Representability theorems for presheaves of spectra

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Introduction

The Brown representability theorem gives a list of conditions for the representablity of a set-valued contravariant functor which is defined on the classical stable homotopy category. It has had many uses through the years, and has long been part of the canon of Algebraic Topology.

It is entirely reasonable to ask for a more general version of the theorem, which would give conditions on a closed model category $\mathcal{N}$ and a contravariant set-valued functor G defined on the homotopy category $\mathrm{Ho}(\mathcal{N})$ so that the functor G is representable. One could call this a Brown representability theorem for $\mathcal{N}$, although some might say that it is a "cohomological" Brown representability result [1], [9].

A result of this type is proved in this paper, and appears as Theorem 19. The conditions for the result are essentially classical: the model category $\mathcal{N}$ must have a set of compact generators, suitably defined, while the functor G should take coproducts to products and should satisfy a Mayer-Vietoris property. Theorem 19 asserts that G is representable under these circumstances. The proof displayed for this result is the standard argument (see also the proof of Theorem 3.1 in [8], or Heller's purely categorical formulation in [2]), albeit translated into the language of model categories. The ideas behind Theorem 19 and its proof are not new.

Multiple settings in which Theorem 19 applies are displayed in the third section, following the proof. The basic message is that there are classical Brown representability results for various stable model structures arising from simplicial presheaves — these include presheaves of spectra, diagrams of spectra, and motivic T-spectra, as well as presheaves of chain complexes — *so long as* the underlying local model structure is defined on a rather forgiving Grothendieck topology, for which a set of compact generators can be defined in a traditional way.

Many of the standard geometric topologies, such as the étale topology, are not so forgiving, and the classical argument for Brown representability does not work for those cases. The problem is the compact generation requirement, which fails because "small" inductive colimits of fibrant objects may not be fibrant in any reasonable sense. This is overcome by using the observation that inductive colimits of fibrant simplicial presheaves are fibrant provided that the inductive systems are large enough, but at the cost of introducing homotopy coherence issues which cannot be addressed within the traditional framework for Brown representability.

Homotopy coherence problems are often solved in the context of simplicial functors between simplicial model categories, and that is what is done here. The main result of this paper, which is Theorem 12, gives conditions on a pointed simplicial model category $\mathcal{M}$ and a (contravariant) simplicial functor $F : \mathcal{M}^{op} \to s\mathbf{Set}_*$ taking values in pointed simplicial sets, which conditions imply that F is sectionwise weakly equivalent to a representable functor $\mathbf{hom}(\ ,Y)$ that is defined by an object Y of $\mathcal{M}$ which is fibrant and cofibrant.

The conditions on the simplicial model category $\mathcal{M}$ are abstractions of the general behaviour of the stable model structure on presheaves of spectra.

The conditions on the functor F are satisfied by representable functors: this functor should preserve weak equivalences between cofibrant objects, and should take homotopy colimits to homotopy inverse limits.

The homotopy colimits condition implies strong forms of both the wedge and Mayer-Vietoris properties that one finds in the conditions for the classical Brown representability theorem, and it gives a way of inductively producing vertices of homotopy inverse limits of big towers. This last device solves the homotopy coherence problem in the formulation and proof of the representability theorem; the technique appears in the proof of Proposition 5, which is the key step in the derivation of the main result.

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1 General assumptions

Suppose that $\mathcal{M}$ is a closed simplicial model category. Write $\mathbf{hom}(A,B)$ for the function complexes associated with simplicial structure of $\mathcal{M}$, and let

$$A \otimes \Delta^n_+ \to B$$

be the morphisms making up their simplices. Here, as usual, L_+ is a simplicial set L with a disjoint base point attached.

Suppose that the model category $\mathcal{M}$ has the following properties:

M1 The category $\mathcal{M}$ has all small limits and colimits, and $\mathcal{M}$ is pointed in the sense that its initial and terminal objects coincide.

M2 The model structure on $\mathcal{M}$ is cofibrantly generated, meaning that there is a set I of cofibrations and a set J of trivial cofibrations, such that $p : X \to Y$ is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all members of I (respectively J). All source objects of morphisms in the set I are cofibrant.

Suppose that $\mathcal{C}$ is a small Grothendieck site. Then the injective model structure on the category $s\,\mathrm{Pre}(\mathcal{C})_*$ of pointed simplicial presheaves on $\mathcal{C}$ satisfies the properties **M1** and **M2**, as do all of its Bousfield localizations. The categories $\mathrm{Spt}(\mathcal{C})$ of presheaves of spectra and $\mathrm{Spt}_\Sigma(\mathcal{C})$ of presheaves of symmetric spectra on the site $\mathcal{C}$ also satisfy these properties. The standard models for the motivic stable category [4] also satisfy properties **M1** and **M2**.

We shall also assume that the category $\mathcal{M}$ satisfies the following condition:

M3 There is a set S of cofibrant objects K such that

a) a map $f : X \to Y$ is a weak equivalence if and only if it induces bijections

$$[K, X] \xrightarrow{\cong} [K, Y]$$

of morphisms in the homotopy category $\mathrm{Ho}(\mathcal{M})$ for all objects K of S.

b) There is an infinite cardinal β such that, if $Y : \gamma \to \mathcal{M}$ is an inductive system of fibrant objects Y_s, $s < \gamma$, then the colimit $\varinjlim_{s<\gamma} Y_s$ is fibrant, and the map

$$\varinjlim_{s<\gamma} \mathbf{hom}(K, Y_s) \to \mathbf{hom}(K, \varinjlim_{s<\gamma} Y_s)$$

is a weak equivalence for all $K \in S$.

The condition **M3** is typical of models for stable homotopy theory. In particular, the shifted suspension spectra $\Sigma^\infty(L \wedge U_+)[n]$ which are associated to finite pointed simplicial sets L and objects U of the small site $\mathcal{C}$ gives such a set S for the stable model structure on presheaves of spectra, while their analogues $F_n(L \wedge U_+)$ for symmetric spectra give the set S for presheaves of symmetric spectra [5]. The same is true for the spectrum and symmetric spectrum models for the motivic stable category [4].

Observe that a statement analogous to **M3b** holds in all categories of pointed simplicial presheaves and their Bousfield localizations for any set of objects S. In all cases where **M3a** holds, the set S is a set of generators for the underlying model category in the sense of [3].

Say that the closed simplicial model category $\mathcal{M}$ *satisfies the conditions* **M*** if it satisfies the conditions **M1**, **M2** and **M3**.

Suppose that I is a small category. Since $\mathcal{M}$ is cofibrantly generated, the category $\mathcal{M}^I$ of I-diagrams, or functors $I \to \mathcal{M}$ and their natural transformations, has a model structure for which a map $X \to Y$ is a fibration (respectively weak equivalence) if all component maps $X_i \to Y_i$ are fibrations (respectively weak equivalences) of $\mathcal{M}$. The cofibrations for the theory, which are called *projective cofibrations*, are those maps which have the left lifting property with respect to all trivial fibrations. This is the projective model structure for the category $\mathcal{M}^I$ of I-diagrams in $\mathcal{M}$.

If the closed simplicial model category $\mathcal{M}$ satisfies conditions **M***, then the projective model structure on the diagram category $\mathcal{M}^I$ satisfies conditions **M***. The "generating set" in this case is the set of objects $L_i(K)$, where $K \in S$ and the functor L_i is left adjoint to the functor $X \mapsto X_i$.

2 The representability theorem

Suppose that $\mathcal{M}$ is a closed simplicial model category which satisfies the properties **M*** of the first section.

We shall be considering functors

$$F : \mathcal{M}^{op} \to s\mathbf{Sets}_*$$

which are defined contravariantly on $\mathcal{M}$, and have the following properties:

F1 The space $F(*)$ is contractible.

F2 The functor F takes weak equivalences $f : A \to B$ between cofibrant objects to weak equivalences $f^* : F(B) \to F(A)$.

F3 Suppose that I is a small category, and that $X : I \to \mathcal{M}$ is a projective cofibrant diagram in $\mathcal{M}$. Then the map

$$F(\varinjlim_i X_i) \to \underleftarrow{\text{holim}}_{\,i} F(X_i)$$

is a weak equivalence. In other words, F should take homotopy colimits to homotopy inverse limits, up to weak equivalence.

Recall that the *homotopy colimit* $\underrightarrow{\text{holim}}_{\,I}\ X$ for a diagram $X : I \to \mathcal{M}$ is defined up to weak equivalence by taking a projective cofibrant model $\tilde{X} \to X$ of X (ie. a weak equivalence with $\tilde{X}$ projective cofibrant), and then one sets

$$\underrightarrow{\text{holim}}_{\,I}\ X = \varinjlim_I\ \tilde{X}.$$

We shall say that a functor F *satisfies the properties* **F*** if it satisfies the properties **F1**, **F2** and **F3**.

The point of this section is to establish conditions on the functor F which guarantee that F is sectionwise equivalent to a representable functor $\mathbf{hom}(\ ,Y)$ with Y fibrant. A natural transformation $f : F \to G$ of functors $\mathcal{M}^{op} \to s\mathbf{Set}_*$ is said to be a *sectionwise equivalence* if the map $f : F(A) \to G(A)$ is a weak equivalence for each cofibrant object A of $\mathcal{M}$

The three conditions are invariant of sectionwise equivalence: if there is a sectionwise equivalence $f : F \to G$, then F satisfies the properties **F*** if and only if they are satisfied by G. Thus, for example, it is harmless to suppose that F takes values in Kan complexes, since the the natural map

$$j : F \to \mathrm{Ex}^{\infty} F$$

arising from Kan's Ex^{∞} construction is a sectionwise weak equivalence.

The three conditions are satisfies by all representable functors $\mathbf{hom}(\ ,Y)$ with Y fibrant. In particular, if $X : I \to \mathcal{M}$ is a projective cofibrant diagram, then the canonical map

$$\mathbf{hom}(\varinjlim_{i} X_i, Y) \to \underleftarrow{\mathrm{holim}}_{\,i}\mathbf{hom}(X_i, Y)$$

is a weak equivalence, because the I^{op}-diagram defined by $i \mapsto \mathbf{hom}(X_i, Y)$ is injective fibrant (see, for example, [6, p.114-5]), so that the functor $\mathbf{hom}(\ ,Y)$ satisfies **F3**. The conditions **F1** and **F2** are easy to verify in this case.

It follows that if a functor F is sectionwise equivalent to a representable functor $\mathbf{hom}(\ ,Y)$ with Y fibrant, then F satisfies conditions **F***.

Example 1. Suppose that $I = \mathrm{Ob}(I)$ is a discrete category on its set of objects, so that it has only identity arrows. A diagram $X : I \to \mathcal{M}$ consists of a set of objects X_i, $i \in I$, and X is projective cofibrant if and only if all of the objects X_i are cofibrant. Then condition **F3** for F in this case asserts that the composite map

$$F(\bigvee_i X_i) \to \prod_i F(X_i)$$

is a weak equivalence. The special case of condition **F3** for discrete diagrams is otherwise known as the *wedge property* for the functor F.

Example 2. Among all diagrams having the shape

$$X_1 \xleftarrow{i_1} X_0 \xrightarrow{i_2} X_2$$

the projective cofibrant ones are the diagrams for which all objects X_i are cofibrant and the two morphisms i_1, i_2 are cofibrations. Then condition **F3** for diagrams of this shape means precisely that the diagram

$$\begin{array}{ccc} F(X_1 \cup_{X_0} X_2) & \longrightarrow & F(X_2) \\ \downarrow & & \downarrow \\ F(X_1) & \longrightarrow & F(X_0) \end{array}$$

is homotopy cartesian. In the presence of condition **F3**, and because all of the objects X_i are cofibrant, this diagram will be homotopy cartesian if just one of the maps i_1, i_2 is a cofibration. This is a strong form of the *Mayer-Vietoris property* for F — see the description of this property property in Section 3.

Example 3. Suppose that γ is an infinite cardinal number, and that $A : \gamma \to \mathcal{M}$ is a directed system indexed by γ. Suppose that the system is projective cofibrant — this means that A_0 is cofibrant, all maps $A_s \to A_{s+1}$ are cofibrations, and that the maps $\varinjlim_{s<t} \to A_t$ are cofibrations for all limit ordinals $t < \gamma$. Observe that the restriction of the diagram A to any limit ordinal $\beta < \gamma$ is a projective cofibrant β-diagram.

Applying the functor F to all A_s, $s < \gamma$ gives a "tower" $F(A) : \gamma^{op} \to s\mathbf{Set}_*$. Suppose that $j : F(A) \to Z$ is an injective fibrant model in the category of γ^{op}-diagrams. Then Z_0 is fibrant, the maps $Z_{s+1} \to Z_s$ are fibrations for all $s < \gamma$, and the maps $Z_t \to \varprojlim_{s<t} Z_s$ are fibrations for all limit ordinals $t < \gamma$.

The assumption that the functor F satisfies condition **F3** means that the map

$$F(\varinjlim_{s<t} A_s) \to \varprojlim_{s<t} Z_s$$

is a weak equivalence for all limit ordinals $t \leq \gamma$.

In the special case that the map $\varinjlim_{s<t} A_s \to A_t$ is an isomorphism for all limit ordinals $t < \gamma$, we could, in the construction of Z, set

$$Z_t = \varprojlim_{s<t} Z_s$$

for all limit ordinals t.

Suppose that X is cofibrant and that $u \in F(Y)_0$ with Y cofibrant. If Y is also fibrant then an evaluation map

$$u^* : [X, Y] \to \pi_0 F(X)$$

can be defined by the assignment $[\alpha] \mapsto [\alpha^*(u)]$. If Y is not fibrant, let $j : Y \to LY$ be a fibrant model (with j a trivial cofibration). Then there is a unique element $[v] \in \pi_0 F(LY)$ such that $[v] \mapsto [u]$ under the isomorphisms $\pi_0 F(LY) \to \pi_0 F(Y)$, and then the *evaluation map* u^* is defined to be the composite

$$[X, Y] \xrightarrow{\cong} [X, LY] \xrightarrow{v^*} \pi_0 F(X).$$

The definition of u^* is independent of the choice of fibrant model LY.

Suppose that Y is a cofibrant object of $\mathcal{M}$. An element u of $F(Y)_0$ is said to be *universal* if the evaluation map

$$u^* : [K, Y] \to \pi_0 F(K)$$

is an isomorphism for all objects $K \in S$.

The near-term goal of the following is to show that every functor F satisfying conditions **F*** has a universal element. Suppose that F satisfies conditions **F*** for the rest of this section.

Lemma 4. *Suppose that A is a cofibrant object of $\mathcal{M}$ and that $u \in F(A)_0$. Then there is a cofibration $i : A \to B$ and an element $v \in F(B)_0$ with $i^*([v]) = [u]$ in $\pi_0 F(A)_0$, and such that in the diagram*

$$\begin{array}{ccc} [K, A] & \xrightarrow{i_*} & [K, B] \\ & \searrow^{u^*} & \downarrow{\scriptstyle v^*} \\ & & \pi_0 F(K) \end{array}$$

the following hold:

1) the map v^* is surjective,

2) if $u^*(\alpha) = u^*(\beta)$ then $i_*(\alpha) = i_*(\beta)$.

Proof. We can suppose that A is fibrant.

Form the coproduct

$$A \vee (\bigvee_\lambda K)$$

over all

$$\lambda \in \pi_0 F(K), \ K \in S.$$

Take the list of all pairs of elements $[\alpha], [\beta] \in [K, A]$ such that $u^*[\alpha] = u^*[\beta]$ in $\pi_0 F(K)$, and choose representatives $\alpha, \beta : K \to A$ for all such pairs of elements. Form the pushout diagram

$$\begin{array}{ccccc} \bigvee_{(\alpha,\beta)} (K \vee K) & \longrightarrow & A & \longrightarrow & A \vee (\bigvee_\lambda K) \\ \downarrow & & & & \downarrow \\ \bigvee_{(\alpha,\beta)} (K \otimes \Delta^1_+) & & \longrightarrow & & B \end{array}$$

and observe that all objects in the diagram are cofibrant.

Write j for the composite cofibration

$$A \to A \vee (\bigvee_\lambda K) \to B.$$

There is an element $[w] \in \pi_0 F(A \vee (\bigvee_\lambda K))$ which restricts to $[u] \in \pi_0 F(A)$ and all $\lambda \in \pi_0 F(K)$, by the wedge property. There is an element $[v] \in \pi_0 F(B)$ which restricts simultaneously to $[w]$ and the sequence $(u^*[\alpha] = u^*[\beta])$, since the diagram

$$\begin{array}{ccc} F(B) & \longrightarrow & F(\bigvee_{(\alpha,\beta)}(K \otimes \Delta^1_+)) \\ \downarrow & & \downarrow \\ F(A \vee (\bigvee_\lambda K)) & \longrightarrow & F(\bigvee_{(\alpha,\beta)}(K \vee K)) \end{array}$$

is homotopy cartesian.

The map $v^* : [K, B] \to \pi_0 F(K)$ is surjective for all K by construction. All pairs of elements $[\alpha], [\beta] \in [K, A]$ such that $u^*[\alpha] = u^*[\beta]$ in $\pi_0 F(K)$ also have the same image in $[K, B]$. □

Proposition 5. *Suppose that A is cofibrant and that u is a vertex of $F(A)$. Then there is a cofibration $i : A \to Y$ with a universal element $v \in F(Y)_0$ such that $i^*([v]) = [u] \in \pi_0 F(A)$.*

Proof. We construct a projective cofibrant diagram $A : \beta \to \mathcal{M}$ together with an inductive fibrant model $j : F(A) \to Z$, by induction on $s < \beta$.

Set $A_0 = A$ and $u_0 = u$. Suppose that $t < \beta$ and that A_s and the maps $j : F(A_s) \to Z_s$ have been defined for $s < t$. Suppose further that vertices $v_s \in Z_s$ have been chosen which are compatible in the sense that if $s' \leq s < t$ then $v_s \mapsto v_{s'}$ under the fibration $p : Z_s \to Z_{s'}$. Suppose that $v_0 = j(u_0)$.

It $t = s + 1$ then the map $F(A_s) \to Z_s$ is a weak equivalence, so there is a vertex $u_s \in F(A_s)$ such that $j_*[u_s] = [z_s] \in \pi_0 Z_s$. Choose a cofibration $i : A_s \to A_{s+1}$ with $u_{s+1} \in F(A_{s+1})$ according to the construction of Lemma 4. Form a diagram

$$\begin{array}{ccc} F(A_{s+1}) & \xrightarrow{j} & Z_{s+1} \\ \downarrow & & \downarrow p \\ F(A_s) & \xrightarrow[j]{} & Z_s \end{array}$$

such that j is a trivial cofibration and p is a fibration. Then $[p(j(u_{s+1}))] = [z_s] \in \pi_0 Z_s$ and p is a fibration so there is a vertex $z_{s+1} \in Z_{s+1}$ such that $p(z_{s+1}) = z_s$ and $[z_{s+1}] = [j(u_{s+1})] \in \pi_0 Z_{s+1}$.

It t is a limit ordinal, set $A_t = \varinjlim_{s<t} A_s$, set $Z_t = \varprojlim_{s<t} Z_s$, and let $j : F(A_t) \to Z_t$ be the canonical map. Then $j : F(A_t) \to Z_t$ is a weak equivalence since F takes homotopy colimits to homotopy inverse limits. Let the vertex z_t be the map $* \to \varinjlim_{s<t} Z_s$ which is determined by all z_s, $s < t$.

Suppose that $Y = \varinjlim_{s<\beta} A_s$, and set $Z_\beta = \varprojlim_{s<\beta} Z_s$. Let z_β be the vertex of Z_β which is defined by all the z_s, $s < \beta$. The natural transformation $j : F(A) \to Z$ induces a weak equivalence

$$F(Y) \xrightarrow{j_*} Z_\beta$$

again by condition **F3**. It follows that there is a vertex $v \in F(Y)$ such that

$$[j_*(v)] = [z_\beta] \in \pi_0 Z_\beta.$$

Then there are commutative diagrams

$$\begin{array}{ccccc} [K, A_s] & \longrightarrow & [K, A_{s+1}] & \longrightarrow & [K, Y] \\ & \searrow^{u_s^*} & & \searrow^{u_{s+1}^*} & \downarrow v^* \\ & & & & \pi_0 F(K) \end{array}$$

and the map v^* is an isomorphism for all $K \in S$. In effect, the function

$$\varinjlim_{s<\gamma} [K, A_s] \to [K, Y]$$

is a bijection for all $K \in S$, since β is sufficiently large. If $i : A \to Y$ is the canonical map, then $i^*([v]) = [u] \in \pi_0 F(A)$. □

Lemma 6. *Suppose that $f : Y \to Y'$ is a morphism of cofibrant objects of $\mathcal{M}$. Suppose that $u \in F(Y)_0$ and $u' \in F(Y')_0$ are universal, and that $f^*([u']) = [u] \in \pi_0 F(Y)$. Then f is a weak equivalence.*

Proof. This is a consequence of the condition **M3a**, together with the commutativity of the diagrams

$$\begin{array}{ccc} [K, Y] & \xrightarrow{f_*} & [K, Y'] \\ & \underset{u^*}{\overset{\cong}{\searrow}} & \Big\downarrow {\scriptstyle \cong\ (u')^*} \\ & & \pi_0 F(K) \end{array}$$

which are associated to all objects $K \in S$. □

Lemma 7. *Suppose given maps*

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\Big\downarrow & & \\ B & & \end{array}$$

of $\mathcal{M}$ such that i is a cofibration and A and Y are cofibrant. Suppose that $u \in F(Y)_0$ is universal and that $x \in F(B)$ satisfies $i^*([x]) = f^*([u]) \in \pi_0 F(A)$. Then f extends to a map $g : B \to Y$ in the homotopy category such that $g^*([u]) = [x] \in \pi_0 F(B)$.

Proof. Form the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\Big\downarrow & & \Big\downarrow{\scriptstyle i_*} \\ B & \longrightarrow & B \cup_A Y \end{array} \qquad (1)$$

Then there is an element $[w] \in \pi_0 F(B \cup_A Y)$ which restricts to $[x] \in \pi_0 F(B)$ and $[u] \in \pi_0 F(Y)$ by the Mayer-Vietoris property. There is a cofibration $j : B \cup_A Y \to Y'$ such that Y' has a universal element $v \in F(Y')_0$ such that $j^*([v]) = [w] \in \pi_0 F(B \cup_A Y)$, by Lemma 4. The composite map

$$Y \xrightarrow{i_*} B \cup_A Y \to Y'$$

is a weak equivalence by Lemma 6, and is therefore an isomorphism in the homotopy category. □

Proposition 8. *Suppose that $F : \mathcal{M}^{op} \to s\mathbf{Set}_*$ is a functor which satisfies conditions* **F*** *. Then there is a cofibrant object Y of $\mathcal{M}$ and a vertex $u \in F(Y)_0$ such that the evaluation map*

$$u^* : [X, Y] \xrightarrow{\cong} \pi_0 F(X)$$

is a bijection for all cofibrant objects X of $\mathcal{M}$.

Proof. By Proposition 5 applied to some $v \in F(*)_0$ (note that $F(*)$ is contractible hence non-empty) there is a cofibrant object Y of $\mathcal{M}$ with a universal element $u \in F(Y)$. We show that the induced map

$$u^* : [X, Y] \to \pi_0 F(X)$$

is a bijection for all cofibrant X.

Suppose that $v \in F(X)_0$. Then applying Lemma 7 to the diagram

$$\begin{array}{ccc} * & \longrightarrow & Y \\ \downarrow & & \\ X & & \end{array}$$

gives a map $g : X \to Y$ in the homotopy category $\mathrm{Ho}(\mathcal{M})$ such that $g^*([u]) = [v]$, so that $u^*[g] = [v]$. It follows that the function u^* is surjective.

To prove the injectivity of u^*, we can suppose that Y is fibrant. Suppose that $u^*[g_0] = u^*[g_1] = [v] \in \pi_0 F(X)$ for maps $g_0, g_1 : X \to Y$. Consider the maps

$$\begin{array}{ccc} X \vee X & \xrightarrow{(g_0, g_1)} & Y \\ {\scriptstyle (d^0, d^1)} \downarrow & & \\ X \otimes \Delta^1_+ & & \end{array}$$

and choose an element $[w] \in \pi_0 F(X \otimes \Delta^1_+)$ which restricts to $[v]$ along the functions induced by the maps $F(d^0) = F(d^1) : F(X \otimes \Delta^1_+) \to F(X)$. Then Lemma 7 says that (g_0, g_1) extends to a map $h : X \otimes \Delta^1 \to Y$ in the homotopy category $\mathrm{Ho}(\mathcal{M})$ such that $h^*([u]) = [w]$. But then g_0 and g_1 represent the same map in $\mathrm{Ho}(\mathcal{M})$. $\square$

Suppose now that

$$F : \mathcal{M}^{op} \to s\mathbf{Set}_*$$

is a *simplicial functor*, and that it satisfies the conditions **F***.

The simplicial functor F associates a simplicial set map $\alpha^* : F(B) \wedge \Delta^n_+ \to F(A)$ to every morphism $\alpha : A \otimes \Delta^n_+ \to B$. In particular there is a map $f_n : F(A \otimes \Delta^n_+) \wedge \Delta^n_+ \to F(A)$ to the identity map on $A \otimes \Delta^n_+$. Taking adjoints of the f_n gives maps

$$f_{n*} : F(A \otimes \Delta^n_+) \to \mathbf{hom}(\Delta^n_+, F(A)),$$

Every map $\theta : \Delta^m \to \Delta^n$ induces a commutative diagram

$$\begin{array}{ccc} F(A \otimes \Delta^n_+) & \xrightarrow{f_{n*}} & \mathbf{hom}(\Delta^n_+, F(A)) \\ {\scriptstyle (1\wedge\theta)^*}\downarrow & & \downarrow{\scriptstyle \theta^*} \\ F(A \otimes \Delta^m_+) & \xrightarrow[f_{m*}]{} & \mathbf{hom}(\Delta^m_+, F(A)) \end{array}$$

It follows that there is a map

$$f : F(A \otimes L) \to \mathbf{hom}(L, F(A))$$

which is natural in pointed simplicial presheaves A and pointed simplicial sets L. The map

$$f : \mathbf{hom}(A \otimes L, Z) \to \mathbf{hom}(L, \mathbf{hom}(A, Z))$$

is a canonical isomorphism.

In simplicial degree 0, maps $f : A \to B$ in $\mathcal{M}$ are identified with maps $\tilde{f} : A \otimes \Delta^0_+ \to B$ via the diagrams

$$\begin{array}{ccc} A \otimes \Delta^0_+ & & \\ {\scriptstyle \cong}\downarrow & \searrow^{\tilde{f}} & \\ A & \xrightarrow[f]{} & B \end{array}$$

and there are commutative diagrams

$$\begin{array}{ccc} F(B) \wedge \Delta^0_+ & \xrightarrow{\tilde{f}^*} & F(A) \\ {\scriptstyle \cong}\downarrow & \nearrow_{f^*} & \\ F(B) & & \end{array}$$

It follows that the map $f_0 : F(A \otimes \Delta^0_+) \wedge \Delta^0_+ \to F(A)$ is the composite of canonical isomorphisms

$$F(A \otimes \Delta^0_+) \wedge \Delta^0_+ \xrightarrow{\cong} F(A \otimes \Delta^0) \xrightarrow{\cong} F(A).$$

It also follows that the adjoint map

$$f_{0*} : F(A \otimes \Delta^0_+) \to \mathbf{hom}(\Delta^0_+, F(A))$$

is an isomorphism.

Lemma 9. *Suppose that the simplicial functor F satisfies conditions* **F***, *and that F takes values in Kan complexes. Then the natural map*

$$f : F(A \otimes L) \to \mathbf{hom}(L, F(A))$$

is a weak equivalence for all pointed simplicial sets L and all cofibrant objects A of $\mathcal{M}$.

Proof. The map

$$f : F(A \otimes \Delta^0_+) \to \mathbf{hom}(\Delta^0_+, F(A))$$

is an isomorphism, and both functors involved in the natural map f preserve weak equivalences, so that all maps

$$f : F(A \otimes \Delta^n_+) \to \mathbf{hom}(\Delta^n_+, F(A))$$

are weak equivalences.

Now proceed by induction on n for the skeleta $\mathrm{sk}_n L$ of L. The pushout squares

$$\begin{array}{ccc} \bigvee_{\sigma \in NL_n} \partial\Delta^n_+ & \longrightarrow & \mathrm{sk}_{n-1}(L) \\ \downarrow & & \downarrow \\ \bigvee_{\sigma \in NL_n} \Delta^n_+ & \longrightarrow & \mathrm{sk}_n(L) \end{array}$$

are mapped to homotopy cartesian diagrams by both functors (recall that F takes homotopy colimits to homotopy inverse limits), and so the maps

$$f : F(A \otimes \mathrm{sk}_n(L)) \to \mathbf{hom}(\mathrm{sk}_n L, F(A))$$

are weak equivalences for all pointed simplicial sets L, and for all $n \geq 0$.

Finally, the space $F(A \otimes L)$ is naturally equivalent to the homotopy inverse limit of the spaces $F(A \otimes \mathrm{sk}_n L)$, and the space $\mathbf{hom}(L, F(A))$ is naturally equivalent to the homotopy inverse limit of the spaces $\mathbf{hom}(\mathrm{sk}_n L, F(A))$. The result follows. □

A morphism

$$u : \mathbf{hom}(\ , Y) \to F$$

of simplicial functors is completely determined by the element $u = u(1_Y) \in F(Y)_0$ such that $u \mapsto *$ under the map $F(Y) \to F(*)$ which is induced by the morphism $* \to Y$. There is a homotopy

$$\omega : \mathbf{hom}(\ , Y) \wedge \Delta^1_+ \to F$$

between such functors u, u' if and only if there is a path $\omega : u \to u'$ in the fibre $\tilde{F}(Y)$ over the base point of the map $F(Y) \to F(*)$. It also follows there is a natural isomorphism

$$\mathbf{Nat}(\mathbf{hom}(\ , Y), F) \cong \tilde{F}(Y)$$

where $\mathbf{Nat}(G, F)$ denotes the function space of natural transformations between simplicial functors G and F.

As noted before, we are entitled to vary the simplicial functor F up to sectionwise equivalence. In particular Kan's Ex^∞-construction $j : X \to \mathrm{Ex}^\infty X$ determines a simplicial functor $\mathrm{Ex}^\infty(F)$ with $\mathrm{Ex}^\infty(F)(A) = \mathrm{Ex}^\infty(F(A))$, and the weak equivalences $j : F(A) \to \mathrm{Ex}^\infty(F(A))$ define a natural morphism

$$j : F \to \mathrm{Ex}^\infty F$$

of simplicial functors. This map j is a sectionwise weak equivalence.

Remark 10. Suppose that G is a simplicial functor $\mathcal{M}^{op} \to s\mathbf{Set}_*$. Suppose in addition that the pointed simplicial set $G(*)$ is contractible. The canonical maps $t : A \to *$ in $\mathcal{M}$ induce cofibrations $t^* : G(*) \to G(A)$, which together determine a map $t^* : G(*) \to G$ of simplicial functors, where $G(*)$ is the constant simplicial diagram on $\mathcal{M}$ associated to the pointed simplicial set $G(*)$. Form the quotient $G/G(*))$ and consider the canonical transformation $p : G \to G/G(*)$. The map p is a sectionwise equivalence since $G(*)$ is contractible, and there is an isomorphism

$$(G/G(*))(*) = G(*)/G(*) \cong *.$$

Kan's Ex^∞ functor preserves points, so the composite

$$G \xrightarrow{p} G/G(*) \xrightarrow{j} \mathrm{Ex}^\infty(G/G(*)) =: \tilde{G}$$

gives a sectionwise equivalence $G \to \tilde{G}$ such that $\tilde{G}$ takes values in Kan complexes and $\tilde{G}(*) = *$.

Lemma 11. *Suppose that $u \in F(Y)_0$ is a universal element which determines a map*

$$u : \mathbf{hom}(\ ,Y) \to F$$

of simplicial functors, where Y is cofibrant and fibrant and F takes values in Kan complexes and satisfies condition **F***. *Then the map u is a sectionwise weak equivalence.*

Proof. We show that the induced map

$$[L, \mathbf{hom}(A, Y)] \xrightarrow{u_*} [L, F(A)]$$

is a bijection for each pointed simplicial set L and cofibrant object A.

In the diagram

$$\begin{array}{ccc} \mathbf{hom}(A \otimes L, Y) & \xrightarrow{u_*} & F(A \otimes L) \\ \downarrow f & & \downarrow f \\ \mathbf{hom}(L, \mathbf{hom}(A, Y)) & \xrightarrow[u_*]{} & \mathbf{hom}(L, F(A)) \end{array}$$

the vertical maps f are weak equivalences by Lemma 9, and the map

$$u : \mathbf{hom}(A \otimes L, Y) \to F(A \otimes L)$$

is an isomorphism in path components by Proposition 8. But then the map

$$u_* : \mathbf{hom}(L, \mathbf{hom}(A, Y)) \to \mathbf{hom}(L, F(A))$$

induces an isomorphism in path components. □

The following is the main result of this section. It is the *representability theorem* of the section title.

Theorem 12. *Suppose that $\mathcal{M}$ is a closed simplicial model category which satisfies conditions* **M***. *Suppose that $F : \mathcal{M}^{op} \to s\mathbf{Sets}_*$ is a simplicial functor which satisfies conditions* **F***. *Then there are sectionwise equivalences*

$$F \xrightarrow{\simeq} \tilde{F} \xleftarrow{\simeq} \mathbf{hom}(\ ,Y),$$

where Y is some fibrant object of $\mathcal{M}$.

Proof. The construction of Remark 10 gives a sectionwise weak equivalence $F \to \tilde{F}$ such that $\tilde{F}$ takes values in Kan complexes and $\tilde{F}(*) = *$. The simplicial functor satisfies the conditions **F***, and therefore has a universal element $u \in \tilde{F}(Y)$ for some cofibrant (and fibrant) object Y of $\mathcal{M}$ by Proposition 5. This element u defines a morphism of simplicial functors

$$u : \mathbf{hom}(\ ,Y) \to \tilde{F}$$

by the construction of the simplicial functor $\tilde{F}$, and the morphism u is a sectionwise weak equivalence by Lemma 11. □

3 Classical Brown representability

Here are the properties that we shall require for the closed model category $\mathcal{N}$ in this section:

N1 The category $\mathcal{N}$ has all small colimits. The initial object $*$ of $\mathcal{N}$ is also terminal, so that $\mathcal{N}$ is a pointed model category.

N2 There is a set S of compact cofibrant objects $\{K\}$ such that a map $f : X \to Y$ is a weak equivalence if and only if it induces a bijection

$$[K,X] \xrightarrow{\cong} [K,Y]$$

of morphisms in the homotopy category $\mathrm{Ho}(\mathcal{N})$, for all objects K in S.

An object K of $\mathcal{N}$ is said to be *compact* if the function

$$\varinjlim_i [K,Y_i] \to [K, \varinjlim Y_i]$$

is a bijection *for all* inductive systems

$$Y_0 \to Y_1 \to Y_2 \to \dots$$

I shall say that a model category $\mathcal{N}$ satisfies the conditions **N*** if it satisfies properties **N1** and **N2**. It is typical to say, under such circumstances, that the model category $\mathcal{N}$ is *compactly generated*, and that the elements K of S are *compact generators* for $\mathcal{N}$.

Remark 13. If $\mathcal{N}$ satisfies the conditions **N***, then inductive colimits preserve weak equivalences.

More specifically, suppose that the map $f : X \to Y$ is a comparison of inductive systems $X, Y : \alpha \to \mathcal{N}$ (where α is any cardinal) such that all maps $f_s : X_s \to Y_s$, $s < \alpha$, are weak equivalences. Then the induced map

$$\varinjlim X_s \to \varinjlim Y_s$$

is a weak equivalence. In effect, there are bijections

$$[K, \varinjlim X_s] \cong \varinjlim [K, X_s] \cong \varinjlim [K, Y_s] \cong [K, \varinjlim Y_s].$$

for all objects K of S.

We shall now consider functors

$$G : \mathcal{N}^{op} \to \mathbf{Sets}_*$$

which take values in pointed sets, and have the following properties:

G1 G takes weak equivalences to bijections.

G2 The set $G(*)$ is the one-point set.

G3 (wedge property) For any coproduct $\bigvee_i X_i$ of a set of cofibrant objects $\{X_i\}$ of $\mathcal{N}$, the canonical induced map

$$G(\bigvee_i X_i) \to \prod_i G(X_i)$$

is a bijection.

G4 (Mayer-Vietoris property) Suppose that the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & B \cup_A X \end{array}$$

is a pushout, where i is a cofibration and all objects are cofibrant. Then the induced function

$$G(B \cup_A X) \to G(B) \times_{G(A)} G(X)$$

is surjective.

I say that a functor $G : \mathcal{N}^{op} \to \mathbf{Sets}_*$ *satisfies the conditions* **G*** if it has the properties **G1** – **G4**.

Example 14. Suppose that Z is an object of $\mathcal{N}$. Then the functor

$$G(X) = [X, Z]$$

which is defined by morphisms in the homotopy category is pointed by the composite $X \to * \to Z$, and satisfies the conditions **G***.

Lemma 15. *Suppose that*

$$Y_0 \xrightarrow{\alpha} Y_1 \xrightarrow{\alpha} \cdots \tag{2}$$

is a countable sequence of maps of $\mathcal{N}$ where all objects Y_i are cofibrant, and that the functor G satisfies the conditions **G***. *Then the canonical function*

$$G(\varinjlim_i Y_i) \to \varprojlim_i G(Y_i)$$

is surjective.

Proof. Take a sequence of cylinder objects

$$\begin{array}{ccc} Y_i \vee Y_i & & \\ {\scriptstyle (d^0,d^1)}\downarrow & \searrow^{\nabla} & \\ Y_i \otimes I & \xrightarrow[s]{\simeq} & Y \end{array}$$

in $\mathcal{N}$, and form the pushout diagram

$$\begin{array}{ccc} \bigvee_i (Y_i \vee Y_i) & \xrightarrow{(1,\alpha)} & \bigvee_i Y_i \\ {\scriptstyle (d^0,d^1)}\downarrow & & \downarrow \\ \bigvee_i (Y_i \otimes I) & \longrightarrow & L \end{array}$$

Then the object L is the telescope construction for the diagram (2), and it is canonically weakly equivalent to the colimit $\varinjlim_i Y_i$.

There is a pullback diagram

$$\begin{array}{ccc} \varprojlim_i G(Y_i) & \longrightarrow & \prod_i G(Y_i) \\ \downarrow & & \downarrow {\scriptstyle (1,\alpha^*)} \\ \prod_i G(Y_i) & \xrightarrow[\Delta]{} & \prod_i (G(Y_i) \times G(Y_i)) \end{array}$$

and there are isomorphisms $G(Y_i) \cong G(Y_i \otimes I)$, so that the map $G(L) \to \varprojlim G(Y_i)$ is surjective by the Mayer-Vietoris property. □

Suppose that Y is an object of $\mathcal{N}$. An element u of $G(Y)$ is said to be *universal* if the evaluation map

$$u^* : [K, Y] \to G(K)$$

defined by $\alpha \mapsto \alpha^*(u)$ is an isomorphism for all compact generators K.

Lemma 16. *Suppose that X is a cofibrant object of $\mathcal{N}$ and that $v \in G(X)$. Suppose that the functor f satisfies the conditions* **G*.** *Then there is a cofibration $i : X \to Y$ such that there is a universal element $u \in G(Y)$ with $i^*(u) = v$.*

Proof. Suppose that Z is a cofibrant object of $\mathcal{N}$ and that $z \in G(Z)$. By using the methods of proof of Lemma 4 one can show that there is a cofibration $j : Z \to Y$ with $w \in G(Y)$ such that $j^*(w) = z$, and such that in the diagram

$$\begin{array}{ccc} [K,Z] & \xrightarrow{j_*} & [K,Y] \\ & \searrow^{z^*} & \downarrow{w^*} \\ & & G(K) \end{array}$$

w^* is surjective, and if $z^*[\alpha] = z^*[\beta]$ then $j_*[\alpha] = j_*[\beta] \in [K,Y]$ for all compact generators K.

Set $Y_0 = X$ and $u_0 = v$. Then there is a countable sequence of cofibrations $j : Y_n \to Y_{n+1}$ with elements $u_n \in G(Y_n)$ such that $j^*(u_{n+1}) = u_n \in G(Y_n)$, and in all diagrams

$$\begin{array}{ccc} [K,Y_n] & \xrightarrow{j_*} & [K,Y_{n+1}] \\ & \searrow^{u_n^*} & \downarrow{u_{n+1}^*} \\ & & G(K) \end{array} \tag{3}$$

the map u_{n+1}^* is surjective, and if $u_n^*[\alpha] = u_n^*[\beta]$ then $j_*[\alpha] = j_*[\beta] \in [K,Y_{n+1}]$.

The function

$$G(Y) = G(\varinjlim Y_n) \to \varprojlim G(Y_n)$$

is surjective by Lemma 15, so that one can pick $u \in G(Y)$ such that u restricts to all u_n. Then there are commutative triangles

$$\begin{array}{ccc} [K,Y_n] & \longrightarrow & [K,Y] \\ & \searrow^{u_n^*} & \downarrow{u^*} \\ & & G(K) \end{array}$$

and $[K,Y] \cong \varinjlim [K,Y_n]$ since all K are compact. The map u^* is a bijection by the construction of the cofibrations j_* in the diagram (3). □

We now have the following analogues of Lemma 6, Lemma 7 and Proposition 8, respectively, with the same proofs.

Lemma 17. *Suppose that G satisfies the conditions* **G*.** *Suppose that $\alpha : Y \to Y'$ is a morphism of cofibrant objects of $\mathcal{N}$. Suppose that $u \in G(Y)$ and $u' \in G(Y')$ are universal, and that $\alpha^*(u') = u$. Then α is a weak equivalence.*

Lemma 18. *Suppose that the functor G satisfies the conditions* **G***. *Suppose given maps*

$$\begin{array}{ccc} A & \xrightarrow{\beta} & Y \\ {\scriptstyle i}\downarrow & & \\ B & & \end{array}$$

of $\mathcal{N}$ such that i is a cofibration and all objects are cofibrant. Suppose that $u \in G(Y)$ is universal and that $x \in G(B)$ satisfies $i^*(x) = \beta^*(u) \in G(A)$. Then β extends to a map $\gamma : B \to Y$ in the homotopy category $\mathrm{Ho}(\mathcal{N})$ such that $\gamma^*(u) = x$.

Theorem 19. *Suppose that $\mathcal{N}$ is a closed model category which satisfies the conditions* **N***. *Suppose that the functor $G : \mathcal{N}^{op} \to \mathbf{Set}_*$ satisfies the conditions* **G***. *Then there is an object Y of $\mathcal{N}$ and a natural bijection*

$$[X, Y] \xrightarrow{\cong} G(X)$$

for all objects X of $\mathcal{N}$.

Theorem 19 is the analogue of the classical Brown representability theorem for compactly generated pointed closed model categories $\mathcal{N}$, such as the category Spt of ordinary spectra.

It applies to a relatively small list of well-behaved categories of presheaves of spectra and symmetric spectra. These include the categories of presheaves of spectra and symmetric spectra on the scheme category $Sch|_S$ where the local stable equivalences are determined by either the Zariski or Nisnevich topologies. In both cases, the collections of shifted suspensions $\Sigma^\infty(K \wedge U)_+[n]$ of objects $K \wedge U_+$ arising from finite pointed simplicial sets K and S-schemes U determine a set of "compact generators", by the Brown-Gersten and Nisnevich descent theorems, respectively (see [7]).

Testing the applicability of Theorem 19 for presheaves of spectra and symmetric spectra amounts, in all cases, to displaying a set S of compact objects which satisfies **N2**.

In particular, Theorem 19 specializes to a Brown representability result for the stable model structure on presheaves of spectra on the smooth Nisnevich site $Sm|_S$ on a scheme S. The objects $K \wedge V_+$ with K a finite pointed simplicial set and V a smooth S-scheme are compact for the motivic model structure on pointed simplicial presheaves because colimits of inductive systems of motivic fibrant simplicial presheaves satisfy motivic descent. This set of objects then determines compact generators $\Sigma^\infty_T(K \wedge V_+)[n]$ and $F_n(K \wedge V_+)$ for the categories of motivic T-spectra and motivic symmetric T-spectra respectively, for all of the standard suspension objects T [4].

In many other cases of interest, however, such as presheaves of spectra on an étale site, we only have Theorem 12, which has stronger conditions and a stronger conclusion.

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