Lecture 006 (April 4, 2005)

Grothendieck topologies

A Grothendieck site is a small category $\mathcal{C}$ equipped with a topology $\mathbf{T}$.

A Grothendieck topology $\mathbf{T}$ consists of a collection of subfunctors

$$R \subset \hom(U), \quad U \in \mathcal{C},$$

called covering sieves, such that the following axioms hold:

1) (base change) If $R \subset \hom(\ ,U)$ is covering and $\phi : V \to U$ is a morphism of $\mathcal{C}$, then the subfunctor

$$\phi^{-1}(R) = \{\gamma : W \to V \mid \phi \cdot \gamma \in R\}$$

is covering.

2) (local character) Suppose that $R, R' \subset \hom(\ ,U)$ are subfunctors and R is covering. If $\phi^{-1}(R')$ is covering for all $\phi : V \to U$ in R, then R' is covering.

3) $\hom(\ ,U)$ is covering for all $U \in \mathcal{C}$

Typically Grothendieck topologies arise from covering families in sites $\mathcal{C}$ having pullbacks. Covering families are sets of functors which generate covering sieves.

Suppose that $\mathcal{C}$ has pullbacks. A topology $\mathbf{T}$ on $\mathcal{C}$ consists of families of sets of morphisms

$$\{\phi_\alpha : U_\alpha \to U\}, \quad U \in \mathcal{C},$$

called covering families, such that the following axioms hold:

1) Suppose that $\phi_\alpha : U_\alpha \to U$ is a covering family and that $\psi : V \to U$ is a morphism of $\mathcal{C}$. Then the collection $V \times_U U_\alpha \to V$ is a covering family for V.

2) If $\{\phi_\alpha : U_\alpha \to V\}$ is covering, and $\{\gamma_{\alpha,\beta} : W_{\alpha,\beta} \to U_\alpha$ is covering for all α, then the family of composites

$$W_{\alpha,\beta} \xrightarrow{\gamma_{\alpha,\beta}} U_\alpha \xrightarrow{\phi_\alpha} U$$

is covering.

3) The family $\{1 : U \to U\}$ is covering for all $U \in \mathcal{C}$.

Examples:

1) X = topological space. $\mathrm{op}\,|_X$ is the poset of open subsets $U \subset X$. A covering family for an open subset U is an open cover $V_\alpha \subset U$.

2) X = topological space. $\mathrm{loc}\,|_X$ is the category of all maps $f : Y \to X$ which are local homeomorphisms. f is a local homeomorphism if each $x \in Y$ has a neighbourhood U such that $f(U)$ is open in X and the restricted map $U \to f(U)$ is a homeomorphism. A morphism of $\mathrm{loc}\,|_X$ is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

where f and f' are local homeomorphisms. A family $\{\phi_\alpha : Y_\alpha \to Y\}$ of local homeomorphisms (over X) is covering if $X = \cup \phi_\alpha(Y_\alpha)$.

3) X = a scheme (topological space with sheaf of rings locally isomorphic to affine schemes $\mathrm{Sp}(R)$). The underlying topology on X is the Zariski topology. $Zar|_X$ is the poset with objects all open subschemes $U \subset X$. A family $V_i \subset U$ is covering if $\cup V_i = U$ (as sets).

$\phi : Y \to X$ is étale at $y \in Y$ if

a) $\mathcal{O}_y$ is a flat $\mathcal{O}_{f(y)}$-module (ϕ is flat at y).

b) ϕ is unramified at y: $\mathcal{O}_y/\mathcal{M}_{f(y)}\mathcal{O}_y$ is a finite separable field extension of $k(f(y))$.

Say that a map $\phi : Y \to X$ is étale if it is étale at every $y \in Y$.

4) S = scheme. The étale site $et|_S$ has as objects all étale maps $\phi : V \to S$ and all diagrams

$$\begin{array}{ccc} V & \longrightarrow & V' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & S & \end{array}$$

for morphisms (with ϕ, ϕ' étale). A covering family for the étale site is a collection of étale morphisms $\phi_i : V_i \to V$ such that $V = \cup \phi_i(V_i)$ as a set. Equivalently every morphism $\mathrm{Sp}(\Omega) \to V$ lifts to some V_i if Ω is a separably closed field.

5) The Nisnevich site $Nis|_S$ has the same underlying category as the étale site, namely all étale maps $V \to S$ and morphisms between them. A Nisnevich cover is a family of étale maps $V_i \to V$ such that every morphism $\mathrm{Sp}(K) \to V$ lifts to some V_i where K is any field.

6) A flat covering family of a scheme S is a set of flat morphisms $\phi_i : S_i \to S$ (ie. mophisms

which are flat at each point) such that $S = \cup\phi_i(S_i)$ as a set (equivalently $\sqcup S_i \to S$ is faithfully flat). $(Sch|_S)_{fl}$ is the "big" flat site. Pick a large cardinal κ; then $(Sch|_S)$ is the category of S-schemes $X \to S$ such that the cardinality of both the underlying point set of X and all sections $\mathcal{O}_X(U)$ of its sheaf of rings are bounded above by κ.

7) There are corresponding big sites $(Sch|_S)_{Zar}$, $(Sch|_S)_{et}$, $(Sch|_S)_{Nis}$, ... and you can play similar games with topological spaces.

8) Suppose that $G = \{G_i\}$ is profinite group such that all $G_j \to G_i$ are surjective group homomorphisms. Write $G = \varprojlim G_i$. A discrete G-set is a set X with G-action which factors through an action of G_i for some i. Write $G - \mathbf{Set}_{df}$ for the category of G-sets which are both discrete and finite. A family $U_i \to X$ in this category is covering if and only if $\sqcup U_i \to X$ is surjective.

9) Suppose that $\mathcal{C}$ is any small category. Say that $R \subset \hom(\ , x)$ is covering if and only if $1_x \in R$. This is the chaotic topology on $\mathcal{C}$.

10) Suppose that $\mathcal{C}$ is a site and that $x \in \mathcal{C}$. Then

the slice category $\mathcal{C}/x$ inherits a topology from $\mathcal{C}$: a collection of maps $y_i \to y \to x$ is covering if and only if the family $y_i \to y$ covers y.

Definitions: Suppose that $\mathcal{C}$ is a Grothendieck site.

1) A presheaf (of sets) on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to$ **Set**. This definition generalizes, of course: if $\mathcal{A}$ is a category, an $\mathcal{A}$-valued presheaf on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathcal{A}$. The set-valued presheaves on $\mathcal{C}$ form a category (morphisms are natural transformation), written $\mathrm{Pre}(\mathcal{C})$. There's no consistency in notation: eg. $s\,\mathrm{Pre}(\mathcal{C})$ denotes presheaves on $\mathcal{C}$ taking values in simplicial sets.

2) A sheaf (of sets) on $\mathcal{C}$ is a presheaf $F : \mathcal{C}^{op} \to$ **Set** such that the canonical map

$$F(x) \to \varprojlim_{y \to x \in R} F(y)$$

is an isomorphism for each covering sieve $R \subset \hom(x,\ )$. Morphisms of sheaves are natural transformations: write $\mathrm{Shv}(\mathcal{C})$ for the corresponding category $\mathrm{Shv}(\mathcal{C})$ is a full subcategory of $\mathrm{Pre}(\mathcal{C})$. One can also speak of sheaves in any complete category.

Exercise: If the topology on $\mathcal{C}$ is defined by a pretopology (so that $\mathcal{C}$ has all pullbacks), then F is a sheaf if and only if all pictures

$$F(x) \to \prod_i F(x_i) \rightrightarrows \prod_{i,j} F(x_i \times_x x_j)$$

arising from covering families are equalizers.

Facts about covering sieves:

1) If $R \subset R' \subset \hom(\ ,x)$ and R is covering then R' is covering. ($v^{-1}(R) = v^{-1}(R')$ for all $v \in R$)

2) If $R, R' \subset \hom(\ ,x)$ are covering then $R \cap R'$ is covering.

Suppose that $R \subset \hom(\ ,x)$ is a sieve, and F is a presheaf on $\mathcal{C}$. Write

$$F(x)_R = \varprojlim_{y \to x \in R} F(y)$$

If $S \subset R$ then there is an obvious map

$$F(x)_R \to F(x)_S$$

Write

$$LF(x) = \varinjlim_R F(x)_R$$

where the colimit is indexed over the filtering diagram of all covering sieves $R \subset \hom(\ ,x)$. Then

$x \mapsto LF(x)$ is a presheaf and there is a natural presheaf map

$$\eta : F \to LF$$

Say that a presheaf G is **separated** if (equivalently)

1) the map $\eta : G \to LG$ is monic in each section, ie. all functions $G(x) \to LG(x)$ are injective

2) Given $\alpha, \beta \in G(x)$, if there is a covering sieve $R \subset \hom(\ ,x)$ such that $\phi^*(\alpha) = \phi^*(\beta)$ for all $\phi \in R$, then $\alpha = \beta$.

Lemma:

1) LF is separated, for all presheaves F.

2) If G is separated then LG is a sheaf.

3) If $f : F \to G$ is a presheaf map and G is a sheaf, then f factors uniquely through a presheaf map $f_* : LF \to G$.

The object L^2F is a sheaf for every presheaf F, and the functor $F \mapsto L^2F$ is left adjoint to the inclusion $\text{Shv}(\mathcal{C}) \subset \text{Pre}(\mathcal{C})$. The unit of the adjunction is the composite

$$F \xrightarrow{\eta} LF \xrightarrow{\eta} L^2F$$

One often writes $\eta : F \to L^2F = \tilde{F}$ for this composite.

Exactness properties:

Fact: The associated sheaf functor preserves all finite limits.

Proof LF is defined by filtered colimits, and finite limits commute with filtered colimits. $\square$

Fact: $\mathrm{Shv}(\mathcal{C})$ is complete and co-complete. Limits are formed sectionwise.

If $X : I \to \mathrm{Shv}(\mathcal{C})$ is a diagram of sheaves, then the colimit in the sheaf category is $L^2(\varinjlim X)$, where $\varinjlim X$ is the presheaf colimit.

Fact: Every monic is an equalizer.

Proof If $A \subset X$ is a subset, then there is an equalizer

$$A \longrightarrow X \underset{*}{\overset{p}{\rightrightarrows}} X/A$$

The same holds for subobjects $A \subset X$ of presheaves, and hence for subobjects of sheaves, since L^2 is exact. $\square$

Fact: If $\theta : F \to G$ in $\mathrm{Shv}(\mathcal{C})$ is both monic and epi, then θ is an isomorphism.

Proof θ appears in an equalizer

$$F \xrightarrow{\theta} G \underset{g}{\overset{f}{\rightrightarrows}} K$$

since θ is monic. θ is an epi, so $f = g$. But then $1_G : G \to G$ factors through θ, giving a section $\sigma : G \to F$. Finally, $\theta\sigma\theta = \theta$ and θ is monic, so $\sigma\theta = 1$. $\square$

Definitions:

1) A presheaf map $f : F \to G$ is a **local epimorphism** if for each $\alpha \in G(x)$ there is a covering $R \subset \hom(\ , x)$ such that $\phi^*(x) = f(y_\phi)$ for all $\phi \in R$.

2) $f : F \to G$ is a **local monic** if given $\alpha, \beta \in F(x)$ such that $f(\alpha) = f(\beta)$, there is a covering $R \subset \hom(\ , x)$ such that $\phi^*(\alpha) = \phi^*(\beta)$ for all $\phi \in R$.

Lemma: The natural map $\eta : F \to LF$ is a local monomorphism and a local epimorphism.

Lemma: Suppose that $f : F \to G$ is a presheaf morphism. Then f induces an isomorphism of associated sheaves if and only if f is both a local epi and a local monic.

A sheaf map $g : E \to F$ is a monic (resp. epi) if and only if it is a local monic (resp. local epi).

Say that a presheaf map $f : F \to G$ which is both a local epi and a local monic is a **local isomorphism**.

Definition: A **Grothendieck topos** is a category $\mathcal{E}$ which is equivalent to a sheaf category $\mathrm{Shv}(\mathcal{C})$ on some Grothendieck site $\mathcal{C}$.

Grothendieck toposes are characterized by exactness properties:

Theorem: (Giraud) A category $\mathcal{E}$ having all finite limits is a Grothendieck topos if and only if it has the following properties:

1) $\mathcal{E}$ has all small coproducts; they are disjoint and stable under pullback
2) every epimorphism of $\mathcal{E}$ is a coequalizer
3) every equivalence relation $R \rightrightarrows E$ in $\mathcal{E}$ is a kernel pair and has a quotient
4) every coequalizer $R \rightrightarrows E \to Q$ is stably exact
5) there is a (small) set of objects which generates $\mathcal{E}$.

Some explanations:

1) $\sqcup_i \; A_i$ is disjoint if all diagrams

$$\begin{array}{ccc} \emptyset & \longrightarrow & A_j \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & \sqcup_i \; A_i \end{array}$$

are pullbacks for $i \neq j$. $\sqcup_i \; A_i$ is stable under pullback if all diagrams

$$\begin{array}{ccc} \sqcup_i B' \times_B A_i & \longrightarrow & \sqcup_i \; A_i \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

are pullbacks.

3) An **equivalence relation** is a monomorphism $m = (m_0, m_1) : R \to E \times E$ such that

a) the diagonal $\Delta : A \to A \times A$ factors through m $(a \sim a)$

b) the composite $R \xrightarrow{m} E \times E \xrightarrow{\tau} E \times E$ factors through m $(a \sim b \Rightarrow b \sim a)$.

c) the map

$$(m_0 m_{0*}, m_1 m_{1*}) : R \times_E R \to R \times R$$

factors through m (transitivity) where the pullback is defined by

$$\begin{array}{ccc} R \times_E R & \xrightarrow{m_{1*}} & R \\ {\scriptstyle m_{0*}}\downarrow & & \downarrow{\scriptstyle m_0} \\ R & \xrightarrow[m_1]{} & E \end{array}$$

The **kernel pair** of a morphism $u : E \to D$ is a pullback

$$\begin{array}{ccc} R & \xrightarrow{m_1} & E \\ {\scriptstyle m_0}\downarrow & & \downarrow{\scriptstyle u} \\ E & \xrightarrow[u]{} & D \end{array}$$

(**Ex.:** every kernel pair is an equivalence relation). A **quotient** for an equivalence relation $(m_0, m_1) : R \to E \times E$ is a coequalizer

$$R \underset{m_1}{\overset{m_0}{\rightrightarrows}} E \longrightarrow E/R$$

4) a coequalizer $R \rightrightarrows E \to Q$ is **stably exact** if the diagram

$$R \times_Q Q' \rightrightarrows E \times_Q Q' \to Q'$$

is a coequalizer for all morphisms $Q' \to Q$.

5) a **generating set** is a set $\{A_i\}$ which detects non-trivial monomorphisms: if a monomorphism $m : P \to Q$ induces bijections $\hom(A_i, P) \to \hom(A_i, Q)$ for all i, then m is an isomorphism.

Exercise: Show that any category $\mathrm{Shv}(C)$ on a site $\mathcal{C}$ satisfies the conditions of Giraud's theorem. The family $L^2 \hom(\ ,x)$, $x \in \mathcal{C}$ is a set of generators.

How Giraud's theorem works:

If A is the set of generators for $\mathcal{E}$ prescribed by Giraud's theorem, let $\mathcal{C}$ be the full subcategory of $\mathcal{E}$ on the set of objects A. A subfunctor $R \subset \hom(\ ,x)$ on $\mathcal{C}$ is covering if the map

$$\bigsqcup_{y \to x \in R} y \to x$$

is an epimorphism of $\mathcal{E}$.

Every object $E \in \mathcal{E}$ represents a sheaf $\hom(\ ,E)$ on $\mathcal{C}$, and a sheaf F on $\mathcal{C}$ determines an object

$$\varinjlim_{\hom(\ ,y) \to F} y$$

of $\mathcal{E}$.

The adjunction

$$\hom(\varinjlim_{\hom(\ ,y) \to F} y, E) \cong \hom(F, \hom(\ ,E))$$

determines an adjoint equivalence between $\mathcal{E}$ and $\mathrm{Shv}(\mathcal{C})$.

Examples:

1) Suppose that G is a sheaf of groups, and let $G - \mathrm{Shv}(\mathcal{C})$ denote the category of all sheaves X admitting G-action, with equivariant maps between them. $G - \mathrm{Shv}(\mathcal{C})$ is a Grothendieck topos, called the **classifying topos** for G, by Giraud's theorem. The objects $G \times \hom(\ ,x)$ form a generating set.

2) If $G = \{G_i\}$ is a profinite group with all transition maps $G_i \to G_j$ epi, then the category $G - \mathbf{Set}_d$ of discrete G-sets is a Grothendieck topos. The finite discrete G-sets form a generating set for this topos, and the site of finite discrete G-sets is "the" site prescribed by Giraud's theorem.