Lecture 010 (December 2, 2005)

Symmetric spaces

For now, $\mathcal{C}$ will be an arbitrary small Grothendieck site.

A *symmetric space* X consists of pointed simplicial presheaves X^n, $n \geq 0$ on $\mathcal{C}$ with symmetric group actions

$$\Sigma_n \times X^n \to X^n$$

A morphism $f : X \to Y$ consists of pointed simplicial presheaf morphisms $X^n \to Y^n$, $n \geq 0$, which respect the symmetric group actions. The category of symmetric spaces will be denoted (following [2], see also [3]) by $s\,\mathrm{Pre}(\mathcal{C})^{\Sigma}_{*}$.

Examples:

1) Suppose that T is a pointed simplicial presheaf. Then the sequence

$$S^0,\ T,\ T \wedge T,\ T^3,\ \dots$$

forms a symmetric space, which will be denoted by S_T.

More generally, any suspension object $\Sigma^{\infty}_{T} K = S_T \wedge K$ associated to a pointed simplicial presheaf K is a symmetric space.

2) Write Γ for the category of finite pointed sets and pointed functions between them. A Γ-*space* is a functor $A : \Gamma \to s\,\mathrm{Pre}(\mathcal{C})_*$ defined on the category of finite pointed sets and taking values in pointed simplicial presheaves. Certainly, the list of objects. Let $\underline{n}$ denote the ordinal number $\mathbf{n}$, pointed by 0. Then certainly the list

$$A(\underline{0}),\ A(\underline{1}),\ A(\underline{2}),\ \dots$$

associated to a Γ-space A forms a symmetric space, but this is not the one used in practice. The symmetric space of interest is the sequence

$$dA(S^0),\ dA(S^1),\ dA(S^2),\ \dots$$

where d means diagonal of a bisimplicial (or multi-simplicial) set. The reason one cares is that any finite pointed sets K, L determine an canonical map

$$K \wedge A(L) \to A(K \wedge L)$$

so it follows that there are canonical maps

$$S^k \wedge A(S^n) \to A(S^{k+n}),$$

which induce pointed simplicial presheaf maps

$$S^k \wedge dA(S^n) \to dA(S^{k+n}). \tag{1}$$

In particular, the list of spaces $dA(S^n)$ (denoted by $A(S)$) has the structure of a spectrum, but more is true: $dA(S^n)$ has an action by the symmetric group Σ_n defined by permuting the factors of

$$S^n = S^1 \wedge \cdots \wedge S^1,$$

and each map (1) is $(\Sigma_k \times \Sigma_n)$-equivariant.

There is a Γ-space $\Phi(S, Y)$ associated to a spectrum Y, defined by

$$\Phi(S, Y)(\underline{n}) = \mathbf{hom}(S^{\times n}, Y).$$

The functor $\Phi(S, \ )$ is right adjoint to the functor $A \mapsto A(S)$, and for suitable model structures these two functors determine an equivalence of the stable homotopy category of connective spectra (ie. $\pi_n X = 0$ for $n < 0$)with the homotopy category of Γ-spaces ... at least for ordinary spectra and ordinary Γ-spaces (ie. not presheaves). Somebody should fill this gap — it should be easy to do.

Much more detail can be found in [1].

There is a basic functorial tensor product construction for symmetric spaces that is essentially the answer to life, the universe and everything.

Given symmetric spaces X and Y, their tensor product $X \otimes Y$ is specified in degree n by

$$(X \otimes Y)^n = \bigvee_{r+s=n} \Sigma_n \otimes_{\Sigma_r \times \Sigma_s} (X_r \wedge Y_s).$$

Here,

$$\Sigma_n \otimes_{\Sigma_r \times \Sigma_s} (X_r \wedge Y_s)$$

has the Σ_n-structure which is induced from the $(\Sigma_r \times \Sigma_s)$-structure on $X_r \wedge Y_s$ along the canonical inclusion

$$i : \Sigma_r \times \Sigma_s \subset \Sigma_{r+s} = \Sigma_n$$

This means that Σ_n-equivariant maps

$$\Sigma_n \otimes_{\Sigma_r \times \Sigma_s} (X_r \wedge Y_s) \to W$$

can be identified with $(\Sigma_r \times \Sigma_s)$-equivariant maps

$$X_r \wedge Y_s \to i_* W,$$

where $i_* W$ denotes restriction of the Σ_n-structure of Z to a $(\Sigma_r \times \Sigma_s)$-structure along the inclusion i.

A map of symmetric spaces $X \otimes Y \to Z$ therefore consists of $(\Sigma_r \times \Sigma_s)$-equivariant maps

$$X_r \wedge Y_s \to i_* Z_{r+s} = Z_{r+s}$$

for all $r, s \geq 0$.

An example of such a map is the canonical map

$$\otimes : S_T \otimes S_T \to S_T$$

which is determined by the canonical isomorphisms

$$S^r \wedge S^s \cong S^{r+s}$$

Write $c_{r,s} \in \Sigma_{r+s}$ for the shuffle which is defined by

$$c_{r,s}(i) = \begin{cases} s+i & i \leq r \\ i-r & i > r \end{cases}$$

It follows that $c_{s,r} c_{r,s} = 1$.

The canonical twist automorphism

$$\tau : X \otimes Y \to Y \otimes X$$

is uniquely determined by the composites

$$X^r \wedge Y^s \xrightarrow{\tau} Y^r \wedge X^s \to (Y \otimes X)^{r+s} \xrightarrow{c_{s,r}} (Y \otimes X)^{r+s}.$$

Note that we have to multiply by the shuffle $c_{s,r}$ to make the composite equivariant for the inclusion $\Sigma_r \times \Sigma_s \subset \Sigma_{r+s}$. One checks that the composites

$$X \otimes Y \xrightarrow{\tau} Y \otimes X \xrightarrow{\tau} X \otimes Y$$

are identities.

The tensor product $(X, Y) \mapsto X \otimes Y$ is symmetric monoidal. The map $\otimes : S_T \otimes S_T \to S_T$ gives S_T the structure of an abelian monoid in the category of symmetric spaces.

A *symmetric T-spectrum* X is a symmetric space with the structure

$$m_X : S_T \otimes X \to X$$

of a module over S_T. This means that X comes equipped with (bonding) maps

$$\sigma_{1,s} : T \wedge X^s \to X^{1+s}$$

such that all composite bonding maps

$$T^r \wedge X^s \to X^{r+s}$$

are equivariant for the inclusion $\Sigma^r \times \Sigma^s \subset \Sigma^{r+s}$. There is an obvious category of such things, which we denote by $\mathrm{Spt}_T^{\Sigma}(\mathcal{C})$.

The category of symmetric T-spectra has a symmetric monoidal smash product. Given symmetric T-spectra X, Y, the *smash product* $X \wedge Y$ is defined by the coequalizer

$$S_T \otimes X \otimes Y \rightrightarrows X \otimes Y \to X \wedge Y.$$

where the arrows $S_T \otimes X \otimes Y \rightrightarrows X \otimes Y$ are

$m_X \otimes Y$ and the composite

$$S_T \otimes X \otimes Y \xrightarrow{\tau \otimes Y} X \otimes S_T \otimes Y \xrightarrow{X \otimes m_Y} X \otimes Y.$$

If X is a symmetric space, then $S_T \otimes X$ has the structure of a symmetric T-spectrum. $S_T \otimes X$ is the free symmetric T-spectrum on X: there are natural bijections

$$\hom_{\mathrm{Spt}_T^{\Sigma}(\mathcal{C})}(S_T \otimes X, Y) \cong \hom_{s\,\mathrm{Pre}(\mathcal{C})_*^{\Sigma}}(X, Y).$$

If K is a pointed simplicial presheaf and $n \geq 0$ there is a symmetric space $G_n K$ with

$$(G_n K)^r = \begin{cases} * & r \neq n \\ \Sigma_n \otimes K = \vee_{\Sigma_n} K & r = n. \end{cases}$$

There is a natural bijection

$$\hom_{s\,\mathrm{Pre}(\mathcal{C})_*^{\Sigma}}(G_n K, W) \cong \hom_{s\,\mathrm{Pre}(\mathcal{C})_*}(K, W^n).$$

It follows that if we define a symmetric T-spectrum $F_n K$ by

$$F_n K = S_T \otimes G_n K,$$

then there is a natural bijection

$$\hom_{\mathrm{Spt}_T^{\Sigma}(\mathcal{C})}(F_n K, Z) \cong \hom_{s\,\mathrm{Pre}(\mathcal{C})_*}(K, Z^n).$$

for all $n \geq 0$, pointed simplicial presheaves K and symmetric T-spectra Z. Note that

$$F_0 K = \Sigma_T^{\infty} K.$$

There is, plainly, a forgetful functor

$$U : \mathrm{Spt}_T^\Sigma(\mathcal{C}) \to \mathrm{Spt}_T(\mathcal{C})$$

which forgets the symmetric group actions. I claim that U has a left adjoint

$$V : \mathrm{Spt}_T(\mathcal{C}) \to \mathrm{Spt}_T^\Sigma(\mathcal{C}).$$

V is constructed inductively, by using the layer filtration L_nX of a T-spectrum X. In effect, for shifted suspension spectra $\Sigma_T^\infty K[-n]$, it must be the case that

$$\begin{aligned}\hom_{\mathrm{Spt}_T^\Sigma(\mathcal{C})}(V\Sigma_T^\infty K[-n], Z) &\cong \hom_{\mathrm{Spt}_T(\mathcal{C})}(\Sigma_T^\infty K[-n], UZ)\\ &\cong \hom_{s\,\mathrm{Pre}(\mathcal{C})_*}(K, Z^n)\end{aligned}$$

for V to be a left adjoint, so that there must be natural isomorphisms

$$V\Sigma_T^\infty K[-n] \cong F_nK.$$

In fact, it's not hard to see at all that $(F_nK)^n = \Sigma_n \otimes K$, and that the map $e : K \to \Sigma_n \otimes K$ given by the inclusion of the summand corresponding to the identity $e \in \Sigma_n$ induces a natural map

$$\eta : \Sigma_T^\infty K[-n] \to UF_nK.$$

The map η satisfies a universal property: given a map $f : K \to Z^n$ where Z is a symmetric T-spectrum, there is a unique map $f_* : F_n K \to Z$ of symmetric T-spectra such that the diagram

$$\begin{array}{ccc} \Sigma^\infty K[-n] & \xrightarrow{\eta} & UF_nK \\ & \searrow^{f_*} & \downarrow{\scriptstyle Uf_*} \\ & & UZ \end{array}$$

commutes.

Recall that the layer filtration $L_nX \subset X$ of a T-spectrum X is constructed by a sequence of pushouts of the form

$$\begin{array}{ccc} \Sigma_T^\infty(T \wedge X^n)[-n-1] & \longrightarrow & L_nX \\ \downarrow & & \downarrow \\ \Sigma_T^\infty X^{n+1}[-n-1] & \longrightarrow & L_{n+1}X \end{array}$$

and so $VL_{n+1}X$ can be inductively specified by the pushouts

$$\begin{array}{ccc} F_{n+1}(T \wedge X^n) & \longrightarrow & VL_nX \\ \downarrow & & \downarrow \\ F_{n+1}X^{n+1} & \longrightarrow & VL_{n+1}X \end{array}$$

and canonical maps $\eta : L_nX \to UVL_nX$. Write $VX = \varinjlim_n VL_nX$. Then the resulting functor

$$V : \mathrm{Spt}_T(\mathcal{C}) \to \mathrm{Spt}_T^\Sigma(\mathcal{C})$$

is left adjoint to the forgetful functor U.

The projective model structure

A map $f : X \to Y$ of symmetric T-spectra is said to be a *level weak equivalence* if all component maps $X^n \to Y^n$ are local weak equivalences of simplicial presheaves.

Say that a map $p : X \to Y$ of symmetric T-spectra is a *projective fibration* if all maps $p : X^n \to Y^n$ are global fibrations of pointed simplicial presheaves. A map $i : A \to B$ is said to be a *projective cofibration* if it has the LLP with respect to all maps which are projective fibrations and level weak equivalences.

Suppose that $A \to B$ is a cofibration of pointed simplicial presheaves. Then $F_nA \to F_nB$ is a projective cofibration for all n. This map is a trivial projective cofibration if $A \to B$ is also a weak equivalence.

Lemma: The category $\mathrm{Spt}_T^{\Sigma}(\mathcal{C})$, with the projective fibrations, projective cofibrations and level weak equivalences, satisfies the conditions for a proper closed simplicial model category. This model structure is cofibrantly generated.

Proof: A map $p : X \to Y$ is a projective fibration if and only if it has the RLP with respect to all maps $F_nA \to F_nB$ which are induced by α-bounded trivial cofibrations $A \to B$ of pointed simplicial presheaves. The map p is a projective trivial fibration if and only if it has the RLP with respect to all maps $F_nC \to F_nD$ which are induced by α-bounded cofibrations $C \to D$ of pointed simplicial presheaves. It follows that any map $f : X \to Y$ of symmetric T-spectra has factorizations

$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle j}\nearrow & & \searrow{\scriptstyle p} & \\ X & & \xrightarrow{\quad f \quad} & & Y \\ & {\scriptstyle i}\searrow & & \nearrow{\scriptstyle q} & \\ & & W & & \end{array}$$

such that p is a projective fibration and j is a projective trivial cofibration which has the LLP with respect to all projective fibrations, and q is a projective trivial fibration and i is a projective cofibration. This proves the factorization axiom **CM5**.

In addition, if $i : A \to B$ is a projective trivial cofibration, then i has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

such that p is a projective fibration and j is a projective trivial cofibration which has the LLP with respect to all fibrations. But then, p is a level equivalence, so the lifting exists in the diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

so that i is a retract of j and therefore has the LLP with respect to all projective fibrations. This proves the lifting axiom **CM4**. The remaining closed model axioms are trivial to verify.

The function complex construction $\mathbf{hom}(X, Y)$ is the obvious one, and the axiom **SM7** follows from the corresponding axiom for the strict model structure on T-spectra.

Every projective cofibration is in the saturation of the maps $F_nA \to F_nB$ which are induced by cofibrations $A \to B$ of pointed simplicial sets. The maps $F_nA \to F_nB$ are all level cofibrations, and so every projective cofibration is a level cofibration. Properness is an easy consequence of this observation together with properness for pointed simplicial presheaves.

The maps $F_nA \to F_nB$ induced by α-bounded cofibrations (respectively α-bounded trivial cofibrations) generate the projective cofibrations (resp. trivial projective cofibrations) of $\mathrm{Spt}_T^{\Sigma}(\mathcal{C})$. □

Corollary: The functor V takes cofibrations (respectively strictly trivial cofibrations) of T-spectra to projective cofibrations (respectively trivial projective cofibrations) of symmetric T-spectra.

Proof: This is an obvious adjunction argument. □

Remark: I presently don't know how to localize the projective model structure on symmetric T-spectra. The problem is that it's not obvious that projective cofibrations $X \to Y$ pull back to projective cofibrations $A \cap X \to A$ over subobjects $A \subset Y$. The corresponding fact for T-spectra is kind of special.

The injective model structure

Lemma 1: Suppose that α is an infinite cardinal such that $\alpha > |\mathcal{C}|$. Suppose given level cofibrations

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

such that the vertical map i is a level equivalence and the object A is α-bounded. Then there is an α-bounded subobject $B \subset Y$ with $A \subset B$ such that the induced map $B \cap X \to B$ is a level equivalence.

Proof: The proof is essentially the same as that of the corresponding result for simplicial presheaves.

In the natural commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Ex}^{\infty} X & & \\ & & & \searrow^{j_i} & \\ {\scriptstyle i}\downarrow & & \downarrow & & \mathrm{Ex}^{\infty} X \times_{\mathrm{Ex}^{\infty} Y} \mathrm{Ex}^{\infty} Y^I \\ & & & \swarrow_{\pi_i} & \\ Y & \longrightarrow & \mathrm{Ex}^{\infty} Y & & \end{array}$$

the map j_i is a sectionwise cofibration and weak equivalence and π_i is a sectionwise fibration, in all levels. Pullback this factorization of $\mathrm{Ex}^{\infty} X \to$

Ex^∞ along the map $Y \to \mathrm{Ex}^\infty Y$ to find a factorization

$$\begin{array}{ccc} X & \xrightarrow{j_Y} & Z_Y \\ {\scriptstyle i}\downarrow & \swarrow {\scriptstyle \pi_Y} & \\ Y & & \end{array}$$

such that j_i is a sectionwise cofibration and a weak equivalence in all levels and π_i is sectionwise fibration in all levels. This construction respects all filtered colimits of maps i, so that

$$Z_Y = \varinjlim_{B \subset Y} Z_B,$$

where $\pi_B : Z_B \to B$ is the corresponding replacement for the map $B \cap X \to B$.

Note that Z_B is α-bounded if B is α-bounded.

The maps $Z_Y^n \to Y^n$ are local equivalences and local fibrations, and therefore have the local RLP with respect to all inclusions $\partial\Delta^m \subset \Delta^m$. Write $B_0 = A$. Then it follows that every lifting problem

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & Z^n_{B_0}(U) \\ \downarrow & & \downarrow \\ \Delta^m & \longrightarrow & B_0^m(U) \end{array}$$

has a local solution over some α-bounded $B' \subset Y$ with $B_0 \subset B'$. There are at most α such lifting problems, so there is an α-bounded $B_1 \subset Y$ with

$B_0 \subset B_1$ such that every lifting problem as above has a local solution over B_1.

Continue inductively to produce a countable sequence

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

such that every lifting problem

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & Z^n_{B_i}(U) \\ \downarrow & & \downarrow \\ \Delta^m & \longrightarrow & B^m_i(U) \end{array}$$

has a local solution over B_{i+1}. Let $B = \cup_i B_i$. Then it follows that the map $\pi_B : Z_B \to B$ is a local trivial Kan fibration, so that the map $B \cap X \to B$ is a local equivalence. □

Say that a map $p : X \to Y$ of symmetric T-spectra is an *injective fibration* if it has the RLP with respect to all level trivial cofibrations.

Lemma 2: A map $\pi : Z \to W$ has the RLP with respect to all level cofibrations if and only if it has the RLP with respect to all α-bounded cofibrations.

Proof: Suppose that π has the RLP with respect to all α-bounded cofibrations, and suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \longrightarrow & W \end{array}$$

where i is a level cofibration. Consider the poset of all partial lifts

$$\begin{array}{ccc} A & \longrightarrow & Z \\ \downarrow & \overset{\theta}{\nearrow} & \\ B' & & \downarrow \\ \downarrow & & \\ B & \longrightarrow & W \end{array}$$

where the composite of the cofibrations $A \subset B' \subset B$ is i. If $B' \neq B$, B is a union of its α-bounded subobjects, and so there is an α-bounded $C \subset B$ such that $B' \neq B' \cup C$. There is a pushout of level cofibrations

$$\begin{array}{ccc} C \cap B' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ C & \longrightarrow & C \cup B' \end{array}$$

The map π has the RLP with respect to all α-bounded cofibrations, such as $C \cap B' \subset C$, so that the partial lift θ can be extended to a partial lift $\theta' : C \cup B' \to Z$. A Zorn's Lemma argument on the poset of partial lifts finishes the argument: the poset of partial lifts is non-empty and inductively ordered, and the maximal elements are full lifts. □

Lemma 3: Suppose that $\pi : Z \to W$ has the RLP property with respect to all cofibrations. Then π is a trivial fibration in all levels.

Proof: π has the RLP with respect to all maps $F_n A \to F_n B$ induced by cofibrations $A \to B$ of pointed simplicial presheaves. □

Lemma 4: Suppose that $p : X \to Y$ has the RLP with respect to all α-bounded level trivial cofibrations and that p is a level equivalence. Then p has the RLP with respect to all cofibrations.

Proof: By Lemma 2 the map p has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i'} & W \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

where π has the RLP with respect to all cofibrations, and i is a level cofibration. The map π is a level equivalence by Lemma 3, so that i' is a level equivalence. Given a diagram

$$\begin{array}{ccc} E & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle i'} \\ & & W \\ & \overset{\theta}{\nearrow} & \Big\downarrow{\scriptstyle \pi} \\ F & \longrightarrow & Y \end{array}$$

with $E \to F$ an α-bounded cofibration, the dotted arrow θ exists, and the image $\theta(F) \subset D$ is α-bounded. By Lemma 1 there is an α-bounded object $D' \subset W$ with $\theta(F) \subset D'$ such that the induced map $D' \cap X \to D'$ is a level equivalence. It follows that any diagram

$$\begin{array}{ccc} E & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle q} \\ F & \longrightarrow & Y \end{array}$$

with $E \to F$ an α-bounded cofibration has a factorization

$$\begin{array}{ccccc} E & \longrightarrow & D' \cap X & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow & \nearrow & \Big\downarrow{\scriptstyle p} \\ F & \longrightarrow & D' & \longrightarrow & Y \end{array}$$

where the dotted arrow exists since $D' \cap X \to D'$ is an α-bounded trivial cofibration.

It follows that p has the RLP with respect to all α-bounded level cofibrations, and hence with respect to all level cofibrations by Lemma 2. $\square$

Lemma 5: Suppose that a map $p : X \to Y$ of symmetric T-spectra has the RLP with respect to all α-bounded level trivial cofibrations. Then p is an injective fibration.

Proof: We show that every level trivial cofibration $i : A \to B$ has the LLP with respect to p.

The map i has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & C \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle q} \\ & & B \end{array}$$

where j is in the saturation of the set of α-bounded level trivial cofibrations and q has the RLP with respect to all α-bounded level trivial cofibrations. The map q is also a level equivalence, and so q has the RLP with respect to all cofibrations by Lemma 4. From the diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & C \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle q} \\ B & \xrightarrow[1]{} & B \end{array}$$

we see that i is a retract of j, and therefore has the advertised left lifting property with respect to p. $\square$

The smash $X \wedge K$ of a symmetric T-spectrum X with a pointed simplicial presheaf K makes perfect sense, and one forms the function complex $\mathbf{hom}(X, Y)$ for symmetric T-spectra X, Y by requiring that

$$\hom(X, Y)_n = \hom(X \wedge \Delta^n_+, Y).$$

Proposition 1: The level weak equivalences, level cofibrations and injective fibrations together give the category $\mathrm{Spt}^\Sigma_T(\mathcal{C})$ of symmetric T-spectra the structure of a proper closed simplicial model category. This model structure is cofibrantly generated, by the α-bounded level cofibrations and the α-bounded level trivial cofibrations.

The model structure for symmetric T-spectra of the Proposition is often called the *injective model structure.*

Localized injective model structures

Suppose that S is a set of level cofibrations of symmetric T-spectra which includes the set J of α-bounded level trivial cofibrations. Suppose that all induced maps

$$(A \wedge \Delta^m_+) \cup (B \wedge \partial\Delta^m_+) \to B \wedge \Delta^m_+$$

are in S for each $A \to B$ in S.

Suppose that α is a cardinal such that $\alpha > |\operatorname{Mor}(\mathcal{C})|$. Suppose also that $\alpha > |B|$ for all morphisms $i : A \to B$ appearing in the set S and that $\alpha > |S|$. Choose a cardinal λ such that $\lambda > 2^\alpha$.

Suppose that $f : X \to Y$ is a morphism of symmetric T-spectra. Define a functorial system of factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_s} & E_s(f) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_s} \\ & & Y \end{array}$$

of the map f indexed on all ordinal numbers $s < \lambda$ as follows:

1) Given the factorization (f_s, i_s) define the factorization (f_{s+1}, i_{s+1}) by requiring that the di-

agram

$$\begin{array}{ccc} \bigvee_{\mathbf{D}} A & \xrightarrow{(\alpha_{\mathbf{D}})} & E_s(f) \\ {\scriptstyle \vee i}\downarrow & & \downarrow \\ \bigvee_{\mathbf{D}} B & \rightarrow & E_{s+1}(f) \end{array}$$

is a pushout, where the wedge is indexed over all diagrams $\mathbf{D}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathbf{D}}} & E_s(f) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f_s} \\ B & \xrightarrow[\beta_{\mathbf{D}}]{} & Y \end{array}$$

with $i : A \to B$ in the set S. Then the map i_{s+1} is the composite

$$X \xrightarrow{i_s} E_s(f) \xrightarrow{g_*} E_{s+1}(f)$$

2) If s is a limit ordinal, set $E_s(f) = \varinjlim_{t<s} E_s(f)$.

Set $E_\lambda(f) = \varinjlim_{s<\lambda} E_s(f)$. Then there is an induced factorization

$$\begin{array}{ccc} X & \xrightarrow{i_\lambda} & E_\lambda(f) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_\lambda} \\ & & Y \end{array}$$

of the map f. Then i_λ is a cofibration. The map f_λ has the right lifting property with respect to the cofibrations $i : A \to B$ in S by a standard argument, since any map $\alpha : A \to E_\lambda(f)$ must factor through some $E_s(f)$ by the choice of cardinal λ.

Write $L(X) = E_\lambda(c)$ for the result of this construction when applied to the canonical map $c : X \to *$. Then we have the following:

Lemma 6:

1) Suppose that $t \mapsto X_t$ is a diagram of level cofibrations indexed by any cardinal $\gamma > 2^\alpha$. Then the natural map

$$\varinjlim_{t<\gamma} L(X_t) \to L(\varinjlim_{t<\gamma} X_t)$$

is an isomorphism.

2) The functor $X \mapsto L(X)$ preserves level cofibrations.

3) Suppose that ζ is a cardinal with $\zeta > \alpha$, and let $\mathcal{F}_\zeta(X)$ denote the filtered system of subobjects of X having cardinality less than ζ. Then the natural map

$$\varinjlim_{Y \in \mathcal{F}_\zeta(X)} L(Y) \to L(X)$$

is an isomorphism.

4) If $|X| < 2^\omega$ where $\omega \geq \alpha$ then $|L(X)| < 2^\omega$.

5) Suppose that U, V are subobjects of a presheaf of spectra X. Then the natural map

$$L(U \cap V) \to L(U) \cap L(V)$$

is an isomorphism.

The proof of Lemma 6 is the same as the corresponding result for T-spectra.

A map is said to be S-injective if it has the RLP with respect to all members of S, and an object X is S-injective if the map $X \to *$ is S-injective. Note that all S-injective objects are injective fibrant. By construction, LX is S-injective.

Say that a morphism $f : X \to Y$ of $\mathrm{Spt}_T^{\Sigma}(C)$ is an L-*equivalence* if it induces a weak equivalence

$$f^* : \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

of simplicial sets for all S-injective objects Z.

Every level equivalence (ie. equivalence for the injective structure) is an L-equivalence.

Lemma: The class of cofibrations which are L-equivalences is closed under pushout.

Proof: Any pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

with i a cofibration induces a pullback diagram of fibrations

$$\begin{array}{ccc} \mathbf{hom}(D,Z) & \longrightarrow & \mathbf{hom}(B,Z) \\ \downarrow & & \downarrow i^* \\ \mathbf{hom}(C,Z) & \longrightarrow & \mathbf{hom}(A,Z) \end{array}$$

and a cofibration $i : A \to B$ is an L-equivalence if and only if the map

$$i^* : \mathbf{hom}(B,Z) \to \mathbf{hom}(A,Z)$$

is a trivial fibration of simplicial sets for all S-injective Z. □

Corollary A: All cofibrations in the saturation of the set S are L-equivalences.

Corollary B: A map $f : X \to Y$ is an L-equivalence if and only if the induced map $f_* : LX \to LY$ is a level equivalence.

Proof: Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & LX \\ f \downarrow & & \downarrow f_* \\ Y & \xrightarrow[j]{} & LY \end{array}$$

Then the horizontal maps j are L-equivalences by the previous Corollary, so that f is an L-equivalence if and only if f_* is an L-equivalence. But f_* is an

L-equivalence if and only if f_* is a level equivalence since LX and LY are injective fibrant. □

Lemma 7: Suppose given a cofibration $i : X \to Y$ which is an L-equivalence, and suppose that $A \subset Y$ is a 2^λ-bounded subobject, where λ is chosen as above. Then there is a 2^λ-bounded subobject $B \subset Y$ with $A \subset B$ and such that the cofibration $B \cap X \to B$ is an L-equivalence.

Proof: Write $B_0 = A$, and set $\kappa = 2^\lambda$.

Consider the diagram

$$\begin{array}{ccc} & & LX \\ & & \downarrow \\ LB_0 & \longrightarrow & LY \end{array}$$

Then the maps are level cofibrations [6.2] and $LX \to LY$ is a strict equivalence by assumption. LB_0 is κ-bounded by [6.4], so there is a κ-bounded subobject $C_1 \subset LY$ with $LB_0 \subset C_1$ such that $C_1 \cap LX \to C_1$ is a strict equivalence, by Lemma 1. Since C_1 is κ-bounded there is a κ-bounded subobject $B_1 \subset Y$ with $B_0 \subset B_1$ such that $C_1 \subset LB_1$ [6.3]. Proceeding inductively we find κ-bounded subobjects

$$C_1 \subset C_2 \subset \dots$$

of LY and κ-bounded subobjects

$$B_0 \subset B_1 \subset B_2 \subset \dots$$

indexed by $i < \kappa$, such that C_s and B_s are defined at limit ordinals s by colimits, and

$$LB_i \subset C_{i+1} \subset LB_{i+1}$$

and $C_i \cap LX \to C_i$ is a level weak equivalence.

Write $B = \varinjlim_{i<\kappa} B_i$. Then B is κ-bounded, and

$$L(B) = \varinjlim_{i<\kappa} L(B_i) = \varinjlim_{i<\kappa} C_i$$

by [6.1] and construction. Also

$$\begin{aligned} L(B \cap X) = L(B) \cap L(X) &= \varinjlim_{i<\kappa} L(B_i) \cap L(X) \\ &\cong \varinjlim_{i<\kappa} C_i \cap L(X) \end{aligned}$$

by [6.1], [6.5] and construction. It follows that the map

$$B \cap X \to B$$

is an L-equivalence. □

Lemma 8: The κ-bounded L-trivial cofibrations generate the full class of L-trivial cofibrations.

Proof: This is (by now) the usual argument.

Suppose that $i : A \to B$ is an L-trivial cofibration. Then i has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

where j is a cofibration in the saturation of the set of κ-bounded L-trivial cofibrations and p has the RLP with respect to all κ-bounded trivial cofibrations. p is also an L-equivalence.

It suffices to show that p has the right lifting property with respect to all κ-bounded cofibrations, for then p has the RLP with respect to all cofibrations, and then one can find the displayed lifting in the diagram

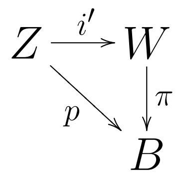
$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

so that i is a retract of j.

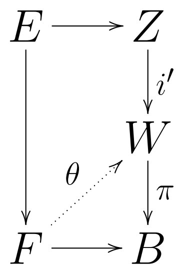
Suppose given a diagram

$$\begin{array}{ccc} E & \longrightarrow & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ F & \longrightarrow & B \end{array} \tag{2}$$

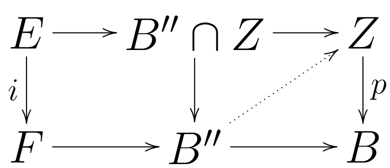
where i is a κ-bounded cofibration. Then p has a factorization



where i' is a cofibration and π has the RLP with respect to all cofibrations. Then the lift exists in the diagram



Then $\theta(F)$ is κ-bounded, so there is a κ-bounded subobject $B'' \subset W$ with $\theta(F) \subset B''$ such that $B'' \cap Z \to B''$ is an L-equivalence, by Lemma Lemma 7. It follows that the diagram (2) has a factorization



and the displayed lift exists by the assumptions on p. □

Say that a map $p : X \to Y$ is an *L-fibration* if it has the RLP with respect to all L-trivial cofibrations.

The following is a consequence of the proof of Lemma 8:

Corollary C: Suppose that $p : X \to Y$ is an L-fibration and an L-equivalence. Then p has the RLP with respect to all cofibrations.

Corollary D: A map $p : X \to Y$ is an L-fibration and an L-equivalence if and only if it is an injective fibration and a level equivalence.

Proof: If p is an injective fibration and a level equivalence, then it is an L-equivalence. Also, p has the RLP with respect to all cofibrations, so it is an L-fibration. $\square$

Theorem: The category $\mathrm{Spt}_T^{\Sigma}(\mathcal{C})$ of symmetric T-spectra, with the classes of cofibrations, L-equivalences and L-fibrations, satisfies the axioms for a left proper closed simplicial model category.

Proof: Every map $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where j is a cofibration and p is an injective fibration and a level equivalence. Then p is an L-fibration and an L-equivalence by Corollary D.

The other factorization axiom follows from Lemma 8: a map $p : X \to Y$ is an L-fibration if and only if it has the RLP with respect to all κ-bounded L-trivial cofibrations.

The lifting axiom **CM4** follows from Corollary C. The other axioms are trivial to verify. $\square$

Lemma: A symmetric T-spectrum X is L-fibrant if and only if it is S-injective.

Proof: An L-fibrant symmetric spectrum is clearly S-injective.

For the converse assertion, one can use the standard argument.

Alternatively, suppose that Z is S-injective and suppose given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & Z \\ {\scriptstyle i}\downarrow & \nearrow & \\ B & & \end{array}$$

where the map i is a cofibration and an L-equivalence. Then the induced map

$$i^* : \mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$$

is a trivial fibration of simplicial sets, and is therefore surjective. Specializing the surjectivity to simplicial degree 0 shows that the dotted arrow exists. □

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