Lecture 003 (April 4, 2005)

Homotopy inverse limits

Write $\underleftarrow{\mathrm{holim}}_I \, X = \mathbf{hom}(B(I/?), X)$.

Note that $(\underleftarrow{\mathrm{holim}}_I \, X)_0 = \hom(B(I/?), X)$ can be identified with the cosimplicial end given by the inverse limit of all diagrams

$$\begin{tikzcd}
 & \Pi_{j_0 \to \dots \to j_n} \; X(j_n)_n \arrow[d, "\theta^*"] \\
\Pi_{i_0 \to \dots \to i_m} \; X(i_m)_m \arrow[r, "\theta_*"'] & \Pi_{j_0 \to \dots \to j_n} \; X(j_n)_m
\end{tikzcd}$$

Lemma: Suppose that $p : X \to Y$ is a pointwise trivial fibration. Then the induced function

$$p_* : \hom(B(I/?), X) \to \hom(B(I/?), Y)$$

is surjective. In other words , $B(I/?)$ is projective cofibrant.

Proof Suppose that $g : B(I/?) \to Y$ is a natural transformation. A lifting $h : B(I/?) \to X$ of Y is constructed by induction over the skeleta of BI. Producing h amounts to finding an element $h(\sigma) \in X(i_n)_n$ for each $\sigma : i_0 \to \dots i_n \in BI_n$ such that $ph(\sigma) = g(\sigma)$ and the obvious compatibility conditions are satisfied.

Write $(\theta, \sigma)_*$ for the map making the diagram

$$\begin{tikzcd}
X(i_{\theta(m)}) \arrow[r, "(\theta,\sigma)_*"] & X(i_n) \\
\Pi_{j_0 \to \cdots \to j_m} \; X(j_m)_m \arrow[u, "pr_{\theta^*\sigma}"] \arrow[r, "\theta_*"'] & \Pi_{i_0 \to \cdots \to i_n} \; X(i_n)_n \arrow[u, "pr_\sigma"']
\end{tikzcd}$$

for each n-simplex $\sigma : i_0 \to \cdots \to i_n$ of BI_n.

All maps $p : X(i) \to Y(i)$ are surjective, so there are vertices $h(i) \in X(i)_0$ such that $ph(i) = g(i)$ for each vertex $i : \mathbf{0} \to I$ of BI.

Suppose that $h(\tau)$ has been defined for all $\tau \in BI_k$ for $k < n$. If $\sigma = s_i(\tau) \in BI_n$ is degenerate, then $h(\sigma)$ is completely determined: $h(s_i\tau) = s_i h(\tau)$. If σ is non-degenerate then $h(\sigma)$ must be compatible the family of faces $d_i(d^i, \sigma)_* h(d_i\sigma)$ and must map to $g(\sigma)$: $h(\sigma)$ is a solution of a lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "(d_i(d^i,\sigma)_*h(d_i\sigma))"] \arrow[d] & X(i_n) \arrow[d, "p"] \\
\Delta^n \arrow[r, "g(\sigma)"'] \arrow[ur, dotted] & Y(i_n)
\end{tikzcd} \qquad \square$$

The following is a consequence of the Lemma and the projective simplicial model structure for $\mathbf{S}^I$:

Corollary: Suppose that $p : X \to Y$ is a pointwise fibration (resp. pointwise triv. fibration).

Then the induced map $\underleftarrow{\text{holim}}_I X \to \underleftarrow{\text{holim}}_I Y$ is a fibration (resp. triv. fibration).

Corollary: Suppose that $f : X \to Y$ is a pointwise weak equivalence of I-diagrams of Kan complexes. Then the induced map $\underleftarrow{\text{holim}}_I X \to \underleftarrow{\text{holim}}_I Y$ is a weak equivalence.

Lemma: Suppose that $j : X \to Z$ is a pointwise weak equivalence, X is a diagram of Kan complexes, and that Z is globally fibrant. Then there is a weak equivalence

$$\underleftarrow{\text{holim}}_I X \simeq \varprojlim_I Z.$$

Proof The map

$$j_* : \underleftarrow{\text{holim}}_I X \to \underleftarrow{\text{holim}}_I Z = \mathbf{hom}(B(I/?), Z)$$

is a weak equivalence by the lemma above. The map

$$\varprojlim_I Z = \mathbf{hom}(*, Z) \to \mathbf{hom}(B(I/?), Z)$$

is a weak equivalence since all spaces $B(I/i)$ are contractible and Z is globally fibrant. $\square$

Examples:

(1) $X : \mathbf{\Delta} \to \mathbf{S}$ pointwise fibrant cosimplicial space, with globally fibrant model $j : X \to Z$. Then

$$\underleftarrow{\text{holim}}_{\mathbf{\Delta}} Z = \mathbf{hom}(B(\mathbf{\Delta}/?), Z) \xleftarrow{\simeq} \mathbf{hom}(\Delta, Z) = \mathbf{Tot}(Z)$$

since $B(\mathbf{\Delta}/\mathbf{n}) \simeq \mathbf{n}$. There is a functor $\mathbf{\Delta}/\mathbf{n} \to \mathbf{n}$ defined by sending $\theta : \mathbf{k} \to \mathbf{n}$ to $\theta(k) \in \mathbf{n}$. This functor is natural in $\mathbf{n}$. All of the displayed spaces are equivalent to $\underleftarrow{\text{holim}}_{\mathbf{\Delta}} X$.

(2) The globally fibrant objects among all diagrams

$$X_1 \to X_0 \leftarrow X_2$$

are those for which X_0 is fibrant and both maps are fibrations. The inverse limit is the pullback $X_1 \times_{X_0} X_2$. To compute the homotopy type, by properness, it is enough to replace one of the maps $X_1 \to X_0$ or $X_2 \to X_0$ by a fibration and then take the pullback.

(3) Suppose that the index category I has an initial object i. Then $\underleftarrow{\text{holim}}_I X \simeq X(i)$. If $i : X \to Z$ is a globally fibrant model, then

$$\underleftarrow{\text{holim}}_I X \simeq \varprojlim_I Z \cong Z(i) \simeq X(i).$$

(4) Suppose that I is a discrete category on a set of objects. Then $\varprojlim_I Z \cong \prod_{i\in I} Z(i)$, and $\underleftarrow{\text{holim}}_I X \cong \prod_i Z(i)$ if $X \to Z$ is a globally fibrant model. **Warning:** the map $\prod_i X(i) \to \prod_i Z(i)$ is not a weak equivalence in general.

(5) The globally fibrant objects among all towers

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots$$

are those objects for which X_0 is fibrant and all arrows are fibrations.

Homotopy colimits, revisited

Fact: $\underrightarrow{\text{holim}}_I X$ is the coend

$$\begin{tikzcd}
B(i/I) \times X_j \arrow[r, "1\times\alpha"] \arrow[d, "\alpha^*\times 1"'] & B(i/I) \times X_i \\
B(j/I) \times X_j &
\end{tikzcd}$$

where $\alpha : j \to i$ varies over the morphisms of I.

Proof: The n-simplices $i_0 \to \cdots \to i_n$ of BI determine n-simplices

$$i_0 \xrightarrow{1} i_0 \to \cdots \to i_n$$

of $B(i/I)$ and hence maps

$$\Delta^n \times X_{i_0} \to B(i_0/I) \times X_{i_0}.$$

Show that, for any Y, the restrictions of maps $B(i/I) \times X_i \to Y$ along the above maps completely determines a map off the co-end in the statement of the Lemma. This family of restrictions determines a family of maps

$$\bigsqcup_{i_0 \to \cdots \to i_n} \Delta^n \times X_{i_0} \to Y$$

which uniquely gives a map $\underrightarrow{\text{holim}}_I\, X \to Y$. $\square$

Lemma: Suppose that $\pi : W \to X$ is a projective cofibrant model of the I-diagram X. Then there is a weak equivalence

$$\varinjlim_{I} W \simeq \underrightarrow{\text{holim}}_I\, X.$$

Proof The induced map

$$\underrightarrow{\text{holim}}_I\, W \to \underrightarrow{\text{holim}}_I\, X$$

is a weak equivalence. Suppose that Y is a Kan complex, and observe (from the "Fact" at the beginning of the Lecture) that there is an isomorphism

$$\mathbf{hom}(\underrightarrow{\text{holim}}_I\, X, Y) \cong \underleftarrow{\text{holim}}_{I^{op}}\, \mathbf{hom}(X, Y).$$

The I^{op}-diagram $\mathbf{hom}(W, Y)$ is globally fibrant, by an adjunction argument, since W is projective cofibrant. In effect, lifting problems of I^{op}-

diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & \mathbf{hom}(W,Y) \\
B \arrow[ur, dotted] &
\end{tikzcd}$$

are equivalent to lifting problems of I-diagrams

$$\begin{tikzcd}
 & \mathbf{hom}(B,Y) \arrow[d, "i^*"] \\
W \arrow[r] \arrow[ur, dotted] & \mathbf{hom}(A,Y)
\end{tikzcd}$$

The map $\underrightarrow{\text{holim}}_I\, W \to \varinjlim_I W$ induces the map

$$\varprojlim_{I^{op}} \text{hom}(W,Y) \to \underleftarrow{\text{holim}}_{I^{op}}\, \mathbf{hom}(W,Y)$$

which is a weak equivalence since $\mathbf{hom}(W,Y)$ is globally fibrant. This is true for all Kan complexes Y, so that $\underrightarrow{\text{holim}}_I\, W \to \varinjlim_I W$ is a weak equivalence. $\square$

Examples:

(1) The projective cofibrant objects among all diagrams

$$A_1 \leftarrow A_0 \to A_2$$

are those diagrams for which both maps are cofibrations. The colimit is the pushout $A_1 \cup_{A_0} A_2$. To compute the homotopy type of the homotopy colimit, it is enough to replace one of the maps in the picture by a cofibration and then take pushout, by a patching argument.

(2) Suppose that the index category I has a terminal object t. Then there is a weak equivalence $\underrightarrow{\mathrm{holim}}_I\, X \simeq X(t)$.

(3) The projective cofibrant objects among all "telescopes"

$$A_0 \to A_1 \to A_2 \to \dots$$

are those for which all maps in the picture are cofibrations.

(4) One way or another, one sees that $BI = \underrightarrow{\mathrm{holim}}_I\, *$. In other words, BI is the homotopy colimit of the diagram $I \to \mathbf{S}$ which takes all objects i to a copy of $\Delta^0 = *$. In an arbitrary cofibrantly generated closed model category $\mathbf{M}$ the diagram category $\mathbf{M}^I$ has both a projective and injective model structure, so that homotopy limits and colimits are always defined. The internal nerve $B_h(I)$ in $\mathbf{M}$ is defined by

$$B_h(I) = \underrightarrow{\mathrm{holim}}_I\, t$$

where t is the constant diagram at the terminal object t of $\mathbf{M}$.

(5) $X =$ bisimplicial set. dX is the co-end of the maps

$$\begin{tikzcd}
X_n \times \Delta^m \arrow[r, "\theta^* \times 1"] \arrow[d, "1 \times \theta"'] & X_m \times \Delta^m \\
X_n \times \Delta^n &
\end{tikzcd}$$

so the last vertex maps $B(\mathbf{\Delta}/\mathbf{n}) \to \Delta^n$ induce a natural map $\underrightarrow{\text{holim}}_{\,\mathbf{\Delta}^{op}}\, X \to d(X)$. If $W \to X$ is a projective cofibrant model, this map is equivalent to $\underrightarrow{\text{holim}}_{\,\mathbf{\Delta}^{op}}\, W \to d(W)$. If Y is a Kan complex, then applying the functor $\mathbf{hom}(\ ,Y)$ to this map induces the map

$$\mathbf{Tot}(\mathbf{hom}(W,Y)) \to \underleftarrow{\text{holim}}_{\,\mathbf{\Delta}}\mathbf{hom}(W,Y)$$

which is a weak equivalence. This is true for all Kan complexes Y, so that $\underrightarrow{\text{holim}}_{\,\mathbf{\Delta}^{op}}\, X \to d(X)$ is a weak equivalence.

Kan extensions

Suppose that $\phi : I \to J$ is a functor between small categories.

ϕ induces a restriction functor $\phi_* : \mathbf{S}^J \to \mathbf{S}^J$ defined by $\phi_*(Y) = Y \cdot \phi$ for $Y : J \to \mathbf{S}$.

Left Kan extensions

ϕ_* has a left adjoint $\phi^* : \mathbf{S}^I \to \mathbf{S}^J$, defined for $X : I \to \mathbf{S}$ by

$$\phi^*(X)(j) = \varinjlim_{\phi(i)\to j} X(i)$$

$\phi^* X$ is the **left Kan extension** of X along ϕ

Example: For $a \in I$, $K \in \mathbf{S}$, $L_a(K)$ is the left Kan extension of the functor $K : \{a\} \to \mathbf{S}$ along the inclusion $\{a\} \subset I$.

$$L_a(K)(i) = \bigsqcup_{a\to i} K$$

The homotopy left Kan extension Li^*X is defined by

$$Li^*X = i^*Z$$

where $Z \to X$ is a projective cofibrant model of X

Examples:

(1) $\underrightarrow{\text{holim}}_I\, X$ is the homotopy left Kan extension of X along the functor $I \to *$.

(2) $L(L_aK) = L_aK$ since every simplicial set K is cofibrant.

$X \mapsto Li^*X$ preserves weak equivalences, as does the restriction functor i_*, and there is a natural bijection

$$[Li^*X, Y] \cong [X, i_*Y]$$

Right Kan extensions

Given $\phi : I \to J$, the restriction functor $\phi_* : \mathbf{S}^J \to \mathbf{S}^I$ has a right adjoint $\phi^! : \mathbf{S}^I \to \mathbf{S}^J$, defined by

$$\phi^! X(j) = \varinjlim_{j \to \phi(i)} X(i)$$

$X \mapsto \phi^! X$ is the right Kan extension functor.

The homotopy right Kan extension $R\phi^! X$ is defined by $R\phi^! X = \phi^! Y$ where $X \to Y$ is a globally fibrant model for X. The functor $X \mapsto R\phi^! X$ preserves weak equivalences, and there is an adjunction bijection

$$[\phi_* Y, X] \simeq [Y, R\phi^! X]$$

Examples:

(1) $\underleftarrow{\mathrm{holim}}_I\, X$ is the homotopy right Kan extension of restriction along the functor $I \to *$.

(2) The right adjoint of the a-sections functor $X \mapsto X(a)$ is the functor $K \mapsto R_a K$ defined by

$$R_a K(i) = \prod_{i \to a} K.$$

The corresponding homotopy right Kan extension is given by sending $i \in I$ to the product

$$\prod_{i \to a} Z$$

where $K \to Z$ is a fibrant model of K.