I$ for some (pointed) simplicial presheaf I.

This result is proved in [2].

The f-local model structures are always left proper, so that point of the Theorem is the right properness assertion.

From the general theory, a T-spectrum X is f-stable fibrant if and only if all pointed simplicial presheaves X^n are f-fibrant and all adjoint bonding maps $X^n \to \Omega_T X^{n+1}$ are f-equivalences.

Motivic stable category: T compact

For motivic homotopy theory, the underlying site is the category $\mathcal{C} = (Sm|_S)_{Nis}$ of smooth schemes of finite type over a scheme S of finite dimension equipped with the Nisnevich topology, and one formally inverts the 0-section

$$f : * \to \mathbb{A}^1$$

as in the previous section for both simplicial presheaves and presheaves of spectra on $\mathcal{C}$ to form the motivic model structures for simplicial presheaves and presheaves of T-spectra, respectively. For presheaves of T-spectra, we generally require that T is compact.

Recall that a T-spectrum Z is motivic fibrant if all level objects Z^n are motivic fibrant simplicial presheaves and all adjoint bonding maps $Z^n \to \Omega_T Z^{n+1}$ are motivic equivalences.

For a motivic fibrant T-spectrum Z, all objects Z^n are, in particular, fibrant for the Nisnevich topology, and all adjoint bonding maps $Z^n \to \Omega_T Z^{n+1}$ are sectionwise equivalences. It follows that every motivic fibrant T-spectrum is stably fibrant for the Nisnevich topology.

A simplicial presheaf Z is motivic fibrant (or injective) if Z is globally fibrant and the 0-section map $* \to \mathbb{A}^1$ induces a trivial global fibration

$$\mathbf{Hom}(\mathbb{A}^1, Z) \to \mathbf{Hom}(*, Z),$$

and the latter requirement is equivalent to the assertion that the maps

$$Z(\mathbb{A}^1 \times U) \to Z(U)$$

are weak equivalences of simplicial sets for all schemes U/S.

Here's the upshot:

Lemma: Suppose given a filtered system $i \mapsto Z_i$ of motivic fibrant simplicial presheaves, and let

$$j : \varinjlim_i Z_i \to F(\varinjlim_i Z_i)$$

be a globally fibrant model for the Nisnevich topology. Then j is a sectionwise weak equivalence and $F(\varinjlim_i Z_i)$ is motivic fibrant.

Proof: $\varinjlim_i Z_i$ satisfies the *cd*-excision property, so that j is a sectionwise weak equivalence by Nisnevich descent. All maps

$$\varinjlim_i Z_i(\mathbb{A}^1 \times U) \to \varinjlim_i Z_i(U)$$

are weak equivalences, so that all maps

$$F(\varinjlim_i Z_i)(\mathbb{A}^1 \times U) \to F(\varinjlim_i Z_i)(U)$$

are weak equivalences. □

This Lemma leads to the following construction of motivic fibrant replacements in T-spectra.

Suppose that X is a T-spectrum, and consider the composition

$$X \xrightarrow{j_{\mathbb{A}^1}} F_{\mathbb{A}^1}X \xrightarrow{\tilde{\eta}} Q_\ell F_{\mathbb{A}^1}X = \varinjlim_k \Omega_T^k F_{\mathbb{A}^1}X[k] \xrightarrow{j} FQ_\ell F_{\mathbb{A}^1}X,$$

where $j : Y \to FY$ is the natural strict fibrant replacement for the Nisnevich topology and $j_{\mathbb{A}^1} : X \to F_{\mathbb{A}^1}X$ is the natural strict motivic fibrant replacement.

Lemma: Suppose that T is compact. Then the map

$$j \cdot \tilde{\eta} \cdot j_{\mathbb{A}^1} : X \to FQ_\ell F_{\mathbb{A}^1}X$$

is a motivic stable equivalence, and the T-spectrum $FQ_\ell F_{\mathbb{A}^1}X$ is motivic fibrant.

Proof: All strict motivic equivalences and all stable equivalences are motivic stable equivalences, so that the map $j \cdot \tilde{\eta} \cdot j_{\mathbb{A}^1}$ is a motivic stable equivalence.

All pointed simplicial presheaves $\Omega_T^k F_{\mathbb{A}^1} X^{n+k}$ are motivic fibrant, so that

$$Q_\ell F_{\mathbb{A}^1} X^n = \varinjlim_k \Omega_T^k F_{\mathbb{A}^1} X^{n+k}$$

satisfies the conditions of the Lemma above. This means that the horizontal maps in all diagrams

$$\begin{array}{ccc} (Q_\ell F_{\mathbb{A}^1} X)^n(U) & \longrightarrow & F(Q_\ell F_{\mathbb{A}^1} X)^n(U) \\ \cong \downarrow & & \downarrow \\ \Omega_T (Q_\ell F_{\mathbb{A}^1} X)^{n+1}(U) & \longrightarrow & \Omega_T (F(Q_\ell F_{\mathbb{A}^1} X)^{n+1})(U) \end{array}$$

are weak equivalences and that $F Q_\ell F_{\mathbb{A}^1} X$ is strictly motivic fibrant. $\square$

Write

$$Q_{\mathbb{A}^1} X = F Q_\ell F_{\mathbb{A}^1} X$$

and

$$\eta_{\mathbb{A}^1} = j \cdot \tilde{\eta} \cdot j_{\mathbb{A}^1} : X \to Q_{\mathbb{A}^1} X.$$

Remark: The use of the Nisnevich descent in the verification that $Q_{\mathbb{A}^1} X$ is motivic fibrant avoids an extremely delicate point: it is not true in general that a motivic equivalence $X \to Y$ of globally fibrant simplicial presheaves induces a motivic equivalence $\Omega X \to \Omega Y$.

We now have the following corollary:

Corollary: A map $f : X \to Y$ of T-spectra is motivic stable equivalence if and only if the induced map $F_{\mathbb{A}^1}X \to F_{\mathbb{A}^1}Y$ is a stable equivalence.

Proof: Consider the diagram

$$\begin{array}{ccc} F_{\mathbb{A}^1}X & \xrightarrow{j\cdot\tilde{\eta}} & FQ_\ell F_{\mathbb{A}^1}X \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ F_{\mathbb{A}^1}Y & \xrightarrow[j\cdot\tilde{\eta}]{} & FQ_\ell F_{\mathbb{A}^1}Y \end{array}$$

The horizontal maps are stable equivalences for the Nisnevich topology, so that one of the vertical maps f_* is a stable equivalence if and only if the other is a stable equivalence. But finally,

$$f_* : Q_{\mathbb{A}^1}X \to Q_{\mathbb{A}^1}Y$$

is a map of stably fibrant T-spectra, and is therefore a stable equivalence if and only if it is a strict equivalence, and this is true if and only if f is a motivic stable equivalence. □

Motivic stable category: $T = S^1 \wedge K$, K compact

This is the context where we have presheaves $\pi_{s,t}X$ of stable homotopy groups for an $(S^1 \wedge K)$-spectrum X. Recall that

$$\pi_{s,t}X(U) = \varinjlim_n [S^{s+n} \wedge K^{t+n}, X^n]_U$$

where the morphisms in the homotopy category are calculated over the S-scheme U.

We have seen that a map $f : X \to Y$ of $(S^1 \wedge K)$-spectra is a stable equivalence if and only if it induces isomorphisms

$$f_* : \tilde{\pi}_{s,t}X \xrightarrow{\cong} \tilde{\pi}_{s,t}Y$$

in sheaves of stable homotopy groups. Here $\tilde{\pi}_{s,t}X$ is the sheaf associated to the presheaf $\pi_{s,t}X$, for the Nisnevich topology.

Alternatively, $f : X \to Y$ is a stable equivalence if and only if all induced maps

$$f_* : \pi_{s,t}X \xrightarrow{\cong} \pi_{s,t}Y$$

of presheaves of stable homotopy groups are isomorphisms. This is a consequence of the compactness of K.

We have also seen that $f : X \to Y$ is a motivic stable equivalence if and only if the map $F_{\mathbb{A}^1}X \to F_{\mathbb{A}^1}Y$ is a stable equivalence for the Nisnevich topology.

The motivic stable homotopy groups of an $(S^1 \wedge K)$-spectrum X are the presheaves stable homotopy groups

$$\pi_{s,t}F_{\mathbb{A}^1}X.$$

Then we have the following:

Lemma: A map $f : X \to Y$ of $(S^1 \wedge K)$-spectra is a motivic stable equivalence if and only if it induces isomorphisms

$$f_* : \pi_{s,t}F_{\mathbb{A}^1}X \xrightarrow{\cong} \pi_{s,t}F_{\mathbb{A}^1}Y$$

in all presheaves of motivic stable homotopy groups.

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a motivic strict fibre sequence, and form the diagram

$$\begin{array}{ccccc} F & \xrightarrow{i} & X & \xrightarrow{p} & Y \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_{\mathbb{A}^1}} \\ F' & \xrightarrow[i']{} & X' & \xrightarrow[p']{} & F_{\mathbb{A}^1}Y \end{array}$$

where j is a strict motivic trivial cofibration, p' is a strict motivic fibration and j_* is the induced map on fibres. The motivic model structure for pointed simplicial presheaves is *proper* (Theorem A), so the map j_* is a strict motivic equivalence and can be identified with a strict motivic fibrant model for F. It follows that the sequence

$$F_{\mathbb{A}^1}F \xrightarrow{i_*} F_{\mathbb{A}^1}X \xrightarrow{p_*} F_{\mathbb{A}^1}Y$$

is a strict fibre sequence of T-spectra, and hence induces a long exact sequence

$$\dots \to \pi_{s,t}F_{\mathbb{A}^1}F \xrightarrow{i_*} \pi_{s,t}F_{\mathbb{A}^1}X \xrightarrow{p_*} \pi_{s,t}F_{\mathbb{A}^1}Y \xrightarrow{\partial} \pi_{s-1,t}F_{\mathbb{A}^1}F \to \dots$$

in presheaves of motivic stable homotopy groups.

Corollary: (of the paragraph above) The motivic stable model structure on $(S^1 \wedge K)$-spectra is proper.

Lemma 1: Every strict motivic fibre sequence is a cofibre sequence, up to motivic stable equivalence.

Proof: Suppose given a strict motivic fibre sequence

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

Then p is a strict fibration for the Nisnevich topology, and so the canonical map

$$X/F \to Y$$

is a stable equivalence of $(S^1 \wedge K)$-spectra, and is therefore a motivic stable equivalence. □

Lemma 2: Suppose that

$$\begin{array}{ccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \end{array}$$

is a comparison of level cofibre sequences of $(S^1 \wedge K)$-spectra. If any two of the maps f_1, f_2 and f_3 are motivic stable equivalences, then so is the third.

Proof: It is harmless to assume that all objects are cofibrant.

From the proof of the corresponding result (Lemma 14) for the stable category for $(S^1 \wedge K)$-spectra in Lecture 007, there are natural stable (hence motivic stable) equivalences It follows that there are natural stable equivalences

$$d(\Sigma_K B^{*,*})[-1] \wedge S^1 \simeq \Sigma_T B[-1] \simeq B.$$

All functors in this picture preserve cofibrant objects and level cofibre sequences. Suppose now that

Z is motivic stably fibrant. Then the diagram

$$\begin{array}{ccccc} \mathbf{hom}(B_3, Z) & \longrightarrow & \mathbf{hom}(B_2, Z) & \longrightarrow & \mathbf{hom}(B_1, Z) \\ {\scriptstyle f_3^*}\downarrow & & \downarrow{\scriptstyle f_2^*} & & \downarrow{\scriptstyle f_1^*} \\ \mathbf{hom}(A_3, Z) & \longrightarrow & \mathbf{hom}(A_2, Z) & \longrightarrow & \mathbf{hom}(A_1, Z) \end{array}$$

is a comparison of fibre sequences of infinite loop spaces. Thus if any two of the vertical maps are (stable) equivalences, then so is the third. □

Lemma 3: Every cofibre sequence is a strict motivic fibre sequence, up to motivic stable equivalence.

Proof: We've seen the argument before:

Suppose given a cofibre sequence

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

and take a factorization

$$\begin{array}{ccc} B & \xrightarrow{j} & Z \\ & {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle p} \\ & & B/A \end{array}$$

where j is a strict motivic equivalence and p is a strict motivic fibration. Let F be the fibre of p. The canonical map $p_* : Z/F \to B/A$ is a (motivic) stable equivalence by Lemma 1, and there is a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & B/A \\ \downarrow & & \simeq \downarrow j & & \downarrow j_* \\ F & \longrightarrow & Z & \longrightarrow & Z/F \end{array}$$

But then $p_* i_* = 1$ so that j_* is a (motivic) stable equivalence. It follows from Lemma 2 that the map $A \to F$ is a motivic stable equivalence. $\square$

Corollary: Every level cofibre sequence

$$A \to B \to B/A$$

has a naturally associated long exact sequence

$$\dots \pi_{s,t} F_{\mathbb{A}^1} A \to \pi_{s,t} F_{\mathbb{A}^1} B \to \pi_{s,t} F_{\mathbb{A}^1}(B/A) \xrightarrow{\partial} \pi_{s-1,t} F_{\mathbb{A}^1} A \to \dots .$$

of presheaves of stable homotopy groups.

Corollary: There are natural isomorphisms

$$\pi_{s+1,t} F_{\mathbb{A}^1}(Y \wedge S^1) \cong \pi_{s,t} F_{\mathbb{A}^1} Y$$

for all $(S^1 \wedge K)$-spectra Y.

Corollary: (additivity) Suppose that X and Y are $(S^1 \wedge K)$-spectra. Then the canonical map

$$X \vee Y \to X \times Y$$

is a stable equivalence.

References

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[3] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, 90:45–143 (2001), 1999.