Lecture 005 (April 4, 2005)

Suppose that A is an I-diagram of abelian groups, and that $K(A,n) \to GK(A,n)$ is a globally fibrant model of $K(A,n)$.

Write $\Gamma_* GK(A,n) = \varprojlim_I GK(A,n)$.

Lemma: There are isomorphisms

$$\pi_i \Gamma_* GK(A,n) \cong \begin{cases} {\varprojlim_I}^{n-i} A & i \leq n \\ 0 & i > n \end{cases}$$

Proof There are short exact sequences

$$0 \to K(A,n-1) \to E(A,n) \to K(A,n) \to 0$$

where $E(A,n)$ is a diagram of simplicial abelian groups which is acyclic in each section. There is a comparison of fibre sequences

$$\begin{array}{ccccc} K(A,n-1) & \longrightarrow & E(A,n) & \longrightarrow & K(A,n) \\ \downarrow & & \downarrow & & \downarrow \\ GK(A,n-1) & \longrightarrow & GE(A,n) & \xrightarrow[p]{} & GK(A,n) \end{array}$$

where the vertical maps are globally fibrant models and p is a global fibration. The global fibration $GE(A,n) \to *$ is a weak equivalence, and has RLP wrt all inclusions $\partial\Delta^n \to \Delta^n$ of constant diagrams, so $\Gamma_* GE(A,n)$ is a contractible simplicial set. The sequence

$$\Gamma_* GK(A, n-1) \to \Gamma_* GE(A, n) \to \Gamma_* GK(A, n)$$

is a fibre sequence since p is a global fibration. Thus

$$\pi_i \Gamma_* GK(A, n) \cong \pi_{i-1} \Gamma_* GK(A, n-1)$$

for $i \geq 1$ while

$$\pi_0 \Gamma_* GK(A, n) \cong [*, K(A, n)] \cong \varprojlim_I{}^n A.$$

Use induction on n $\square$

Postnikov towers

Suppose that X is a Kan complex (aka. fibrant simplicial set).

For $x, y \in X_m$ write $x \overset{n}{\sim} y$ if $x|_{\mathrm{sk}_n \Delta^m} = y|_{\mathrm{sk}_n \Delta^m}$. There are commutative diagrams

$$\begin{array}{ccc} \Delta^k & \xrightarrow{\theta} & \Delta^m \\ \uparrow & & \uparrow \\ \mathrm{sk}_n \Delta^k & \longrightarrow & \mathrm{sk}_n \Delta^m \end{array}$$

for all ordinal number maps $\theta : \mathbf{k} \to \mathbf{n}$. Thus, if $x \overset{n}{\sim} y$ then $\theta^*(x) \overset{n}{\sim} \theta^*(y)$. Thus the assignment $m \mapsto P_n X_m = X_m / \overset{n}{\sim}$ respects simplicial structure, and there is a natural simplicial set map

$$\pi_n : X \to P_n X.$$

There are also canonical induced maps $P_{n+1}X \to P_n X$ making the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\pi_{n+1}} & P_{n+1}X \\ & {\scriptstyle \pi_n} \searrow & \downarrow \\ & & P_n X \end{array}$$

Facts: (1) $\pi_n : X \to P_n X$ is a Kan fibration.

(2) π_n induces isomorphisms $\pi_i X \to \pi_i P_n X$ for $0 \leq i \leq n$, and $\pi_j P_n X = 0$ for $j > n$.

(3) The map $P_{n+1}X \to P_n X$ is a fibration with fibre $K(\pi_{n+1}X, n+1)$.

(4) $X = \varprojlim_n P_n X$.

Proof of (1) Consider the lifting problems

$$\begin{array}{ccc} \Lambda^m_k & \xrightarrow{y} & X \\ \downarrow & \nearrow & \downarrow \pi_n \\ \Delta^m & \xrightarrow[{[x]}]{} & P_n X \end{array}$$

$\pi_n : X_r \to P_n X_r$ is a bijection if $r \leq n$, so the lift exists if $m \leq n$.

If $m = n + 1$ then any representative x of $[x] : \Delta^{n+1} \to X$ restricts to y on $\Lambda^{n+1}_k \subset \mathrm{sk}_n \Delta^{n+1}$.

If $m > n + 1$, then $\mathrm{sk}_n \Lambda^m_k = \mathrm{sk}_n \Delta^m$, so any extension $z : \Delta^m \to X$ of $y : \Lambda^m_k \to X$ satisfies $[z] = [x]$ in $P_n X$. $\square$

Suppose $X : I \to \mathbf{S}$ is a diagram of Kan complexes, and form the tower

$$P_0 X \leftarrow P_1 X \leftarrow P_2 X \leftarrow \dots$$

of pointwise fibrations. Take a globally fibrant model

$$\begin{array}{ccccccc} P_0X & \longleftarrow & P_1X & \longleftarrow & P_2X & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ GP_0X & \xleftarrow[p]{} & GP_1X & \xleftarrow[p]{} & GP_2X & \xleftarrow[p]{} & \cdots \end{array}$$

(all vertical maps weak equivs, GP_0X globally fibrant, all p global fibrations).

Then

(1) $\varprojlim GP_nX$ is globally fibrant, and $X \to \varprojlim GP_nX$ is a weak equivalence (non-trivial in other contexts).

(2) The fibre of $p : GP_{n+1}X \to GP_nX$ is a globally fibrant model for $K(\pi_{n+1}X, n+1)$ which will be denoted by $GK(\pi_{n+1}X, n)$.

What are we trying to do in general?

Calculate

$$\pi_n \Gamma_* Y \cong [\Gamma^* S^n, Y]_* \cong [\Gamma^* S^n, X]_* \cong \pi_n \underleftarrow{\text{holim}}\ X$$

(maps in the pointed homotopy category) where $X \to Y$ is a globally fibrant model of an I-diagram X. These are the "hypercohomology" groups for I determined by the non-abelian object X.

NB: $S^n = \Delta^n / \partial \Delta^n$ and $\Gamma^* : \mathbf{S} \to \mathbf{S}^I$ is the constant diagram functor.

$$\pi_n \Gamma_* Y = \pi_*(\Gamma^* S^n, Y) \cong [\Gamma^* S^n, Y]_*$$

since Y is globally fibrant. It doesn't matter which globally fibrant model $X \to Y$ we use, so Y could be $\underleftarrow{\lim}\ GP_n X$.

$$\underleftarrow{\lim}\ \Gamma_* GP_n X \cong \Gamma_* \underleftarrow{\lim}\ GP_n X$$

The tower of fibrations

$$\Gamma_* GP_0 X \leftarrow \Gamma_* GP_1 X \leftarrow \Gamma_* GP_2 X \leftarrow \dots$$

determines a calculational device, namely the "descent spectral sequence" for $\pi_* \underleftarrow{\text{holim}}\ X$.

Towers of fibrations

Suppose

$$Y_0 \xleftarrow{p} Y_1 \xleftarrow{p} Y_2 \xleftarrow{p} \dots$$

is a fibrant tower of pointed simplicial sets (all maps fibrations, Y_0 Kan complex). Let F_n be the fibre of $p : Y_n \to Y_{n-1}$. Consider the picture

$$\begin{array}{ccccccc}
\pi_{t-s+1}Y_s & & & & \pi_{t-s}Y_{s+1} & \overset{\partial}{\dashrightarrow} & \pi_{t-s-1}F_{s+2} \\
\downarrow p & & & & \vdots\, p & & \\
\pi_{t-s+1}Y_{s-1} & \xrightarrow{\partial} & \pi_{t-s}F_s & \xrightarrow{i} & \pi_{t-s}Y_s & & \\
\vdots\, p & & & & \downarrow p & & \\
\pi_{t-s+1}Y_{s-2} & & & & \pi_{t-s}Y_{s-1} & \xrightarrow{\partial} & \pi_{t-s-1}F_s
\end{array}$$

Here's a definition

$$\mathcal{E}_r^{s,t} = i^{-1}(\mathrm{Im}(p^{r-1}))/\partial(\ker(p^{r-1})), \quad t \geq s.$$

NB: Mod out by action of $\ker(p^{r-1})$ if $t - s = 0$.

Then we obtain differentials $d_r : \mathcal{E}_r^{s,t} \to \mathcal{E}_r^{s+r,t+r-1}$, $d_r^2 = 0$ with $\mathcal{E}_{r+1}^{s,t} \cong$ homology of complex

$$\mathcal{E}_r^{s-r,t-r+1} \xrightarrow{d_r} \mathcal{E}_r^{s,t} \xrightarrow{d_r} \mathcal{E}_r^{s+r,t+r-1}$$

We have spectral sequence $\{\mathcal{E}_r^{s,t}\}$ "converging" to $\pi_{t-s} \varprojlim Y_n$. This is the Bousfield-Kan spectral sequence for a tower of fibrations.

Here are some of the most useful results (which don't depend on convergence!):

Lemma: Suppose that $f : X \to Y$ is a morphism of fibrant towers such that

1) the induced map $\mathcal{E}_r^{s,t}(X) \to \mathcal{E}_r^{s,t}(Y)$ is a bijection for all s, t,

2) $\mathcal{E}_r^{s,s}(Y) = *$ for all s.

Then the induced map $\varprojlim X \to \varprojlim Y$ is a weak equivalence.

Proof Exercise: think about $r = 1$ first.

Lemma: Suppose that X is a fibrant tower and that $\mathcal{E}_r^{s,t} = *$ for $0 \leq t - s \leq k$. Then $\varprojlim X_n$ is k-connected.

The hypotheses of the Lemma imply that

$$\varprojlim \pi_i X_n = * = \varprojlim{}^1 \pi_{i+1} X_n$$

for $0 \leq i \leq k$. Use the Milnor exact sequence.

Descent spectral sequence

Consider the fibrant tower

$$\Gamma_* GP_0 X \xleftarrow{p} \Gamma_* GP_1 X \xleftarrow{p} \dots$$

The fibre F_n of $p : \Gamma_* GP_n X \to \Gamma_* GP_{n-1} X$ is a equivalent to $\Gamma_* GK(\pi_n X, n)$. Thus

$$\mathcal{E}_1^{s,t} = \pi_{t-s} F_s \cong \pi_{t-s} \Gamma_* GK(\pi_s X, s) \cong \varprojlim{}^{s-(t-s)} \pi_s F_s$$

for $0 \leq t - s \leq s$ and is 0 in other degrees.

Thomason's re-indexing trick: $E_{r+1}^{s,t} = \mathcal{E}_r^{t,2t-s}$ gives

$$E_2^{s,t} = \varprojlim{}^s \; \pi_t X$$

with differentials

$$d_r = d_{r-1} : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

Example: The Bousfield-Kan spectral sequence for a cosimplicial space X

$$E_2^{s,t} = H^s_\Delta \pi_t X \Rightarrow \pi_{t-s} \operatorname{\underleftarrow{holim}} X \cong \pi_{t-s} \mathbf{Tot}(X).$$

Bousfield and Kan write $\pi^s \pi_t X$ for $H^s_\Delta \pi_t X$.

Note: re-indexing preserves total degree $t - s$.