Pointed torsors

J.F. Jardine

February 2, 2010

Introduction

This paper gives a characterization of homotopy fibres over trivial torsors of the inverse image maps

$$\pi^* : B(H - \mathbf{Tors}) \to B(\pi^* H - \mathbf{tors})$$

of torsor categories which are induced by geometric morphisms $\pi : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ between Grothendieck toposes. In this generality, H is a sheaf of groupoids on the site $\mathcal{C}$, an $H-\mathbf{tors}$ stands for its associated category of torsors, or the groupoid of global sections of the associated stack [3].

These homotopy fibres are characterized as nerves of, equivalently, suitably defined categories of pointed torsors and pointed cocycles. The equivalence between pointed torsors and pointed cocycles (Lemma 2) is a refinement of the equivalence between categories of torsors and cocycles which is displayed in [4]. These general results appear in the first section of this paper.

Section 2 contains some first applications of these principles for simplicial sheaves on "big" and "small" étale sites for schemes S. The results are prototypical in that they hold for any standard algebraic geometric topology on a category of schemes. In particular, it is shown that the homotopy type of the groupoid of pointed torsors is independent of the size of the underlying site (see the discussion leading to Remark 6), and that the homotopy fibre of the inverse image

$$f^* : B(H - \mathbf{tors}) \to B(f^* H - \mathbf{tors})$$

for an object $f : T \to S$ in the big site is the nerve of a category of cocycles in the slice category of simplicial sheaves under T (Remark 7).

In Section 3, we specialize to classifying toposes $\mathcal{B}G$ of profinite groups $G = \{G_i\}$ (and hence to finite étale sites), and to the canonical stalk $\pi^* : \mathcal{B}G \to \mathbf{Set}$ for such a group G. In this case, $\pi^* H$ is just a groupoid, the base change map has the form

$$\pi^* : B(H - \mathbf{tors}) \to B\pi^* H,$$

and we specify the homotopy fibres F_x over the objects x of $\pi^* H$. A precise calculation can be achieved in this case, in that F_x is a union of contractible spaces, indexed on morphisms $C(G_i) \to BH$ defined on the Čech resolutions

$C(G_i)$ associated to the component groups G_i of G, up to refinement (Lemma 12 and Corollary 11, respectively).

In particular, if S is a connected Noetherian scheme, and $\pi_1(S, y)$ is the Grothendieck fundamental group of S at some geometric point y, and H is a constant group, then $\pi_0(F)$ of the fibre F over the base point of the classifying space BH is the set of homomorphisms $\pi_1(S, y) \to H$ taking values in H.

The sets $\pi_0(F_x)$ collectively give the best candidates for étale non-abelian *pointed* H^1 invariant with coefficients in a sheaf of groupoids H. The desire to make sense of this concept, and the calculations for profinite groups displayed here, motivated the results in this paper.

1 Pointed torsors

Suppose that $\mathcal{C}$ is a small Grothendieck site.

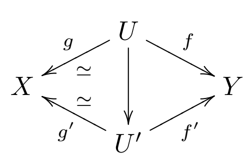
We shall use the injective model structure on the category $s\,\mathrm{Shv}(\mathcal{C})$ of simplicial sheaves on $\mathcal{C}$ throughout this paper. The cofibrations for this structure are the monomorphisms, the weak equivalences are the local weak equivalences, and the fibrations are the injective fibrations. The local weak equivalences are those simplicial sheaf maps $f : X \to Y$ which induce weak equivalences $X_x \to Y_x$ of simplicial sets in all stalks if stalks are available, but more generally are those maps which induce isomorphisms in all possible sheaves of homotopy groups. The injective fibrations are defined by a right lifting property with respect to all trivial cofibrations. Injective fibrations are also called global fibrations in the literature.

The injective model structure has a list of attributes: it is a cofibrantly generated proper closed simplicial model structure, such that weak equivalences are closed under finite products. The associated homotopy category $\mathrm{Ho}(s\,\mathrm{Shv}(\mathcal{C}))$ is a non-abelian derived category for the sheaf category on the site $\mathcal{C}$.

For simplicial sheaves X and Y, the *cocycle category* $h(X, Y)$ as objects consisting of simplicial sheaf maps

$$X \xleftarrow[\simeq]{g} U \xrightarrow{f} Y, \tag{1}$$

and morphisms consisting of commutative diagrams



It is a basic property [4] of cocycle categories (since the injective model structure is proper and has the property that weak equivalences are closed under finite products) that the assignment which takes a cocycle (1) to the morphism fg^{-1} in the homotopy category of simplicial sheaves induces a bijection

$$\pi_0 h(X, Y) \cong [X, Y]$$

between the set of path components of $h(X,Y)$ (equivalently the path component set $\pi_0 Bh(X,Y)$ of the nerve $Bh(X,Y)$) and the set $[X,Y]$ of morphisms in the homotopy category $\mathrm{Ho}(s\,\mathrm{Shv}(\mathcal{C}))$.

Suppose that H is a sheaf of groupoids on the site $\mathcal{C}$.

Recall [4] that an *H-torsor* is a H-diagram $\pi : X \to \mathrm{Ob}(H)$ in sheaves such that the induced map $\underrightarrow{\mathrm{holim}}_H\ X \to *$ is a weak equivalence. A morphism of H-diagrams is an H-equivariant map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & \mathrm{Ob}(H) & \end{array}$$

The sheaf map $f : X \to Y$ is necessarily a weak equivalence, and hence a sheaf isomorphism. It follows that the category of H-torsors and the H-equivariant maps between them form a groupoid, which will be denoted by $H - \mathbf{tors}$. This groupoid is the groupoid of global sections of a presheaf of groupoids $H - \mathbf{Tors}$, which is the stack associated to H — see [4].

Standard examples of torsors include the representable functors $H(x,\ )$ and $H(\ ,x)$ associated to a global section x of the object sheaf $\mathrm{Ob}(H)$. There is an isomorphism

$$H(x,\ ) \xrightarrow{f} H(\ ,x)$$

of H-torsors which is defined in sections by taking the morphism $x \xrightarrow{\alpha} y$ to $y \xrightarrow{\alpha^{-1}} x$. These are the trivial torsors. In general, an H-torsor morphism (ie. a trivialization over x)

$$f : H(x,\ ) \to X$$

is completely determined by the global section $f(1_x)$ of X, which section maps to x under the structure map $X \to \mathrm{Ob}(H)$. Observe that the structure map $H(x,\ ) \to \mathrm{Ob}(H)$ is defined in sections by sending the morphism $x \xrightarrow{\alpha} y$ to the target object y.

An H-torsor $X \to \mathrm{Ob}(H)$ consists of $H(U)$-diagrams $X(U) : H(U) \to \mathbf{Set}$, $U \in \mathcal{C}$, with $x \mapsto X(U)(x)$, so that

$$X(U) = \bigsqcup_{x \in \mathrm{Ob}(H(U))} X(U)(x)$$

as a presheaf. The sheaf theoretic homotopy colimit $\underrightarrow{\mathrm{holim}}_H\ X$ is weakly equivalent to a point by the definition of H-torsor, so that there is a functorially assigned cocycle

$$* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_H\ X \to BH$$

for each H-torsor X. It follows that there is a functor

$$\underrightarrow{\mathrm{holim}}_H : H - \mathbf{tors} \to h(*, BH).$$

The homotopy colimit functor $\underrightarrow{\mathrm{holim}}_H$ has a left adjoint

$$\mathrm{pb} : h(*, BH) \to H - \mathbf{tors}$$

that is defined by sheafifying a presheaf, which presheaf is defined in sections by taking path components $\pi_0(\mathrm{pb}_x)$ of spaces pb_x. These spaces pb_x are defined by the pullback diagrams

$$\begin{array}{ccc} \mathrm{pb}_x & \longrightarrow & V(U) \\ \downarrow & & \downarrow \\ B(H(U)/x) & \longrightarrow & BH(U) \end{array}$$

where x ranges through the set $\mathrm{Ob}(H(U))$ of objects of $H(U)$.

Suppose that the functor $\pi : \mathcal{C} \to \mathcal{D}$ is a site morphism. Then the inverse image functor

$$\pi^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$$

is exact and preserves local weak equivalences of simplicial sheaves. In particular, if $X \to \mathrm{Ob}(H)$ is a H-torsor, then the induced object $\pi^* X \to \mathrm{Ob}(\pi^* H)$ is a $\pi^* H$-torsor. It follows that the inverse image functor π^* induces a functor

$$\pi^* : H - \mathbf{tors} \to \pi^* H - \mathbf{tors}.$$

Suppose that $x : * \to \mathrm{Ob}(\pi^* H)$ is a global section of the sheaf of groupoids $\pi^* H$. A *pointed H-torsor over x* is an H-torsor $X \to \mathrm{Ob}(H)$ together with a fixed lifting

$$\begin{array}{ccc} & & \pi^* X \\ & \overset{z}{\nearrow} & \downarrow \\ * & \xrightarrow[x]{} & \mathrm{Ob}(\pi^* H) \end{array}$$

A morphism of pointed H-torsors over x is a morphism of H-torsors which respects the liftings. Write $H - \mathbf{tors}_x$ for the corresponding groupoid.

Lemma 1. *The groupoid $H - \mathbf{tors}_x$ of pointed H-torsors over x is the homotopy fibre of the map*

$$\pi^* : H - \mathbf{tors} \to \pi^* H - \mathbf{tors}.$$

Proof. In the category of groupoids, the homotopy fibre F_x of the morphism

$$i_* : H - \mathbf{tors} \to \pi^* H - \mathbf{tors}$$

over the torsor $\pi^* H(x, \ )$ has objects consisting of all $\pi^* H$-torsor morphisms $\pi^* H(x, \ ) \to \pi^* X$. The morphisms of this groupoid are the commutative diagrams

$$\begin{array}{ccc} & \nearrow & \pi^* X \\ \pi^* H(x, \ ) & & \downarrow \theta_* \\ & \searrow & \pi^* Y \end{array}$$

where $\theta : X \to Y$ is a morphism of H-torsors.

A global section $z : * \to \pi^*X$ of the torsor π^*X extends to a unique H-torsor morphism

$$z_* : \pi^*H(x,\ ) \to \pi^*X,$$

where x denotes the image of z in the global sections of the sheaf $\mathrm{Ob}(\pi^*H)$.

It follows that the assignment which takes the pointed torsor $* \xrightarrow{z} \pi^*X$ to the torsor morphism $\pi^*H(x,\ ) \xrightarrow{z_*} \pi^*X$ defines an isomorphism

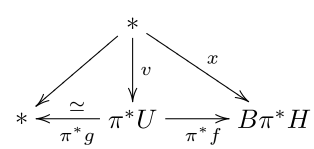
$$F_x \cong H - \mathbf{tors}_x$$

of groupoids. □

A *pointed H-cocycle over x* is a cocycle

$$* \xleftarrow[\simeq]{g} U \xrightarrow{f} BH$$

together with a morphism

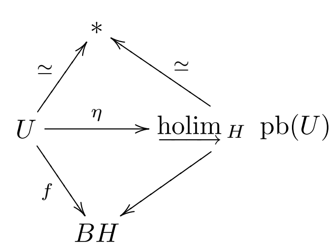


of π^*H-cocycles. A morphism of pointed H-cocycles over x is a morphism of cocycles which respects choices of sections. Write $h(*, BH)_x$ for the corresponding category.

The unit of the adjunction

$$\mathrm{pb} : h(*, BH) \leftrightarrows H - \mathbf{tors} : \underrightarrow{\mathrm{holim}}_H$$

is a natural cocycle morphism



If the object π^*U is pointed by a morphism $v : * \to \pi^*U$ over x, then the composite sheaf map

$$* \xrightarrow{v} \pi^*U_0 \xrightarrow{\pi^*(\eta)} \pi^*(\underrightarrow{\mathrm{holim}}_H\ \mathrm{pb}(U))_0 = \pi^*\,\mathrm{pb}(U)$$

gives $\pi^*\,\mathrm{pb}(U)$ a base point over x. It follows that the pullback functor restricts to a functor

$$\mathrm{pb} : h(*, BH)_x \to H - \mathbf{tors}_x.$$

Also, if the torsor $X \to \mathrm{Ob}(H)$ is pointed by a map

$$* \xrightarrow{z} \pi^*(X) = \pi^*(\underrightarrow{\mathrm{holim}}_H\ X)_0$$

over x, then the global section z defines a map

$$* \xrightarrow{z} \pi^*(\underrightarrow{\mathrm{holim}}_H\ X)$$

which gives the canonical cocycle

$$* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_H X \to BH$$

the structure of a pointed cocycle over x. Thus, the canonical cocycle construction restricts to a functor

$$\underrightarrow{\mathrm{holim}}_H : H - \mathbf{tors}_x \to h(*, BH)_x.$$

We then have the following:

Lemma 2. *The pullback and homotopy colimit functors induce an adjoint pair of functors*

$$\mathrm{pb} : h(*, BH)_x \leftrightarrows H - \mathbf{tors}_x : \underrightarrow{\mathrm{holim}}_H.$$

Corollary 3. *The nerve $Bh(*, BH)_x$ is weakly equivalent to the homotopy fibre BF_x of the simplicial set map*

$$\pi^* : B(H - \mathbf{tors}) \to B(\pi^* H - \mathbf{tors}).$$

over the trivial torsor $H(x,\ )$.

2 Étale sites

Suppose that S is scheme and let $(Sch|_S)_{et}$ be the big site of S-schemes $X \to S$, equipped with the étale topology. The inclusion $i : et|_S \subset (Sch|_S)_{et}$ of the standard étale site in the big site is a site morphism, and induces a geometric morphism

$$i : \mathrm{Shv}(Sch|_S)_{et} \to \mathrm{Shv}(et|_S).$$

The corresponding direct image functor

$$i_* : \mathrm{Shv}(Sch|_S)_{et} \to \mathrm{Shv}(et|_S).$$

is defined by composition with, or restriction along, the inclusion functor i. The inverse image functor i^* is exact, and therefore preserves trivial cofibrations of simplicial sheaves. The restriction functor i_* is also exact (this is uncommon for direct images) and therefore preserves local weak equivalences of simplicial sheaves. The canonical map

$$\eta : F \to i_* i^* F$$

is an isomorphism for all sheaves F on the étale site $et|_S$.

Suppose that X is a simplicial sheaf on the big site $(Sch|_S)_{et}$. Then restriction along i preserves weak equivalences and therefore defines a functor

$$i_* : h(*, X) \to h(*, i_*X)$$

which sends the cocycle $* \xleftarrow{\simeq} U \xrightarrow{f} X$ to the cocycle $* \xleftarrow{\simeq} i_*U \xrightarrow{i_*f} i_*X$. There is a functor

$$\tilde{i} : h(*, i_*X) \to h(*, X)$$

which sends the cocycle $* \xleftarrow{\simeq} V \xrightarrow{\alpha} i_*X$ to the cocycle $* \xleftarrow{\simeq} i^*V \xrightarrow{\alpha_*} X$, where α_* is the adjoint of α. It is easy to show that the functor $\tilde{i}$ is left adjoint to the functor i_*. We have therefore proved the following:

Lemma 4. *The restriction functor* $i_* : \mathrm{Shv}(Sch|_S)_{et} \to \mathrm{Shv}(et|_S)$ *induces a weak equivalence*

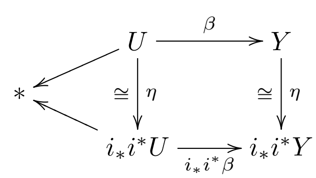
$$i_* : Bh(*, X) \xrightarrow{\simeq} Bh(*, i_*X).$$

for all simplicial sheaves X *on the big site* $(Sch|_S)_{et}$.

The inverse image functor induces a functor

$$i^* : h(*, Y) \to h(*, i^*Y)$$

for simplicial sheaves Y on the étale site $et|_S$. The composite functor i_*i^* sends the cocycle $* \xleftarrow{\simeq} U \xrightarrow{\beta} Y$ to the cocycle $* \xleftarrow{\simeq} i_*i^*U \xrightarrow{i_*i^*\beta} i_*i^*Y$, and there is a commutative diagram



It follows that the composite

$$i_*i^* : Bh(*, Y) \to Bh(*, i_*i^*Y)$$

is homotopic to the isomorphism

$$\eta_* : Bh(*, Y) \xrightarrow{\cong} Bh(*, i_*i^*Y)$$

which is defined by composition with η. Now we can show the following:

Corollary 5. *The inverse image functor*

$$i^* : \mathrm{Shv}(et|_S) \to \mathrm{Shv}(Sch|_S)_{et}$$

induces a weak equivalence

$$i^* : Bh(*, Y) \to Bh(*, i^*Y)$$

for all simplicial sheaves Y *on the étale site* $et|_Y$.

Proof. The composite map $i_* i^*$ is a weak equivalence by the discussion above, and the map i_* is a weak equivalence by Lemma 4. $\square$

Suppose that the scheme homomorphism $f : T \to S$ is an object of the big site $(Sch|_S)_{et}$. The map f induces a site morphism

$$\tilde{f} : (Sch|_S)_{et} \to (Sch|_T)_{et}$$

which is defined by pullback along f in the obvious way, and hence induces a geometric morphism

$$f : \mathrm{Shv}(Sch|_T)_{et} \to \mathrm{Shv}(Sch|_S)_{et},$$

for which the direct image functor f_* is defined by composition with $\tilde{f}$. The inverse image functor

$$f^* : \mathrm{Shv}(Sch|_S)_{et} \to \mathrm{Shv}(Sch|_T)_{et}$$

is defined by composition with the functor $f : Sch|_T \to Sch|_S$ which is given by composition with $f : T \to S$.

The diagram of direct image functors

$$\begin{array}{ccc} \mathrm{Shv}(Sch|_T)_{et} & \xrightarrow{i_*} & \mathrm{Shv}(et|_T) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{Shv}(Sch|_S)_{et} & \xrightarrow[i_*]{} & \mathrm{Shv}(et|_S) \end{array}$$

commutes, so that there is a canonical isomorphism $\gamma : i^* f^* \xrightarrow{\cong} f^* i^*$. It follows that there is a homotopy commutative diagram

$$\begin{array}{ccccc} Bh(*, X) & & \xrightarrow[\simeq]{i^*} & & Bh(*, i^* X) \\ {\scriptstyle f^*}\big\downarrow & & & & \big\downarrow{\scriptstyle f^*} \\ Bh(*, f^* X) & & & & Bh(*, f^* i^* X) \\ & {\scriptstyle i^*}\searrow^{\simeq} & & {}^{\cong}\swarrow{\scriptstyle \gamma_*} & \\ & & Bh(*, i^* f^* X) & & \end{array} \tag{2}$$

for each simplicial sheaf X on the étale site $et|_S$.

Remark 6. The existence of the diagram (2) means that the small and big site versions of f^* (the vertical maps in the diagram) have weakly equivalent homotopy fibres.

It follows in particular that if H is a sheaf of groupoids on the étale site $et|_S$ and $x \in \mathrm{Ob}(H)(T)$ is a global section of $f^* H$, then there is a weak equivalence

$$Bh(*, BH)_x \simeq Bh(*, Bi^* H)_x,$$

and hence (by Lemma 2) a weak equivalence

$$B(H - \mathbf{tors}_x) \simeq B(i^* H - \mathbf{tors}_x)$$

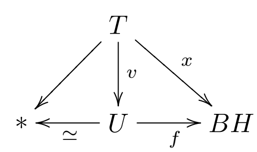
relating the respective categories of pointed torsors for the small and big sites.

Suppose now that H is a sheaf of groupoids on the big site $(Sch|_S)_{et}$, and suppose that the S-scheme $f : T \to S$ is an object of that site. Suppose that $x \in \mathrm{Ob}(f^* H)$ is a global section.

In general, for any sheaf X on $(Sch|_s)_{et}$, a global section of $f^* X$ is a section of $X(T)$, and there is a natural bijection

$$\hom(*, f^* X) \cong \hom(T, X),$$

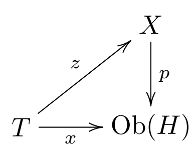
where T denotes the sheaf represented by the object $f : T \to S$ It follows that the objects of the category $h(*, BH)_x$ of pointed cocycles over x can be identified with commutative diagrams



(3)

where the bottom row forms a cocycle over S. From this point of view, a morphism of pointed cocycles is a cocycle morphism which respects all choices of sections.

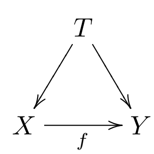
A pointed H-torsor over x can be identified with an H-torsor $p : X \to \mathrm{Ob}(H)$ with a commutative diagram



(4)

A morphism of pointed torsors over x is then an H-torsor morphism which respects sections over T.

Remark 7. The slice category $T/s\,\mathrm{Shv}(Sch|_S)_{et}$ has a model structure, for which a morphism



is a weak equivalence (respectively cofibration, fibration) if the simplicial sheaf map $f : X \to Y$ is a local weak equivalence (respectively cofibration, injective fibration) of simplicial sheaves. In this category, a cocycle

$$t \xleftarrow{\simeq} v \xrightarrow{f} x$$

is exactly a pointed cocycle

$$\begin{array}{ccccc} & & T & & \\ & \overset{t}{\swarrow} & \downarrow{\scriptstyle v} & \overset{x}{\searrow} & \\ * & \xleftarrow[\simeq]{} & U & \xrightarrow[f]{} & BH \end{array}$$

as described above, and the category $h(*, BH)_x$ of pointed cocycles can be identified with the cocycle category $H(t, x)$ for the slice category.

Weak equivalences are closed under finite products in the slice category, and its model structure is right proper. Theorem 1 of [4] therefore says that there is a canonical bijection

$$\pi_0 h(*, BH)_x = \pi_0 H(t, x) \cong [t, x],$$

where $[t, x]$ denotes morphisms from t to x in the homotopy category

$$\mathrm{Ho}(T/s\,\mathrm{Shv}(Sch|_S)_{et}).$$

3 Profinite groups

Suppose that the group-valued functor $G : I \to \mathbf{Grp}$ is a profinite group. This means that I is left filtered (any two objects i, i' have a common lower bound, and any two morphisms $i \rightrightarrows j$ have a weak equalizer), and that all of the constituent groups G_i, $i \in I$, are finite. I shall also insist that all of the transition homomorphisms $G_i \to G_j$ in the diagram are surjective.

Example 8. The standard example is the absolute Galois group $G(k)$ of a field k. One takes all finite Galois extensions L/k inside an algebraically closed field Ω containing k in the sense that one has a fixed imbedding $i : k \to \Omega$, and the Galois extensions are commutative diagrams of field homomorphisms

$$\begin{array}{ccc} k & \xrightarrow{i} & \Omega \\ & \searrow \quad \nearrow & \\ & L & \end{array}$$

where L is a finite Galois extension of k. These are the objects of a right filtered category, for which the morphisms $L \to L'$ respect structure, and the contravariant functor $G(k)$ which associates the Galois group $G(L/k)$ to each of these pictures is the absolute Galois group.

Generally, suppose that G is a profinite group, and let $G - \mathbf{Set}_{df}$ be the category of finite discrete G-sets, as in [2].

Recall that a discrete G-set is a set F equipped with an action

$$G \times F \to G_i \times F \to F,$$

where $G = \varprojlim_i G_i$ (note the abuse of notation), and a morphism of discrete G-sets is a G-equivariant map.

The category $G - \mathbf{Set}_{df}$ has a topology for which the covering families are the G-equivariant surjections $U \to V$. A presheaf F on $G - \mathbf{Set}_{df}$ is a sheaf for this topology if and only if

1) F takes disjoint unions to products, and

2) each canonical map $G_i \to G_i/H$ induces a bijection

$$F(G_i/H) \xrightarrow{\cong} F(G_i)^H.$$

It follows in particular that the topology is subcanonical: every finite discrete G-set X represents a sheaf $\hom(\ , X)$. The resulting sheaf category

$$\mathcal{B}G := \mathrm{Shv}(G - \mathbf{Set}_{df})$$

is the classifying topos for the profinite group G.

Let

$$\pi : G - \mathbf{Set}_{df} \to \mathbf{Set}$$

be the functor which takes a finite discrete G-set to its underlying set. Every set X represents a sheaf $\pi_* X$ on $G - \mathbf{Set}_{df}$ with

$$\pi_* X(U) = \hom(\pi(U), X).$$

The left adjoint π^* of the corresponding functor π_* has the form

$$\pi^* F = \varinjlim \ F(G_i), \tag{5}$$

by a cofinality argument.

Recall (for example, from [2]), that the list $\{\pi\}$ consisting of the geometric morphism π alone is an "adequate" collection of points for the classifying topos $\mathcal{B}G$. Thus, for example, a simplicial sheaf morphism $f : X \to Y$ on $G - \mathbf{Set}_{df}$ is a local weak equivalence if and only if the induced map $\pi^* X \to \pi^* Y$ is a weak equivalence of simplicial sets.

Suppose that H is a sheaf of groupoids on $G - \mathbf{Sets}_{df}$ and that x is an object of the groupoid $\pi^* H$.

As above, let $H - \mathbf{tors}_x$ denote the category of H-torsors pointed by x, let $h(*, BH)_x$ be the category of pointed H-cocycles over x, and recall from Lemma 2 that pullback and homotopy colimit define an adjunction

$$\mathrm{pb} : h(*, BH)_x \leftrightarrows H - \mathbf{tors}_x : \underrightarrow{\mathrm{holim}}_H.$$

It follows that there is a homotopy equivalence

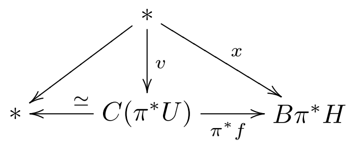
$$Bh(*, BH)_x \simeq B(H - \mathbf{tors}_x)$$

of the associated nerves.

A pointed Čech cocycle over x is a cocycle

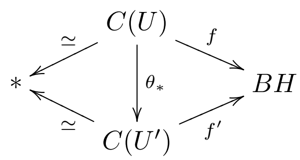
$$* \xleftarrow{\simeq} C(U) \xrightarrow{f} BH$$

together with a morphism

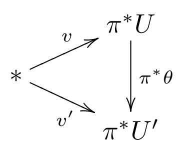


of π^*H-cocycles, where $C(U)$ is the Čech resolution associated to an epimorphism $U \to *$ in $\mathcal{B}G$.

A morphism of pointed Čech cocycles over x is a sheaf morphism $\theta : U \to U'$ which induces a morphism of cocycles



which morphism preserves base points in the sense that the diagram



commutes. Write $h_{Cech}(*, BH)_x$ for the category of Čech cocycles over x.

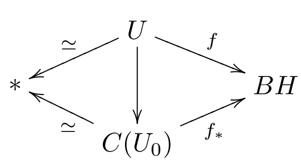
Lemma 9. *The inclusion functor*

$$i : h_{Cech}(*, BH)_x \to h(*, BH)_x$$

is fully faithful, and induces a bijection

$$\pi_0 h_{Cech}(*, BH)_x \xrightarrow{\cong} \pi_0 h(*, BH)_x.$$

Proof. In fact, the inclusion functor i has a left adjoint. The unit of the adjunction is a canonical cocycle morphism



which exists since $C(U_0)$ is the fundamental groupoid of the simplicial sheaf U. □

Write $h_G(*, BH)_x$ for the subcategory of $h_{Cech}(*, BH)_x$ whose objects are the pointed Čech cocycles

$$* \xleftarrow{\simeq} C(G_i) \xrightarrow{f} BH$$

such that $\pi^* f(e_i) = x$ in $\mathrm{Ob}(\pi^* H)$. Here,

$$e_i \in \pi^* G_i = \varinjlim_j \hom(G_j, G_i)$$

is the element which is represented by the identity homomorphism $1 : G_i \to G_i$.

If $\phi : G_j \to G_i$ is a structure homomorphism of the profinite group G then $e_j \mapsto e_i$ under the function $\pi^*\phi : \pi^* G_j \to \pi^* G_i$. A morphism of $h_G(*, BH)_x$ is a structure homomorphism $\phi : G_j \to G_i$ for G which respects cocycles.

Lemma 10. *The inclusion functor*

$$i : h_G(*, BH)_x \subset h_{Cech}(*, BH)_x$$

induces a weak equivalence

$$i_* : Bh_G(*, BH)_x \xrightarrow{\simeq} Bh_{Cech}(*, BH)_x.$$

Proof. Suppose that (f, v) is an object of $h_{Cech}(*, BH)_x$, where $f : C(U) \to BH$ is a cocycle and $v \in \pi^* U$. The element v corresponds to a map $v_* : G_i \to U$ for some G_i, and $v_*(e_i) = z$. It follows that the category $i/(f, v)$ is non-empty.

The category $i/(f, v)$ is also left filtered since $\pi^* U$ is defined by the filtered colimit

$$\pi^* U = \varinjlim_i\ U(G_i).$$

The category $i/(f, v)$ is therefore non-empty and left filtered for all objects (f, v) of the category $h_{Cech}(*, BH)_x$. The desired result then follows from Quillen's Theorem B [1, IV.5.6]. □

Corollary 11. *Suppose that G is a profinite group, H is a sheaf of groupoids on $G - \mathbf{Set}_{df}$, and that x is an object of the stalk groupoid $\pi^* H$. Then there is an isomorphism*

$$\pi_0 Bh(*, BH)_x \cong \varinjlim_i\ \hom(C(G_i), BH)_x$$

where $\hom(C(G_i), BH)_x$ *is the set of groupoid morphisms $f : C(G_i) \to H$ such that $\pi^* f(e_i) = x$.*

The weak equivalence

$$\mathrm{pb} : Bh(*, BH)_x \xrightarrow{\simeq} B(H - \mathbf{tors}_x)$$

is a fibrant model for the space $Bh(*, BH)_x$ in simplicial sets, since the pointed torsor category $H - \mathbf{tors}_x$ is a groupoid. Every automorphism θ of a pointed torsor

$$\begin{array}{ccc} & & \pi^* X \\ & \overset{z}{\nearrow} & \downarrow \\ * & \xrightarrow[x]{} & \mathrm{Ob}(\pi^* H) \end{array}$$

induces a diagram of torsor morphisms

$$\begin{array}{ccc} & \overset{z_*}{\nearrow} & \pi^* X \\ \pi^* H(x,\ ) & \cong \ \cong & \downarrow{\scriptstyle \pi^*(\theta)} \\ & \underset{z_*}{\searrow} & \pi^* X \end{array}$$

so that $\pi^*(\theta)$ is the identity. But π^* is a fully faithful functor, so that θ is the identity as well. We have shown the following

Lemma 12. *The canonical simplicial set map*

$$Bh(*, BH)_x \to \pi_0 Bh(*, BH)_x$$

is a weak equivalence for all $x \in \pi^* \mathrm{Ob}(H)$, for all sheaves of groupoids H on the site $G - \mathbf{Sets}_{df}$.

Example 13. If $\Gamma^* K$ is the constant sheaf on some group K, and $*$ is the unique object of the group $K = \pi^* \Gamma^* K$, then there is an isomorphism

$$\pi_0 Bh(*, B\Gamma^* K)_* \cong \varinjlim_i \ \hom(G_i, K).$$

where $\hom(G_i, K)$ is the set of group homomorphisms $G_i \to K$.

Suppose that H is a sheaf of groupoids on the étale site $et|_k$ for a field k (or on the finite étale site for k — the two sheaf categories are equivalent). Translating the canonical stalk $\pi^* : \mathcal{B}G(k) \to \mathbf{Set}$ to the sheaf category on $et|_k$ through the standard equivalence [5] gives a functor

$$\pi^* : \mathrm{Shv}(et|_k) \to \mathbf{Set}$$

which is defined for a sheaf F by

$$\pi^* F = \varinjlim_L \ F(L),$$

where L varies through the finite Galois extensions L/k in Ω, or more specifically through the objects of the defining diagram for the absolute Galois group $G(k)$, as in Example 8.

Take an element $x \in \pi^* \operatorname{Ob}(H)$. It is a consequence of Corollary 3, Lemma 9 and Lemma 10 that the homotopy fibre $Bh(*, BH)_x$ of the canonical stalk map

$$\pi^* : B(H - \mathbf{tors}) \to B(\pi^* H - \mathbf{tors})$$

over the torsor $\pi^* H(x, \ )$ has its path component set specified by

$$\pi_0 Bh(*, BH)_x \cong \varinjlim_L \hom(C(\operatorname{Sp}(L)), BH)_x,$$

where the colimit is indexed over finite Galois extensions L of k in Ω, and $\hom(C(\operatorname{Sp}(L)), BH)_x$ consists of simplicial sheaf maps (or morphisms of sheaves of groupoids)

$$f : C(\operatorname{Sp}(L)) \to BH \tag{6}$$

such that the degree 0 part $f : \operatorname{Sp}(L) \to \operatorname{Ob}(H)$ represents $x \in \pi^* \operatorname{Ob}(H)$. Insofar as there are isomorphisms

$$EG(L/k) \times_{G(L/k)} \operatorname{Sp}(L) \cong C(\operatorname{Sp}(L))$$

the maps f above can be rewritten as maps

$$f : EG(L/k) \times_{G(L/k)} \operatorname{Sp}(L) \to BH$$

which satisfy the section condition in simplicial degree 0.

It is a consequence of Lemma 12 that the map

$$Bh(*, BH)_x \to \pi_0 Bh(*, BH)_x$$

is a weak equivalence.

Finally, as in Example 13, if $\Gamma^* K$ is the constant sheaf on the étale site $et|_k$ on some group K, then $\pi^* B\Gamma^* K \cong BK$, and the homotopy fibre $Bh(*, B\Gamma^* K)_*$ of the map

$$\pi^* : B(\Gamma^* K - \mathbf{tors}) \to BK$$

has

$$\pi_0 Bh(*, B\Gamma^* K)_* \cong \varinjlim_L \hom(G(L/k), K).$$

Similar observations hold for sheaves of groupoids on the finite étale site of a connected Noetherian scheme S. In particular, *pointed* non-abelian étale H^1 of S with coefficients in a constant group is pro-represented by the Grothendieck fundamental group in the categorical sense.

References

[1] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[2] J. F. Jardine. *Generalized Étale Cohomology Theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.

[3] J. F. Jardine. Stacks and the homotopy theory of simplicial sheaves. *Homology Homotopy Appl.*, 3(2):361–384 (electronic), 2001. Equivariant stable homotopy theory and related areas (Stanford, CA, 2000).

[4] J. F. Jardine. Cocycle categories. In *Algebraic Topology*, volume 4 of *Abel Symposia*, pages 185–218. Springer, Berlin Heidelberg, 2009.

[5] James S. Milne. *Étale Cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.