Lecture 007 (April 4, 2005)

Geometric morphisms

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck sites. A **geometric morphism** $f : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ consists of functors $f_* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ and $f^* : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ such that f^* is left adjoint to f_* and f^* preserves finite limits.

f^* is often called the inverse image functor, while f_* is called the direct image functor. f^* is exact; f_* is usually not exact, and hence has higher derived functors.

Examples

1) Suppose $f : X \to Y$ is a continuous map of topological spaces. Pullback along f induces a functor $\mathrm{op}\,|_Y \to \mathrm{op}\,|_X$: $U \subset Y \mapsto f^{-1}(U)$. Open covers pull back to open covers, so if F is a sheaf on X then composition with the pullback gives a sheaf f_*F on Y with $f_*F(U) = F(f^{-1}(U))$. The resulting functor $f_* : \mathrm{Shv}(\mathrm{op}\,|_X) \to \mathrm{Shv}(\mathrm{op}\,|_Y)$ is the direct image

   The left Kan extension $f^p : \mathrm{Pre}(\mathrm{op}\,|_Y) \to \mathrm{Pre}(\mathrm{op}\,|_X)$ is defined by

$$f^pG(V) = \varinjlim G(U)$$

where the colimit is indexed over all diagrams

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & U \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The category $\mathrm{op}\,|_Y$ has all products (ie. intersections), so the colimit is filtered and $G \mapsto f^pG$ commutes with finite limits. The inverse image functor $f^* : \mathrm{Shv}(\mathrm{op}\,|_Y) \to \mathrm{Shv}(\mathrm{op}\,|_X)$ is defined by $f^*(G) = L^2f^p(G)$. The resulting pair of functors forms a geometric morphism $f : \mathrm{Shv}(\mathrm{op}\,|_X) \to \mathrm{Shv}(\mathrm{op}\,|_Y)$.

2) A point of $\mathrm{Shv}(\mathcal{C})$ is a geometric morphism $\mathbf{Set} \to \mathrm{Shv}(\mathcal{C}$. Every point $x \in X$ of a topological space X determines a continuous map $\{x\} \subset X$ and hence a geometric morphism

$$\mathbf{Set} \cong \mathrm{Shv}(\mathrm{op}\,|_{\{x\}}) \xrightarrow{x} \mathrm{Shv}(\mathrm{op}\,|_X)$$

The set

$$x^*F = \varinjlim_{x \in U} F(U)$$

is the **stalk** of F at x

3) Suppose that $f : X \to Y$ is a morphism of schemes. Etale maps (resp. covers) are stable under pullback, and so there is a functor

$\mathrm{et}\,|_Y \to \mathrm{et}\,|_X$ defined by pullback, and if F is a sheaf on $\mathrm{et}\,|_X$ then there is a sheaf f_*F on $\mathrm{et}\,|_Y$ defined by $f_*F(V \to Y) = f(X \times_Y V \to X)$ The restriction functor $f_* : \mathrm{Pre}(\mathrm{et}\,|_X) \to \mathrm{Pre}(\mathrm{et}\,|_Y)$ has a left adjoint f^p defined by

$$f^p G(U \to X) = \varinjlim G(V)$$

where the colimit is indexed over all diagrams

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where both vertical maps are étale. The colimit is filtered, essentially because étale maps are stable under pullback. The inverse image functor $f^* : \mathrm{Shv}(\mathrm{et}\,|_Y) \to \mathrm{Shv}(et|_X)$ is defined by

$$f^*F = L^2 f^p F.$$

and so f induces a geometric morphism f : $\mathrm{Shv}(et|_X) \to \mathrm{Shv}(et|_Y)$.

A morphism of schemes $f : X \to Y$ induces a geometric morphism $f : \mathrm{Shv}(?|_X) \to \mathrm{Shv}(?|_Y)$ and/or $f : (Sch|_X)_? \to (Sch|_Y)_?$ for all of the geometric topologies (eg. Zariski, flat, Nisnevich, qfh, ...), by similar arguments.

4) Suppose that k is a field. Any scheme map $x : \mathrm{Sp}(k) \to X$ induces a geometric morphism

$$\mathrm{Shv}(et|_k) \to \mathrm{Shv}(et|_X)$$

If k happens to be separably closed, then there is an equivalence $\mathrm{Shv}(et|_k) \simeq \mathbf{Set}$ and the resulting geometric morphism $x : \mathbf{Set} \to \mathrm{Shv}(\mathrm{et}\,|_X)$ is called a geometric point of X. The inverse image functor

$$F \mapsto f^*F = \varinjlim_{\begin{tikzcd} & U \arrow[d] \\ \mathrm{Sp}(k) \arrow[ur] \arrow[r, "x"'] & X \end{tikzcd}} F(U)$$

is the stalk of F at x.

5) Suppose that S and T are topologies on a site $\mathcal{C}$ so that $S \subset T$. In other words, T has more covers than S and hence refines S. Then every sheaf for T is a sheaf for S; write

$$\pi_* : \mathrm{Shv}(\mathcal{C}, T) \subset \mathrm{Shv}(\mathcal{C}, S)$$

for the corresponding inclusion. The associated sheaf functor for the topology T gives a left adjoint π^* for the inclusion functor π_*, and of course π^* preserves finite limits.

Example: there is a geometric morphism $\mathrm{Shv}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C})$ determined by the inclusion of the sheaf category in the presheaf category and the associated sheaf functor.

Say that a Grothendieck topos $\mathrm{Shv}(\mathcal{C})$ **has enough points** if there is a set of geometric morphisms $x_i : \mathbf{Set} \to \mathrm{Shv}(\mathcal{C})$ such that the induced morphism

$$\mathrm{Shv}(\mathcal{C}) \xrightarrow{(x_i^*)} \prod_i \mathbf{Set}$$

is faithful.

Lemma: Suppose that $f : \mathrm{Shv}(\mathcal{D}) \to \mathrm{Shv}(\mathcal{C})$ is a geometric morphism. Then the following are equivalent:

a) $f^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D})$ is faithful.

b) f^* reflects isomorphisms

c) f^* reflects epimorphisms

d) f^* reflects monomorphisms

Proof Suppose that f^* is faithful, ie. that $f^*(g_1) = f^*(g_2)$ implies that $g_1 = g_2$. Suppose that $m : F \to G$ is a morphism of $\mathrm{Shv}(\mathcal{C})$ such that $f^*(m)$ is monic. If $m \cdot f_1 = m \cdot f_2$ then $f^*(f_1) = f^*(f_2)$ so $f_1 = f_2$. The map m is therefore monic. Similarly

f^* reflects epimorphisms and hence isomorphisms.

Suppose that f^* reflects epimorphisms and suppose given $g_1, g_2 : F \longrightarrow G$ such that $f^*(g_1) = f^*(g_2)$. $g_1 = g_2$ if and only if their equalizer $e : E \longrightarrow F$ is an epimorphism. But f^* preserves equalizers and reflects epimorphisms, so e is an epi and $g_1 = g_2$. The other arguments are similar. $\square$

Frames and Boolean algebras: basic definitions

A **lattice** L is a partially ordered set which has all finite coproducts $x \vee y$ and all finite products $x \wedge y$.

A lattice L has 0 and 1 if it has an initial and terminal object, respectively.

A lattice L is said to be **distributive** if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all x, y, z.

A **complement** for x is a lattice L with 0 and 1 is an element a such that $x \vee a = 1$ and $x \wedge a = 0$.

If L is also distibutive the complement, if it exists, is unique: if b is another complement for x, then

$$\begin{aligned} b = b \wedge 1 &= b \wedge (x \vee a) = (b \wedge x) \vee (b \wedge a) \\ &= (x \wedge a) \vee (b \wedge a) = (x \vee b) \wedge a = a \end{aligned}$$

In this case write $\neg x$ for the complement of x.

A **Boolean algebra** B is a distributive lattice with 0 and 1 in which every element has a complement.

A lattice L is said to be **complete** if it has all small limits and colimits (aka. all small meets and joins).

A **frame** P is a lattice which has all small joins (and all finite meets) and which satisfies an infinite distributive law

$$U \wedge (\bigvee_i V_i) = \bigvee_i (U \wedge V_i)$$

Examples:

1) The poset $\mathcal{O}(T)$ of open subsets of a topological space T is a frame. Every continuous map $f : S \to T$ induces a morphism of frames $f^{-1} : \mathcal{O}(T) \to \mathcal{O}(S)$, defined by $U \mapsto F^{-1}(U)$.
2) The power set $\mathcal{P}(I)$ of a set I is a complete Boolean algebra.

Every frame A has a canonical Grothendieck topology: a family $y_i \leq x$ is covering if $\vee_i y_i = x$. Write $\mathrm{Shv}(A)$ for the corresponding sheaf category.

Every complete Boolean algebra B is a frame, and therefore has an associated sheaf category $\mathrm{Shv}(B)$.

Example: Suppose that I is a set. Then there is an equivalence

$$\mathrm{Shv}(\mathcal{P}(I)) \simeq \prod_{i\in I} \mathbf{Set}$$

Any set I of points $x_j : \mathbf{Set} \to \mathrm{Shv}(\mathcal{C}$ assembles to give a geometric morphism

$$x : \mathrm{Shv}(\mathcal{P}(I)) \to \mathrm{Shv}(\mathcal{C}).$$

The direct image functor

$$x_* : \prod_i \mathbf{Set} \to \mathrm{Shv}(\mathcal{C})$$

is defined by $x_*(A_i) = \prod_i x_{i*}(A_i)$. $\mathrm{Shv}(\mathcal{C})$ has enough points if and only if there is a geometric morphism x as above such that the inverse image functor x^* is faithful.

Complete Boolean algebras

Lemma: Suppose that F is a sheaf of sets on a complete Boolean algebra $\mathcal{B}$. Then the poset $Sub(F)$ of subobjects of F is a complete Boolean algebra.

Proof $Sub(F)$ is a frame, by an argument on the presheaf level. It remains to show that every object $G \in Sub(F)$ is complemented. The obvious candidate for $\neg G$ is

$$\neg G = \bigvee_{H \wedge G = \emptyset} H$$

and we need to show that $G \vee \neg G = F$.

Every $K \leq \hom(\ ,A)$ is representable:

$$K = \varinjlim_{\hom(\ ,B) \to K} \hom(\ ,B) = \hom(\ ,C)$$

where

$$C = \bigvee_{\hom(\ ,B) \to K} B \in \mathcal{B}$$

Consequence: $Sub(\hom(\ ,A)) \cong Sub(A)$ is a complete Boolean algebra. If $C \leq A$ then $\neg C = \neg C \wedge A$.

Consider all diagrams

$$\begin{tikzcd}
\phi^{-1}(G) \arrow[r] \arrow[d] & G \arrow[d] \\
\hom(\ ,A) \arrow[r, "\phi"'] & F
\end{tikzcd}$$

There is an induced pullback

$$\begin{tikzcd}
\phi^{-1}(G) \vee \neg\phi^{-1}(G) \arrow[r] \arrow[d, "\cong"'] & G \vee \neg G \arrow[d] \\
\hom(\ ,A) \arrow[r, "\phi"'] & F
\end{tikzcd}$$

F is a union of its representables (all ϕ are monic since all $\hom(\ ,A)$ are subobjects of the terminal sheaf), so $G \vee \neg G = F$. $\square$

Lemma: Suppose that B is a complete Boolean algebra. Then every epimorphism $\pi : F \to G$ in $\mathrm{Shv}(B)$ has a section.

The Lemma asserts that the sheaf category on a complete Boolean algebra satisfies the Axiom of Choice.

Proof Consider the family of lifts

$$\begin{tikzcd}
 & F \arrow[d, "\pi"] \\
N \arrow[ur] \arrow[r, "\leq"'] & G
\end{tikzcd}$$

This family is non-empty, because every $x \in G(1)$ restricts along some covering $B \leq 1$ to a family of elements x_B which lift to $F(B)$; all maps $\hom(\ ,B) \to G$ are monic.

Zorn's lemma implies that the family of lifts has maximal elements. Suppose that N is maximal and that $\neg N \neq \emptyset$. Then there is an $x \in \neg N(C)$ for some C, and there is a covering $B' \leq C$ such that $x_{B'} \in N(B')$ lifts to $F(B')$ for all members of the cover. Then $N \wedge \hom(\ ,B') = \emptyset$ so the lift extends to a lift on $N \vee \hom(\ ,B')$, contradicting the maximality of N. $\square$

Definition: A **Boolean localization** for $\mathrm{Shv}(\mathcal{C})$ is a geometric morphism $p : \mathrm{Shv}(B) \to \mathrm{Shv}(\mathcal{C})$ such that p^* is faithful (or, in topos-speak, that p is surjective).

Theorem: (Barr) Boolean localizations exist for every Grothendieck topos $\mathrm{Shv}(\mathcal{C})$.