Lecture 011 (December 2, 2005)

Stable homotopy theory of symmetric spectra

The stable model structure for symmetric T-spectra (and the corresponding stable homotopy category) arises from formally inverting the set S of level cofibrations which is generated over the set J of α-bounded level trivial cofibrations by the set of cofibrations obtained by applying the functor V to the cofibrant replacements of the maps

$$\Sigma_T^\infty T[-1-n] \to S_T[-n],$$

where S_T is the sphere spectrum for the T-spectrum category.

We also assume that if the cofibration $A \to B$ is in S and $K \to L$ is an α-bounded cofibration of pointed simplicial presheaves, then the induced cofibration

$$(B \wedge K) \cup (A \wedge L) \to B \wedge L$$

is in the set S.

Say that an L-equivalence for this theory is a *stable equivalence*, and that an L-fibration is a *stable fibration*. The cofibrations for this theory are the level cofibrations.

Note that an object Z is stably fibrant if and only if the Z is S-injective. Furthermore, Z is S-injective if and only if Z is injective and the underlying T-spectrum UZ is stably fibrant.

There is a natural isomorphism

$$\mathbf{hom}(VA, Z) \cong \mathbf{hom}(A, UZ),$$

and it follows that every stable equivalence $A \to B$ of T-spectra induces a stable equivalence $VA \to VB$ of symmetric T-spectra. Note that V also preserves level equivalences. The stable model structure for T-spectra is cofibrantly generated, and it follows that every map $f : X \to Y$ of symmetric T-spectra has a natural factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & X_s \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p_s} \\ & & Y \end{array}$$

where i is a stably trivial cofibration and Up_s is a stable fibration. Applying this construction to the map $X \to *$ determines a natural stably trivial cofibration $i : X \to X_s$ such that UX_s is stably fibrant. Finally, consider the composite

$$X \xrightarrow{i} X_s \xrightarrow{j} IX_s$$

where $j : X_s \to IX_s$ is the natural injective model. Then IX_s is of course injective and the map $j : X_s \to IX_s$ is a level equivalence so that UIX_s is stably fibrant. The composite ji is a stable equivalence, and therefore determines a natural stably fibrant model construction the category of symmetric T-spectra.

It follows that a map $f : X \to Y$ of symmetric T-spectra is a stable equivalence if and only if the induced map $IX_s \to IY_s$ is a level equivalence. This is exactly what is meant by a stable equivalence of symmetric T-spectra in the original sources [1], [2], [3].

The stable model structure for symmetric T-spectra given here *does not* coincide with any of those appearing in the literature, because the fibrations and cofibrations are not the same. All cofibrations of symmetric T-spectra are cofibrations for the present model structure, and the fibrations are a little more restrictive. In the original stable model structure for symmetric T-spectra — call it the HSS stable model structure for Hovey-Shipley-Smith — a map $p : X \to Y$ is a fibration of symmetric T-spectra if and only if the underlying map Up is a fibration of T-spectra. This is not quite true here: it will be shown that a map

$p : X \to Y$ of symmetric T-spectra is a stable fibration if and only if it is an injective fibration *and* it restricts to a stable fibration Up of T-spectra. We will see that this whole discussion requires extra assumptions for the basic results to be derived; in particular, T should be a suspension $S^1 \wedge K$ where K is compact up to equivalence.

We begin by describing a construction for symmetric spectra which has no analogue for spectra, namely a natural map $\tilde{\sigma} : X \to \Omega_T X[1]$ where $\Omega_T X$ is a real (not fake) loop space.

Suppose that K is a pointed simplicial presheaf and X is a symmetric T spectrum. Then $\Omega_K X = \mathbf{Hom}(K, X)$ is the symmetric T-spectrum with

$$\Omega_K X^n = \mathbf{Hom}(K, X^n).$$

The symmetric group actions and the bonding maps are defined by their adjoints. Specifically, if $\alpha \in \Sigma_n$, then

$$\alpha : \mathbf{Hom}(K, X^n) \to \mathbf{Hom}(K, X^n)$$

is the unique map such that the diagram

$$\begin{array}{ccc} \mathbf{Hom}(K, X^n) \wedge K & \xrightarrow{\alpha \wedge K} & \mathbf{Hom}(K, X^n) \wedge K \\ \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} \\ X^n & \xrightarrow[\alpha]{} & X^n \end{array}$$

commutes, where ev denotes the standard evaluation (ie. counit) map. Similarly, the bonding map

$$S^p \wedge \mathbf{Hom}(K, X^n) \xrightarrow{\sigma} \mathbf{Hom}(K, X^{p+n})$$

is the unique map such that the diagram

$$\begin{array}{ccc} S^p \wedge \mathbf{Hom}(K, X^n) \wedge K & \xrightarrow{\sigma \wedge K} & \mathbf{Hom}(K, X^n) \wedge K \\ {\scriptstyle S^p \wedge ev} \downarrow & & \downarrow {\scriptstyle ev} \\ S^p \wedge X^n & \xrightarrow[\sigma]{} & X^{p+n} \end{array}$$

commutes.

Suppose that X is a symmetric T-spectrum and that $n > 0$. The symmetric spectrum $X[n]$ has $X[n]^k = X^{n+k}$, and $\alpha \in \Sigma_k$ acts on $X[n]^k$ as the element $1 \oplus \alpha \in \Sigma_{n+k}$. The bonding map $\sigma : S^p \wedge X[n]^k \to X[n]^{p+k}$ is defined to be the composite

$$S^p \wedge X^{n+k} \xrightarrow{\sigma} X^{p+n+k} \xrightarrow{c(p,n) \oplus 1} X^{n+p+k}$$

Every element $\gamma \in \Sigma_n$ induces an isomorphism $\gamma \oplus 1 : X^{n+k} \to X^{n+k}$, and all diagrams

$$\begin{array}{ccccc} S^p \wedge X^{n+k} & \xrightarrow{\sigma} & X^{p+n+k} & \xrightarrow{c(p,n) \oplus 1} & X^{n+p+k} \\ {\scriptstyle S^p \wedge (\gamma \oplus 1)} \downarrow & & \downarrow {\scriptstyle (1 \oplus \gamma \oplus 1} & & \downarrow {\scriptstyle \gamma \oplus 1 \oplus 1} \\ X^p \wedge X^{n+k} & \xrightarrow[\sigma]{} & X^{p+n+k} & \xrightarrow[c(p,n) \oplus 1]{} & X^{n+p+k} \end{array}$$

commute. It follows that each $\gamma \in \Sigma_n$ induces a natural morphism of symmetric spectra $\gamma : X[n] \to X[n]$.

The map $\tilde{\sigma} : X^n \to \Omega_T X[1]^n = \Omega_T X^{1+n}$ is the adjoint of the bonding map $T \wedge X^n \to X^{1+n}$, in that the diagram

$$\begin{array}{ccccc} T \wedge X^n & \xrightarrow{T\wedge\tilde{\sigma}} & T \wedge \Omega_T X^{1+n} & \xrightarrow[\cong]{\tau} & \Omega_T X^{1+n} \wedge T \\ & \searrow^{\sigma} & & & \downarrow ev \\ & & & & X^{1+n} \end{array}$$

commutes. One shows that the diagram

$$\begin{array}{ccc} T^p \wedge X^n & \xrightarrow{T^p\wedge\tilde{\sigma}} & T^p \wedge \Omega^T X^{1+n} \\ \sigma \downarrow & & \downarrow \sigma \\ X^{p+n} & \xrightarrow[\tilde{\sigma}]{} & \Omega_T X^{1+p+n} \end{array}$$

commutes by checking adjoints.

Suppose that X is a symmetric T-spectrum. Define a system $k \mapsto Q^k_\Sigma X$, $k \geq 0$ by specifying that

$$Q^k_\Sigma X = \Omega^k_T IX[k],\ k \geq 0.$$

In particular, $Q^0_\Sigma X = IX$, where $j_\Sigma : X \to IX$ is the natural choice of injective model for X. There is a natural map $Q^k_\Sigma X \to Q^{k+1}_\Sigma X$ given by the map

$$\Omega^k_T \tilde{\sigma}[k] : \Omega^k_T IX[k] \to \Omega^k_T \Omega_T IX[1][k].$$

Set $Q_\Sigma X = I(\varinjlim_k Q^k_\Sigma X)$, and write $\eta : X \to$

$Q_\Sigma X$ for the natural composite

$$X \xrightarrow{j_\Sigma} IX = Q^0_\Sigma X \to \varinjlim_k Q^k_\Sigma X \xrightarrow{j_\Sigma} I(\varinjlim_k Q^k_\Sigma X).$$

Lemma: Suppose that $j : Y \to FY$ is the natural strictly fibrant model for T-spectra Y. Then there is a natural strict equivalence $\theta : FUX \to UIX$.

Proof: Recall that the injective structure on symmetric T-spectra is cofibrantly generated, and that the trivial cofibrations for the structure are generated by the α-bounded level trivial cofibrations. The class of trivial cofibrations for the strict structure on $\mathrm{Spt}_T(\mathcal{C})$ is generated by the cofibrations

$$\Sigma^\infty_T A[-n] \to \Sigma^\infty_T B[-n]$$

which are induced by the α-bounded trivial cofibrations $A \to B$. Applying the functor V to all such cofibrations gives the α-bounded level trivial cofibrations $F_n A \to F_n B$, up to isomorphism.

Recall that, in general, $IY = \varinjlim_{s<\lambda} I_s Y$, where $I_s Y \to I_{s+1} Y$ is defined by the pushout

$$\begin{array}{ccc} C & \longrightarrow & I_s Y \\ \downarrow & & \\ D & & \end{array}$$

for all diagrams arising from α-bounded level trivial cofibrations $C \to D$, and

$$I_t Y = \varinjlim_{s<t} I_s Y$$

at limit ordinals t.

The object FX for T-spectra X has a very similar definition:

$$FX = \varinjlim_{s<\lambda} F_s X$$

where $F_{s+1}X$ is obtained by pushing out $F_s X$ along generating trivial cofibrations and $F_t X = \varinjlim_{s<t} F_s X$ at limit ordinals $t < \lambda$.

I claim that there is a map of systems

$$\theta_s : F_s U X \to U I_s X$$

which consists of strict weak equivalences.

Suppose that θ_s exists. Then all diagrams

$$\begin{array}{ccccc} \Sigma^\infty A[-n] & \longrightarrow & F_s U X & \xrightarrow{\theta_s} & U I_s X \\ {\scriptstyle i}\downarrow & & & & \\ \Sigma^\infty B[-n] & & & & \end{array}$$

determine commutative diagrams

$$\begin{array}{ccc} F_n A & \longrightarrow & I_s X \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ F_n B & \longrightarrow & I_{s+1} X \end{array}$$

where i_* is an α-bounded level trivial cofibration. The induced diagrams

$$\begin{array}{ccccc} \Sigma^\infty A[-n] & \longrightarrow & F_sUX & \xrightarrow{\theta_s} & UI_sX \\ {\scriptstyle i}\downarrow & & & & \downarrow \\ \Sigma^\infty B[-n] & & \longrightarrow & & UI_{s+1}X \end{array}$$

therefore induce a map $\theta_{s+1} : F_{s+1}UX \to UI_{s+1}X$ such that the diagram

$$\begin{array}{ccc} F_sUX & \xrightarrow{\theta_s} & UI_sX \\ \downarrow & & \downarrow \\ F_{s+1}UX & \xrightarrow[\theta_{s+1}]{} & UI_{s+1}X \end{array}$$

commutes. It follows that θ_{s+1} is a strict equivalence. Filtered colimits preserve strict equivalences, and the forgetful functor U preserves filtered colimits, and so one concludes that the induced maps

$$\theta_t : F_tUX \to UI_tX$$

are strict equivalences at limit ordinals t. The statement of the Lemma follows. □

Lemma 1: Suppose that T is compact up to equivalence. Suppose that $f : X \to Y$ is a map of symmetric T-spectra such that the induced map $UX \to UY$ is a stable equivalence of T-spectra. Then f is a stable equivalence.

Proof: In level n, the $\varinjlim_k Q_\Sigma^k X$ is the filtered colimit of the system

$$IX^n \xrightarrow{\sigma} \Omega_T IX^{n+1} \xrightarrow{\Omega_T\sigma} \Omega_T^2 IX^{n+2} \xrightarrow{\Omega_T^2\sigma} \dots$$

The comparison $FUX \to UIX$ induces a commutative diagram

$$\begin{array}{ccccccc} FUX^n & \longrightarrow & \Omega_T FUX^{n+1} & \longrightarrow & \Omega_T^2 FUX^{n+2} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ UIX^n & \longrightarrow & \Omega_T UIX^{n+1} & \longrightarrow & \Omega_T^2 UIX^{n+2} & & \cdots \end{array}$$

in which all the vertical maps are weak equivalences of pointed simplicial presheaves. It follows that there are induced natural weak equivalences

$$\begin{array}{ccc} \varinjlim_k \Omega_T^k FUX^{n+k} & \xrightarrow[\simeq]{j} & QUX^n \\ \simeq\downarrow & & \\ \varinjlim_k \Omega_T^k UIX^{n+k} & \xrightarrow[Uj_\Sigma]{\simeq} & UQ_\Sigma X^n \end{array}$$

Thus if $f : X \to Y$ induces a stable equivalence Uf, meaning a level equivalence $QUX \to QUY$, then the map of symmetric T-spectra $f_* : Q_\Sigma X \to Q_\Sigma Y$ is a level equivalence.

If a symmetric T-spectrum Z is stably fibrant then all objects $Q_\Sigma^k Z$ are stably fibrant and all maps $Z \to Q_\Sigma^k Z$ are level equivalences. It follows that

$Q_{\Sigma}^{k}Z$ is stably fibrant and that the natural map $Z \to Q_{\Sigma}Z$ is a level equivalence.

Finally take a stably fibrant model $X \to LX$ for a symmetric T-spectrum X and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & LX \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\simeq\,\eta} \\ Q_{\Sigma}X & \longrightarrow & Q_{\Sigma}LX \end{array}$$

The indicated maps are stable equivalences, so that X is a natural retract of $Q_{\Sigma}X$ in the stable homotopy category. Thus, if $f : X \to Y$ induces a stable equivalence $UX \to UY$, then the induced map $Q_{\Sigma}X \to Q_{\Sigma}Y$ is a level and hence stable equivalence, so that f is a stable equivalence. □

Corollary 2: Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence, and suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level fibre sequence of symmetric T-spectra. Then the canonical map $X/F \to Y$ is a stable equivalence.

Proof: The induced map $U(X/F) \to UY$ is a stable equivalence of T-spectra by Lemma 13 of Lecture 007. □

Lemma 3: Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Suppose given a comparison of cofibre sequences

$$\begin{array}{ccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & B_1/A_1 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ A_2 & \longrightarrow & B_2 & \longrightarrow & B_2/A_2 \end{array}$$

Then if any two of f_1, f_2, f_3 are stable equivalences, then so is the third.

Proof: There is a natural isomorphism

$$\Omega_T Z[1] \cong \Omega_{S^1}\Omega_K Z[1]$$

and the canonical map $Z \to \Omega_T Z[1]$ is a level equivalence if Z is stably fibrant. It follows that the induced diagram

$$\begin{array}{ccccc} \mathbf{hom}(B_2/A_2, Z) & \longrightarrow & \mathbf{hom}(B_2, Z) & \longrightarrow & \mathbf{hom}(A_2, Z) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{hom}(B_1/A_1, Z) & \longrightarrow & \mathbf{hom}(B_1, Z) & \longrightarrow & \mathbf{hom}(A_1, Z) \end{array}$$

is a comparison of fibre sequences of infinite loop spaces for each stably fibrant object Z, and so if any two of the vertical maps is a weak equivalence then so is the third. $\square$

Corollary 4: Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Suppose that

$i : A \to B$ is a cofibration of symmetric T-spectra, and take a factorization

$$\begin{array}{ccc} B & \xrightarrow{j} & Z \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle p} \\ & & B/A \end{array}$$

such that j is a cofibration and a level equivalence and p is an injective fibration. Let F be the fibre of p. Then the induced map $A \to F$ is a stable equivalence.

Proof: It follows from Lemma 15 of Lecture 007 that the map $UA \to UF$ is a stable fibration. $\square$

Corollary 5: Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Then the stable structure for symmetric T-spectra is proper.

Proof: Suppose given a pullback diagram

$$\begin{array}{ccc} W & \xrightarrow{f_*} & X \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & Y \end{array}$$

such that p is a level fibration with fibre F and f is a stable equivalence. Then the diagram above may be replaced up to stable equivalence by the comparison of cofibre sequences

$$\begin{array}{ccc} F & \xrightarrow{1} & F \\ \downarrow & & \downarrow \\ W & \xrightarrow{f_*} & X \\ \downarrow & & \downarrow \\ W/F & \xrightarrow[\simeq]{} & X/F \end{array}$$

by Corollary 2. But then f_* is a stable equivalence by Lemma 3. □

Lemma 6: Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Suppose that a map $p : X \to Y$ is a stable equivalence and that $Up : UX \to UY$ is a stable fibration of T-spectra. Then p is a level weak equivalence.

Proof: The map p is a level fibration. Let F be the fibre of p and consider the fibre sequence

$$F \xrightarrow{i} X \xrightarrow{p} Y.$$

The canonical map $X/F \to Y$ induces a stable equivalence $U(X/F) \to UY$ of T-spectra by Corollary 2, so that $X/F \to Y$ is a stable equivalence of symmetric T-spectra by Lemma 1. It also follows that the map $\pi : X \to X/F$ is a stable equivalence of symmetric T-spectra. The comparison of cofibre sequences

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & X/F \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X/F & \xrightarrow[1]{} & X/F \end{array}$$

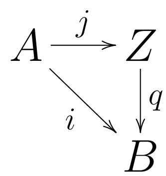
implies that the map $F \to *$ is a stable equivalence of symmetric T-spectra by Lemma 3, so that F is levelwise contractible since it is stably fibrant.

But then $UX \to U(X/F)$ is a stable equivalence of T-spectra, so that $Up : UX \to UY$ is a stable equivalence of T-spectra as well as a stable fibration. It follows that $Up : UX \to UY$ is a level equivalence. $\square$

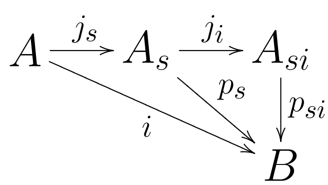
Lemma: Suppose that $T = S^1 \wedge K$, where K is compact up to equivalence. Then an injective fibration $p : X \to Y$ of symmetric T-spectra is a stable fibration if and only if $Up : UX \to UY$ is a stable fibration of T-spectra.

Proof: If $p : X \to Y$ is a stable fibration, then p is an injective fibration, and $Up : UX \to UY$ is a stable fibration. The last claim follows from the observation that the functor V preserves stable equivalences and cofibrations.

Suppose that $i : A \to B$ is a cofibration and a stable equivalence. Then i has a factorization



where q is an injective fibration such that Uq is a stable fibration, and j is a cofibration which is a stable equivalence and has the left lifting property with respect to all such maps. In effect, there are two factorizations



where p_s is a map such that Up_s is a stable fibration and j_s is a stably trivial cofibration which has the LLP with respect to all maps q such that Uq is a stable fibration. The map j_i is a level trivial cofibration and p_{si} is an injective fibration. But then Up_{si} is a strict fibration which is strictly equivalent to a stable fibration, so that Up_{si} is a stable fibration. Set $q = p_{si}$ and $j = j_i j_s$.

But then q is also a stable equivalence, so it is a level weak equivalence by the Lemma. Thus, q is a trivial injective fibration, and therefore has the RLP with respect to all cofibrations. It follows that i is a retract of j and therefore has the LLP with respect to all injective fibrations p such that Up is a stable fibration.

It follows that every S-injective fibration is a stable fibration. □

Say that $p : X \to Y$ is an HSS-fibration if $Up : UX \to UY$ is a stable fibration. Say that the map $i : A \to B$ is an HSS-cofibration if it has the LLP with respect to all maps which are stable equivalences and HSS-fibrations.

Lemma 6 implies that every map $p : X \to Y$ which is both an HSS fibration and a stable equivalence must be a level equivalence. It follows that the class of HSS cofibrations includes all maps $F_nA \to F_nB$ which are induced by α-bounded cofibrations $A \to B$ of pointed simplicial presheaves.

Proposition: Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Then the category $\mathrm{Spt}^{\Sigma}_T(\mathcal{C})$, with the classes of stable equivalences, HSS-fibrations and HSS-cofibrations as defined above, satisfies the axioms for a proper closed simplicial model category.

Proof: The stable model structure on T-spectra is cofibrantly generated. It follows that every map f :

$X \to Y$ of symmetric T-spectra has factorizations

$$\begin{array}{ccccc} & & Z & & \\ & \overset{j}{\nearrow} & & \overset{p}{\searrow} & \\ X & & \xrightarrow{\quad f \quad} & & Y \\ & \underset{i}{\searrow} & & \underset{q}{\nearrow} & \\ & & W & & \end{array}$$

where p is an HHS stable fibration and j is a stable equivalence which has the LLP with respect to all HHS fibrations, and i is an HSS cofibration and q is an HSS fibration such that Uq is a level trivial stable fibration (the latter by Lemma 6). It follows in particular that j is an HSS cofibration and that q is a stable equivalence.

Suppose that the map $i : A \to B$ is an HSS cofibration and a stable equivalence. The i has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ & \underset{i}{\searrow} & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

such that j is an HSS cofibration which has the LLP with respect to all HSS fibrations and is a stable equivalence, and p is an HSS fibration. Then p is a stable equivalence by Lemma 6. It follows that i is a retract of j. $\square$

The smash product

Recall that $S_T \otimes X$ is the free symmetric spectrum associated to symmetric space X.

Lemma: Suppose that Y is a symmetric spectrum. Then there is a canonical isomorphism of symmetric spectra

$$Y \wedge (S_T \otimes X) \cong Y \otimes X.$$

Proof: The composite

$$Y \otimes S_T \otimes X \xrightarrow{\tau \otimes X} S_T \otimes Y \otimes X \xrightarrow{m} Y \otimes X$$

induces a natural map $Y \wedge (S_T \otimes X) \to Y \otimes X$. The unit of S_T induces a map of symmetric spaces $X \to S_T \otimes X$ which then ineduces a map of symmetric spectra $Y \otimes X \to Y \wedge (S_T \otimes X)$. These two maps are inverse to each other. □

Lemma: There is a natural bijection

$$\hom(X \otimes G_n(S^0), Y) \cong \hom(X, Y[n])$$

for morphisms of symmetric T-spectra.

Proof: There is a natural bijection

$$\hom(G_n S^0 \otimes X, Y) \cong \hom(X, Y[n])$$

of maps of symmetric spaces. One checks that this adjunction respects symmetric spectrum structures. □

Corollary: There is a natural isomorphism

$$F_n(A) \wedge F_m(B) \cong F_{n+m}(A \wedge B).$$

Proof: There are isomorphisms

$$\begin{aligned} F_nA \wedge F_mB &\cong (S_T \otimes (G_n(S^0) \wedge A)) \wedge (S_T \otimes (G_m(S^0) \wedge B)) \\ &\cong (S_T \otimes G_n(S^0) \otimes G_m(S^0)) \wedge (A \wedge B) \end{aligned}$$

There are isomorphisms of maps of symmetric spectra

$$\begin{aligned} \hom(S_T \otimes G_n(S^0) \otimes G_m(S^0), Y) &\cong \hom(S_T \otimes G_n, Y[m]) \\ &\cong \hom(S_T, Y[m+n]). \end{aligned}$$

It follows that

$$S_T \otimes G_n(S^0) \otimes G_m(S^0) \cong S_T \otimes G_{m+n}(S^0)$$

as symmetric spectra, and the desired result follows. □

Corollary: The functor $X \mapsto X[n]$ preserves injective fibrations and trivial injective fibrations.

Proof: The functor $Y \mapsto Y \otimes G_n(S^0)$ preserves level cofibrations and level equivalences. □

Theorem A: Suppose that $i : A \to B$ is a projective cofibration and that $j : C \to D$ is a level cofibration of symmetric T-spectra. Then the map

$$(i,j)_* : (B \wedge C) \cup_{(A \wedge C)} (A \wedge D) \to B \wedge D$$

is a level cofibration. If i and j are both projective cofibrations then $(i,j)_*$ is a projective cofibration. If j is a stable equivalence, then $(i,j)_*$ is a stable equivalence.

Proof: For most of these statements, it suffices to assume that the projective cofibration i is a map $F_n A' \to F_n B'$ which is induced by a cofibration $i : A' \to B'$ of pointed simplicial presheaves.

If $p : X \to Y$ is an injective fibration then the induced map

$$\mathbf{Hom}(B',X)[n] \to \mathbf{Hom}(A',X)[n] \times_{\mathbf{Hom}(A',Y)[n]} \mathbf{Hom}(B',Y)[n] \tag{1}$$

is an injective fibration which is trivial if p is trivial. This statement in the case when p is trivial implies that the map

$$(i,j)_* : (F_n B' \wedge C) \cup_{(F_n A' \wedge C)} (F_n A' \wedge D) \to F_n B' \wedge D$$

is a level cofibration.

The fact that the map (1) is an injective fibration implies that the map $(i,j)_*$ is a level weak equivalence if j is a level weak equivalence.

If $C \to D$ is a projective cofibration, then it can be approximated by cofibrations of the form $F_m C' \to F_m D'$, and then the map $(i,j)_*$ is the result of applying the functor F_{n+m} to the cofibration

$$(B' \wedge C') \cup_{(A' \wedge C')} (A' \wedge D') \to B' \wedge D'$$

of pointed simplicial presheaves.

If j is a stably trivial level cofibration, then it is in the saturation of the set S of maps generated by all α-bounded level trivial cofibrations $E \to F$ and all maps

$$F_1 T \wedge F_n S^0 \to S_T \wedge F_n(S^0)$$

induced by the canonical map $F_1 T \to S_T$, subject to the requirement that the map (the "tensor")

$$(C \wedge F) \cup (D \wedge E) \to D \wedge F$$

is in S for all $i : C \to D$ in S and all α-bounded cofibrations $E \to F$ of pointed simplicial sets. If the map i determines a stably trivial cofibration

$$(C \wedge F_n B) \cup (D \wedge F_n A) \to D \wedge F_n B,$$

for all cofibrations $A \to B$ of pointed simplicial presheaves, and if $E \to F$ is an α-bounded cofibration of pointed simplicial presheaves then the cofibration determined by

$$(C \wedge F) \cup (D \wedge E) \to D \wedge F$$

and the map $F_n A \to F_n B$ is the map determined by $i : C \to D$ and the map determined by applying F_n to the inclusion

$$(F \wedge A) \cup (E \wedge B) \to (F \wedge B).$$

We therefore only have to show that "tensoring" the generators of S with maps $F_n A \to F_n B$ gives stable equivalences.

The tensor of $F_n A \to F_n B$ with a map $F_1 T \wedge F_m S^0 \to S_T \wedge F_m S^0$ is the same as the tensor of the map $F_1 T \wedge F_{n+m}(S^0) \to S_T \wedge F_{n+m}(S^0)$ with the map $A \to B$, and this is in S. We have already seen that tensoring with $F_n A \to F_n B$ does the right thing for level trivial cofibrations. □

Lemma B: Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Suppose that $i : A \to B$ is a projective cofibration and that $j : C \to D$ is a level cofibration of symmetric T-spectra. Then the map $(i, j)_*$ is a stable equivalence if i is a stable equivalence.

Proof: The cofibre of $(i, j)_*$ is the smash $B/A \wedge D/C$. The quotient B/A is projective cofibrant, and so there is a level weak equivalence

$$B/A \wedge K \to B/A \wedge D/C,$$

where $K \to D/C$ is a projective cofibrant model for D/C. Then the stably trivial cofibration $* \to B/A$ induces a stable equivalence

$$* \cong * \wedge K \to B/A \wedge K.$$

Thus, the cofibre $B/A \wedge D/C$ of $(i,j)_*$ is stably trivial, so that $(i,j)_*$ is a stable equivalence by Lemma 3. □

Corollary C: If $f : X \to Y$ is a stable equivalence and A is projective cofibrant, then the induced map $f \wedge A : X \wedge A \to Y \wedge A$ is a stable equivalence.

Proof: Any stably trivial cofibration $j : C \to D$ induces a stable equivalence $A \wedge C \to A \wedge D$, by Theorem A. The morphism $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

such that j is a cofibration and p is a trivial injective fibration. The map p has a section $\sigma : Y \to Z$ since all symmetric T-spectra are cofibrant for the injective structure, but then σ is a stably trivial cofibration so that $A \wedge \sigma : A \wedge Y \to A \wedge X$ is a stable equivalence, and so $A \wedge p : A \wedge Z \to A \wedge Y$ is a stable equivalence. The map j is also a stably trivial cofibration, so that $A \wedge j : A \wedge X \to A \wedge Z$ is a stable equivalence. □

Corollary D: Suppose that $i : A \to B$ and $j : C \to D$ are projective cofibrations. Then the induced map

$$(i, j)_* : (B \wedge C) \cup_{(A \wedge C)} (A \wedge D) \to B \wedge D$$

is a projective cofibration which is stably trivial if either i or j is a stable equivalence.

Take away the adjective "projective" in the statement of Corollary D, and you have the description of what it means for a model structure to be monoidal, subject to having a symmetric monoidal smash product. Example: the pointed simplicial set (or presheaf) category with the obvious smash product is monoidal.

Corollary E: Suppose that $T = S^1 \wedge K$ where K is compact up to equivalence. Then the HSS-stucture on the category $\mathrm{Spt}^{\Sigma}_{T}(\mathcal{C})$ of symmetric T-spectra is monoidal.

Proof: The cofibrations for the HSS-structure are the projective cofibrations. □

Here's an issue, perhaps: the HSS-structure on symmetric T-spectra is monoidal when it exists, which is so far only in the case where T is a suspension of an object which is compact up to equivalence. On the other hand, the stable structure for symmetric T-spectra and Corollary D both obtain in extreme generality, and there is a universal description of a derived tensor project: set

$$X \wedge_{\Sigma} Y = X' \wedge Y'$$

where $\pi_X : X' \to X$ and $\pi_Y : Y' \to Y$ are projective cofibrant models for X and Y respectively. The standard description of a monoidal model structure may not be exactly the right thing.

References

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