Lecture 002 (April 4, 2005)

(2) **The injective structure**

Say that $f : X \to Y$ in $\mathbf{S}^I$ is a weak equivalence if all maps $f : X(a) \to Y(a)$ are weak equivalences of $\mathbf{S}$.

The cofibrations for this structure are the (pointwise) inclusions.

(Global) fibrations are those maps which have the RLP wrt all trivial cofibrations.

Suppose that α is an infinite cardinal such that $|\operatorname{Mor}(I)| < \alpha$. Say that an object $X \in \mathbf{S}^I$ is α-bounded if all $|X_n(a)| < \alpha$. Say that a map $f : X \to Y$ (typically a cofibration) is α-bounded if both X and Y are α-bounded.

Lemma: Suppose given a diagram of cofibrations

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

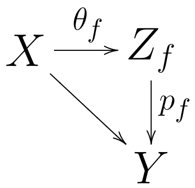
where i is a trivial cofibration and A is λ-bounded, where $\lambda \geq \alpha$. Then there is an λ-bounded subobject $B \subset Y$ with $A \subset B$ such that the map $B \cap X \to B$ is a weak equivalence.

The statement of the Lemma is often called a bounded cofibration condition.

Proof

(1) Y is a union of its λ-bounded subcomplexes: $|L_iK| < \alpha$ if $|K| < \alpha$.

(2) There is a functorial fibration replacement construction in simplicial sets which preserve filtered colimits, ie. there is a natural diagram

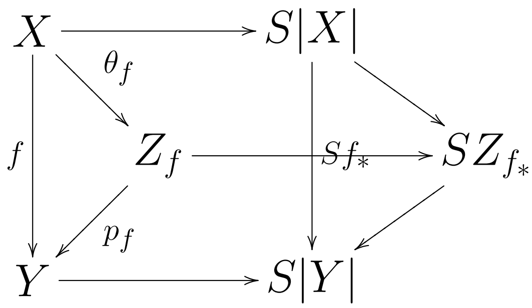


with θ_f a weak equivalence and p_f a Kan fibration, and this construction preserves filtered colimits in f.

The corresponding construction exists for topological spaces: use

$$X \xrightarrow{s_*} Z_f = X \times_Y Y^I \to Y^I \xrightarrow{d_1} Y$$

and form pullback diagrams



f is a weak equivalence iff p_f is a trivial fibration iff p_f has the RLP wrt all $\partial\Delta^n \subset \Delta^n$

(3) Consider the lifting problem

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & Z_j(a) & \longrightarrow & Z_i(a) \\ \downarrow & \nearrow & \downarrow p_j & \nearrow & \downarrow p_i \\ \Delta^n & \longrightarrow & A(a) & \longrightarrow & Y(a) \end{array}$$

where j is the inclusion $A \cap X \to A$ and $a \in I$. The indicated solid arrow lift exists, so there is a λ-bounded subcomplex $D \subset Y$ such that the lift exists over D. Do this for the λ-bounded collection of all such problems, to find a λ-bounded $A_1 \subset Y$ such that all lifting problems

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & Z_j(a) & \longrightarrow & Z_{j_1}(a) \\ \downarrow & \nearrow & \downarrow p_j & \nearrow & \downarrow p_{j_1} \\ \Delta^n & \longrightarrow & A(a) & \longrightarrow & A_1(a) \end{array}$$

are solved over A_1 in the indicated sense. j_1 is the inclusion $A_1 \cap X \to A_1$.

Repeat the construction for A_1 to find an ascending chain of λ-bounded subcomplexes of Y

$$A \subset A_1 \subset A_2 \subset \dots$$

such that all lifting problems over A_i are solved over A_{i+1}. Then all lifting problems over $B = \cup_i A_i$ are solved over B, and B is λ-bounded. $\square$

Choose a cardinal $\lambda \geq \alpha$ as in the statement of the Lemma, and observe that the collection of all λ-bounded trivial cofibrations $A \to B$ in $\mathbf{S}^I$ forms a set.

Lemma: A map $p : X \to Y$ is a global fibration iff it has the RLP wrt all λ-bounded trivial cofibrations.

Proof

Suppose that p has the RLP wrt all λ-bounded trivial cofibrations, and suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where i is a trivial cofibration. Consider the collection of all partial lifts

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \\ D & & \downarrow{\scriptstyle p} \\ \downarrow & & \\ B & \longrightarrow & Y \end{array}$$

where $A \to D$ and $D \to B$ are trivial cofibrations.

Suppose $D \neq B$. Then there is a λ-bounded subcomplex $E \subset B$ such that $D \neq D \cup E$ and such that $E \cap D \to D$ is a trivial cofibration. The diagram of inclusions

$$\begin{array}{ccc} E \cap D & \longrightarrow & D \\ \downarrow & & \downarrow \\ E & \longrightarrow & E \cup D \end{array}$$

is a pushout and E is λ-bounded, so that the partial lift $D \to X$ extends to $E \cup D \to X$.

Finish the proof with a Zorn's lemma argument, indexed on the partial lifts. $\square$

Lemma: Every map $f : X \to Y$ in $\mathbf{S}^I$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where j is a trivial cofibration and p is a global fibration.

Proof This is a classic transfinite small object argument.

Pick a cardinal $\gamma > 2^\lambda$. We shall define a system of factorizations

$$\begin{array}{ccc} X & \xrightarrow{j_s} & Z_s \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p_s} \\ & & Y \end{array}$$

which is functorial in $s < \gamma$ as follows:

(1) Given Z_s define Z_{s+1} by the pushout diagram

$$\begin{array}{ccc} \sqcup_D A_D & \longrightarrow & Z_s \\ \downarrow & & \downarrow \\ \sqcup_D B_D & \longrightarrow & Z_{s+1} \end{array}$$

where the pushout is indexed over the set of all lifting problems

$$\begin{array}{ccc} A_D & \longrightarrow & Z_s \\ \downarrow & & \downarrow p_s \\ B_D & \longrightarrow & Y \end{array}$$

with $A_D \to B_D$ a λ-bounded trivial cofibration. Then the lifting problems induce a map $p_{s+1} : Z_{s+1} \to Y$, and the map i_{s+1} is the composite

$$X \xrightarrow{i_s} Z_s \to Z_{s+1}$$

(2) If t is a limit ordinal and Z_s is defined for $s < t$, set $Z_t = \varinjlim_{s<t} Z_s$.

Set $Z = \varinjlim_{s<\gamma} Z_s$, and let $i : X \to Z$ and $p : Z \to Y$ be the induced maps.

Trivial cofibrations are closed under pushout and filtered colimits, so that i is a trivial cofibration.

If A is λ-bounded, then any map

$$A \to Z = \varinjlim_{s<\gamma} Z_s$$

factors through some inclusion $Z_s \to Z$, for otherwise A has too many elements. Thus p has the right lifting property with respect to all λ-bounded trivial cofibrations, and is therefore a global fibration. $\square$

Theorem: With these definitions (cofibration = monomorphism, weak equivalence = pointwise weak equivalence), the category $\mathbf{S}^I$ satisfies the axioms for a proper closed simplicial model category.

Proof: The proof is essentially an exercise, once the (nasty) factorization axiom has been proved in the Lemma above.

For the other factorization axiom, note that $p : X \to Y$ is a trivial fibration if it has the RLP wrt all inclusions $Y \subset L_a(\Delta^n)$ and hence with respect to all inclusions. A small object argument then implies that any $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

where i is a cofibration and q has the RLP wrt all cofibrations — this finishes the proof of **CM5**.

If f is a fibration and a weak equivalence, then i is a trivial cofibration, and the lift exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ W & \xrightarrow[q]{} & Y \end{array}$$

Then f is a retract of q and hence has the RLP wrt all cofibrations, proving **CM4**.

Note that if $p : X \to Y$ is a global fibration, then all simplicial set maps maps $p : X(a) \to Y(a)$ are Kan fibrations — reason: $L_a\Lambda^n_k \to L_a\Delta^n$ is a trivial cofibration. This implies the properness. □

Example: Bisimplicial sets $\mathbf{S}^2 = \mathbf{S}^{\mathbf{\Delta}^{op}}$. The projective structure is the old Bousfield-Kan model structure for bisimplicial sets. The injective structure is the Reedy structure.