Lecture 001 (April 4, 2005)

Bisimplicial sets

Bisimplicial set $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{Set}$. $(\mathbf{p}, \mathbf{q}) \mapsto X(p, q)$.

Morphism $f : X \to Y$ of bisimplicial sets is natural transformation. $\mathbf{S}^2$ = category of bisimplicial sets.

Examples: (1) $\Delta^{p,q}$ is represented by the pair $(\mathbf{p}, \mathbf{q})$. Maps $\Delta^{p,q} \to X$ classify bisimplices $X(p, q)$.

(2) K, L, simplicial sets $K \tilde{\times} L(p, q) = K_p \times L_q$. $\Delta^{p,q} = \Delta^p \tilde{\times} \Delta^q$.

$X_n = X(n, *)$ simplicial set in horizontal degree n.

Given $\theta : \mathbf{m} \to \mathbf{n}$, write $\theta_h^* = (\theta, 1)^* : X_n \to X_m$,
write $\theta_v^* = (1, \theta) : X(*, n) \to X(*, n)$
$dX_n = X(n, n)$ diagonal simplicial set

Fact: $f : X \to Y$ map of bisimplicial sets such that all simplicial set maps $X_n \to Y_n$ are weak equivalences, then $f_* : dX \to dY$ is a weak equivalence of simplicial sets.

Bisimplicial abelian groups

$A : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{Ab}$ bisimplicial abelian group.

$\mathbf{Tot}(A)_n = \oplus_{p+q=n} \ A(p,q)$.

For $x \in A(p,q)$,

$$\begin{aligned} \partial_h(x) &= \sum_{i=0}^{p} (-1)^i d_{i,h}(x) \\ \partial_v(x) &= \sum_{i=0}^{q} (-1)^{i+p} d_{i,v}(x) \\ \partial(x) &= \partial_h(x) + \partial_v(x) \end{aligned}$$

Theorem (Generalized Eilenberg-Zilber thm) $X =$ bisimplicial set. There is a natural chain homotopy equivalence $f : \mathbf{Tot}(\mathbb{Z}X) \to d\mathbb{Z}X$, $g : d\mathbb{Z}X \to \mathbf{Tot}(\mathbb{Z}X)$.

Special case (Eilenberg-Zilber theorem), $\mathbb{Z}(X \tilde{\times} Y)$: $\mathbf{Tot}(\mathbb{Z}(X) \otimes \mathbb{Z}(Y)) \simeq \mathbb{Z}(X \times Y)$.

$\mathbb{Z}X$ is a colimit of its bisimplices $\mathbb{Z}\Delta^{p,q} \to \mathbb{Z}X$, so there is short exact sequence

$$\bigoplus_{\Delta^{r,s} \to \Delta^{p,q} \to X} \mathbb{Z}\Delta^{r,s} \to \bigoplus_{\Delta^{p,q} \to A} \mathbb{Z}\Delta^{p,q} \to \mathbb{Z}X \to 0$$

$\mathbf{Tot}$ and d are right exact, induce f_*, g_* relating $\mathbf{Tot}(\mathbb{Z}X)$, $d\mathbb{Z}X$.

Consequence: X has two spectral sequences converging to $H_*(dX)$.

Homotopy colimits (first take)

$X : I \to \mathbf{Set}$, I small:

$E_I X$ = translation category associated to X: Objects $(x, i), x \in X(i)$, Morphisms $\alpha : (x, i) \to (y, j)$ are $\alpha : i \to j$ in I such that $\alpha_*(x) = y$.

$BE_I X_n$: $(x_0, i_0) \to (x_1, i_1) \to \cdots \to (x_n, i_n)$ completely determined by $i_0 \to i_1 \to \cdots \to i_n$ in BI_n and $x_0 \in X(i_0)$.

$$BE_I X_n = \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0)$$

with

$$\begin{tikzcd}
X(i_0) \arrow[r] \arrow[d, "in_\sigma"'] & X(i_{\theta(0)}) \arrow[d, "in_{\theta^*\sigma}"] \\
\sqcup_{i_0 \to \cdots \to i_n} X(i_0) \arrow[r, "\theta^*"'] & \sqcup_{j_0 \to \cdots \to j_m} X(j_0)
\end{tikzcd}$$

for $\theta : \mathbf{m} \to \mathbf{n}$.

There is canonical functor: $\pi : E_I X \to I$, $(x, i) \mapsto i$, with induced map $\pi : BE_I X \to BI$.

$X : I \to \mathbf{S}$: $E_I X$ is a simplicial category, and $BE_I X$ is bisimplicial set with vertical simplicial set

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0)$$

in horizontal degree n.

Homotopy colimit $\underrightarrow{\text{holim}}_I\ X = d(BE_IX)$ (but also $= BE_IX$).

Facts: (1) There is natural map $\pi : \underrightarrow{\text{holim}}_I\ X \to BI$.

(2) If $f : X \to Y$ is a natural transformation of I-diagrams such that $f : X(i) \to Y(i)$ is a weak equivalence for all $i \in I$ then

$$f_* : \underrightarrow{\text{holim}}_I\ X \to \underrightarrow{\text{holim}}_I\ Y$$

is a weak equivalence.

(3) There is canonical map $\underrightarrow{\text{holim}}_I\ X \to \varinjlim_I\ X$ induced by bisimplicial set map

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0) \to \varinjlim_I\ X.$$

Examples:

(1) $p : X \to Y$ simplicial set map. Recall $\mathbf{\Delta}/Y$, simplex category for Y. There is a functor $p^{-1} : \mathbf{\Delta}/Y \to \mathbf{S}$ defined by taking $\sigma : \Delta^n \to Y$ to

$$p^{-1}(\sigma) = \Delta^n \times_Y X.$$

Theorem (Quillen) The canonical map

$$\underrightarrow{\text{holim}}_{\mathbf{\Delta}/Y}\, p^{-1} \to \varinjlim_{\mathbf{\Delta}/Y}\, p^{-1} \cong X$$

is a weak equivalence.

Proof Take $x \in X_n$, and consider category C_x of all pairs (σ, y), $y \in p^{-1}(\sigma)_n$, such that $y \mapsto x \in X_n$.

$E_{\mathbf{\Delta}/Y} p_n^{-1}$ is a disjoint union of such categories: C_x is fibre over x for canonical functor

$$E_{\mathbf{\Delta}/Y} p_n^{-1} \to X_n$$

The element $(i_n, x) \in p^{-1}(p(x))_n$ is initial in C_x, so C_x is contractible, and the displayed functor is a weak equivalence for each n. $\square$

Remark: This theorem is the basis for the construction of the Serre spectral sequence.

(2) $f : C \to D$ functor between small categories. Recall definition of slice categories f/d, $d \in D$. $d \mapsto B(f/d)$ is a functor taking values in $\mathbf{S}$. The functors $f/d \to C$ defined by taking $f(c) \to d$ to c define a map

$$\underrightarrow{\operatorname{holim}}_{d \in D} B(f/d) \to BC.$$

Theorem: (Quillen) This map is a weak equivalence.

Proof:

$\underrightarrow{\operatorname{holim}}_{d \in D} B(f/d)$ is the diagonal of a bisimplicial set whose (p, q)-bisimplices are all collections

$$(c_0 \to \cdots \to c_p \in BC_p, d_0 \to \cdots \to d_q \in BD_q, f(c_p) \to d_0)$$

and canonical map picks off $c_0 \to \cdots \to c_p$. The fibre over this simplex is $B(f(c_p)/D)$ and $f(c_p)/D$ has an initial object. □

Remark: This result is one of the steps in the proof of Quillen's Theorem B.

(3) Suppose that $f : X \to BD$ is a simplicial set map, and write $f^{-1}(d)$ for the pullback

$$\begin{tikzcd}
f^{-1}(d) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B(D/d) \arrow[r] & BD
\end{tikzcd}$$

Theorem The induced map

$$\underrightarrow{\text{holim}}_{\, d\in D}\ f^{-1}(d) \to X$$

is a weak equivalence.

Proof: Suppose $x \in X_n$. The fibre over x of the simplicial set map

$$\underrightarrow{\text{holim}}_{\, d\in D}\ f^{-1}(d)_n \to X_n$$

consists of strings of arrows

$$a_0 \to \cdots \to a_n \to b_0 \to \cdots \to b_q$$

in D such that $f(x) = a_0 \to \cdots \to a_n$. This fibre can therefore be identified with $B(a_n/D)$, which is contractible. $\square$

Remark: This result is part of the comparison between I-diagrams and simplicial sets fibred over BD.

Some model structures

Consider the category $\mathbf{S}^I$ of I-diagrams in simplicial sets, where I is a fixed small category. There are two basic model structure on this category:

(1) **Bousfield-Kan structure (projective structure)**

A map $f : X \to Y$ of I-diagrams is a (projective, pointwise ...) fibration if all maps $f : X(i) \to Y(i)$ are fibrations of simplicial sets.

$f : X \to Y$ is a (pointwise) weak equivalence if all $f : X(i) \to Y(i)$ are weak equivalences of simplicial sets.

Cofibrations are maps which have the left lifting property with respect to trivial fibrations.

$p : X \to Y$ is a trivial fibration iff all $p : X(i) \to Y(i)$ have the right lifting property wrt all $\partial\Delta^n \subset \Delta^n$.

For $i \in I$, simplicial set K, let $L_iK : I \to \mathbf{S}$ be defined by

$$L_i(K)(j) = \bigsqcup_{i \to j} K.$$

L_iK is the left Kan extension of K along the inclusion $\{i\} \subset I$, and

(1) $p : X \to Y$ is a fibration iff p has the RLP wrt all $L_i(\Lambda^n_k) \subset L_i\Delta^n$.

(2) $p : X \to Y$ is a fibration and a weak equivalence if and only if p has the RLP wrt all $L_i\partial\Delta^n \subset L_i\Delta^n$.

Consequence: Any cofibration $A \subset B$ of simplicial sets induces a cofibration $L_iA \to L_iB$ for each $i \in I$. Note that all of these maps are pointwise monomorphisms. The functors L_i also preserve trivial cofibrations.

Theorem With these definitions, the category $\mathbf{S}^I$ has the structure of a proper closed simplicial model category.

Proof exercise

$\hom(X, Y)_n = \hom(X \times \Delta^n, Y)$

$X \times \Delta^n$ is the product of X with the constant functor on the simplicial set Δ^n in $\mathbf{S}^I$. Such abuses will be common.