Lecture 008 (February 6, 2005)

This file contains an "elementary" proof of the Barr theorem that says that every Grothendieck topos has a Boolean cover.

The proof is in two steps, just as in the literature (eg. Mac Lane-Moerdijk):

1) Show that every Grothendieck topos has a localic cover (Diaconescu's theorem).
2) show that every localic topos has a Boolean cover.

We begin with the latter.

Boolean covers

Recall that a frame F is a lattice which has all small joins and satisfies an infinite distributive law.

Recall also that every frame A has a canonical Grothendieck topology: say that a family $y_i \leq x$ is covering if $\vee_i y_i = x$. Write $\mathrm{Shv}(A)$ for the corresponding sheaf category.

Say that a Grothendieck topos B is **localic** if it is equivalent to $\mathrm{Shv}(A)$ for some frame A.

Theorem: [Mac Lane-Moerdijk, p.488] A Grothendieck topos B is localic if and only if it is equivalent to Shv(P) for some topology on a poset P.

Proof (culture) The corresponding frame is the poset of subobjects of the terminal object $1 = *$. These subobjects generate B. Use Giraud's theorem. $\square$

A morphism $f : A \to B$ of frames is a poset morphism which preserves structure, ie. preserves all finite meets and all infinite joins, hence preserves both 0 and 1.

Lemma: Every frame morphism $f : A \to B$ has a right adjoint $f_* : B \to A$.

Proof Set $f_*(y) = \vee_{f(x) \leq y} \ x$. $\square$

Suppose that $i : P \to B$ is a morphism of frames. Then precomposition with i determines a functor $i_* : \mathbf{Shv}(B) \to \mathbf{Shv}(P)$, since i preserves covers.

The left adjoint $i^* : \mathbf{Shv}(P) \to \mathbf{Shv}(B)$ of i_* associates to a sheaf F the sheaf i^*F, which is the sheaf associated to the presheaf i^pF, where

$$i^pF(x) = \varinjlim_{x \to i(y)} F(y).$$

Note that the colimit is filtered since i preserves meets.

Lemma: The presheaf $i^p F$ is separated.

Proof Suppose that $\alpha, \beta \in i^p F(x)$ map to the same element in $i^* F(x)$. Then there is a covering family $z_j \leq x$ such that x, y restrict to the same element of $i^p F(z_j)$ for all i. Suppose that $\alpha, \beta \in F(y)$ for some fixed $x \leq i(y)$. Then for each i there is a commutative diagram of relations

$$\begin{array}{ccc} z_j & \longrightarrow & i(v_j) \\ \downarrow & & \downarrow \\ x & \longrightarrow & i(y) \end{array}$$

such that α and β restrict to the same element of $F(v_j)$. But then α and β restrict to the same element of $F(\vee v_j)$ and $\vee z_j = x$ and there is a commutative diagram

$$\begin{array}{ccc} x & \longrightarrow & i(\vee v_j) \\ & \searrow & \downarrow \\ & & i(y) \end{array}$$

Lemma: Suppose that the frame morphism $i : P \to B$ is a monomorphism. Then the functor $i^* : \mathbf{Shv}(P) \to \mathbf{Shv}(B)$ is faithful.

Proof By the previous Lemma, it is enough to show that the canonical map $\eta : F \to i_* i^p F$ is a monomorphism of presheaves for all sheaves F on

P. For then $\eta : F \to i_* i^* F$ is monic, and so i^* is faithful (exercise).

The map $\eta : F(y) \to \varinjlim_{i(y) \leq i(z)} F(z)$ is the canonical map into the colimit which is associated to the identity map $i(y) \leq i(y)$ of B.

Note that i is monic, so that $x = i_* i(x)$ for all $x \in P$, where i_* is the right adjoint of $i : P \to B$. Thus, $i(y) \leq i(z)$ if and only if $y \leq z$, so that category of all morphisms $i(y) \leq i(z)$ has an initial object, namely the identity on $i(y)$. The map

$$\eta : F(y) \to \varinjlim_{i(y) \leq i(z)} F(z)$$

is therefore an isomorphism for all y. $\square$

Suppose that P is a frame and $x \in P$. Write P_x for the subobject of P consisting of all y such that $x \leq y$. P_x is a frame with initial object x and terminal object 1. There is a frame morphism

$$\phi_x : P \to P_x$$

defined by $\phi_x(w) = x \vee w$.

For $x \in P$, write

$$\neg x = \bigvee_{x \wedge y = 0} y$$

Note that $x \wedge \neg x = 0$ so that $x \leq \neg\neg x$.

Suppose that Q is a frame and that $x \in Q$. Write

$$\neg x = \bigvee_{x \wedge y = 0} y$$

Note that $x \wedge \neg x = 0$ so that there is a relation (morphism)

$$\eta : x \leq \neg\neg x$$

for all $x \in Q$; this relation is natural in x. Further, the relation η induces the relation $\neg\eta : \neg\neg\neg x \leq \neg x$, while we have the relation $\eta : \neg x \leq \neg\neg\neg x$ for $\neg x$. It follows that the relation

$$\eta : \neg x \leq \neg\neg\neg x$$

is an equality (isomorphism) for all $x \in Q$.

Define a subposet $\neg\neg Q$ of Q by

$$\neg\neg Q = \{y \in Q |\ y = \neg\neg y\}$$

There is a diagram of relations

$$\begin{array}{ccc} x \wedge y & \longrightarrow & \neg\neg(x \wedge y) \\ \downarrow & \swarrow & \\ (\neg\neg x) \wedge (\neg\neg y) & & \end{array}$$

Thus, the element $x \wedge y$ is a member of $\neg\neg Q$ if both x and y are in $\neg\neg Q$, for in that case the vertical map in the diagram is an isomorphism. If the set of objects x_i are members of $\neg\neg Q$, then the element $\neg\neg(\vee_i x_i)$ is their join in $\neg\neg Q$. It follows that the poset $\neg\neg Q$ is a frame, and that the assigment $x \mapsto \neg\neg x$ defines a frame morphism

$$\gamma : Q \to \neg\neg Q.$$

Lemma: $\neg\neg Q$ is a complete Boolean algebra.

Proof Observe that

$$y \leq \neg z \Leftrightarrow y \wedge z = 0 \Leftrightarrow z \leq \neg y.$$

It follows that

$$\neg(\bigvee \neg y_i) = \wedge \neg\neg y_i = \wedge y_i,$$

giving the completeness. Also, x is complemented by $\neg x$ in $\neg\neg Q$ since

$$x \vee (\neg x) = \neg\neg x \vee \neg\neg\neg x = \neg(\neg x \wedge x) = \neg 0 = 1.$$

Consider the composite morphism

$$\omega : P \to \prod_{x \in P} P_x \to \prod_{x \in P} \neg\neg P_x$$

The product $\Pi_x \neg\neg P_x$ is a complete Boolean algebra.

Lemma: The map ω is a monomorphism.

Proof Also if $x \leq y$ then $\neg\neg\phi_x(y) = 0$ in P_x implies that $y \vee x = x$ in P_x and hence $y = x$ in P. Thus, if $x \leq y$ and $y \neq x$ then x and y have distinct images in $\Pi_x \neg\neg P_x$.

Suppose that y and z are distinct elements of P. Then $y \neq y \vee z$ or $z \leq y$ and $z \neq y$. Then $\omega(y) \neq \omega(y) \vee \omega(z)$ or $\omega(z) \leq \omega(y)$ and $\omega(z) \neq \omega(y)$. The assumption that $\omega(z) = \omega(y)$ contradicts both possibilities, so $\omega(z) \neq \omega(y)$. □

Corollary: Every frame P admits an imbedding $i : P \to B$ into a complete Boolean algebra.

We have proved the following:

Theorem: Suppose that P is a frame. Then there is a complete Boolean algebra B, and a topos morphism $i : \mathrm{Shv}(B) \to \mathrm{Shv}(P)$ such that the inverse image functor $i^* : \mathrm{Shv}(P) \to \mathrm{Shv}(B)$ is faithful.

A geometric morphism i as in the statement of the Theorem is called a Boolean cover of $\mathrm{Shv}(P)$.

Diaconescu's theorem

Suppose that $\mathcal{C}$ is a (small) Grothendieck site. Write $\mathbf{St}(\mathcal{C})$ for the poset of all finite strings

$$\sigma : x_n \to \cdots \to x_0$$

where $\tau \leq \sigma$ if τ extends σ to the left in the sense that τ is of the form

$$y_k \to \cdots \to y_{m+1} \to x_n \to \cdots \to x_0$$

There is a functor $\pi : \mathbf{St}(\mathcal{C}) \to \mathcal{C}$ which is defined by $\pi(\sigma) = x_n$ for σ as above.

If $R \subset \hom(\ ,\sigma)$ is a sieve of $\mathbf{St}(\mathcal{C})$, then $\pi(R) \subset \hom(\ ,x_n)$ is a sieve of $\mathcal{C}$. In effect, if $\tau \leq \sigma$ is in R and $z \to y_k$ is morphism of $\mathcal{C}$, then the string

$$\tau_* : z \to y_k \to \cdots \to y_{m+1} \to x_n \to \dots x_0$$

refines τ and the relation $\tau_* \leq \tau$ maps to $z \to y_k$.

Say that a sieve $R \subset \hom(\ ,\sigma)$ is covering if $\pi(R)$ is a covering sieve of $\mathcal{C}$. Then $\mathbf{St}(\mathcal{C})$ acquires the structure of a Grothendieck site.

Fact: Suppose that F is a sheaf of sets on $\mathcal{C}$. Then $\pi^*(F) = F \cdot \pi$ is a sheaf on $\mathbf{St}(\mathcal{C})$.

Lemma: The functor $F \mapsto \pi^*(F)$ is faithful.

Proof For $x \in \mathcal{C}$ let $\{x\}$ denote the corresponding string of length 0. Then $\pi^* F(\{x\}) = F(x)$. If sheaf morphisms $f, g : F \to G$ on $\mathcal{C}$ induce maps $f_*, g_* : \pi^*(F) \to \pi^*(G)$ such that $f_* = g_*$, then $f_* = g_* : \pi^* F(\{x\}) \to \pi^* G(\{x\})$ for all $x \in \mathcal{C}$. But this means that $f = g$. $\square$

Lemma: The functor π^* preserves local epis and local monics of presheaves.

Proof Suppose $m : P \to Q$ is a local monomorphism of presheaves on $\mathcal{C}$. This means that if $m(\alpha) = m(\beta)$ for $\alpha, \beta \in P(x)$ there is a covering $\phi_i : y_i \to x$ such that $\phi_i^*(\alpha) = \phi_i^*(\beta)$ for all ϕ_i.

Suppose that $\alpha, \beta \in \pi^* P(\sigma)$ such that $m_*(\alpha) = m_*(\beta)$ in $\pi^* Q(\sigma)$. Then $\alpha, \beta \in P(x_n)$ and $m(\alpha) = m(\beta) \in Q(x_n)$. There is a covering $\phi_i : y_i \to x_n$ such that $\phi_i^*(\alpha) = \phi_i^*(\beta)$ for all ϕ_i. But then α, β map to the same element of

$$\pi^* P(y_i \to x_n \to \cdots \to x_0)$$

for all members of a cover of σ.

Suppose that $p : P \to Q$ is a local epimorphism of presheaves on $\mathcal{C}$. Then for all $\alpha \in Q(x)$ there is a covering $\phi_i : y_i \to x$ such that $\phi_i^*(\alpha)$ lifts to an element of $P(y_i)$ for all i. Given $\alpha \in \pi^* Q(\sigma)$, $\alpha \in Q(x_n)$, and there is a cover $\phi_i : y_i \to x_n$ such that $\phi_i^*(\alpha)$ lifts to $P(y_i)$. It follows that there is a cover of σ such that the image of α in

$$\pi^* Q(y_i \to x_n \to \cdots \to x_0)$$

lifts to

$$\pi^* P(y_i \to x_n \to \cdots \to x_0)$$

for all members of the cover. □

Lemma: The functor

$$\pi^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathbf{St}(\mathcal{C}))$$

preserves all small colimits.

Proof Suppose that $A : I \to \mathrm{Shv}(\mathcal{C})$ is a small diagram of sheaves. Write $\varinjlim_i A$ for the presheaf colimit, and let

$$\eta : \varinjlim A \to L^2(\varinjlim A)$$

be the natural associated sheaf map. The map η is a local epi and a local monomorphism. The functor π^* plainly preserves presheaf colimits, and there is a diagram

$$\begin{array}{ccc} \pi^*(\varinjlim A) & \xrightarrow{\pi^*(\eta)} & \pi^*(L^2(\varinjlim A)) \\ \cong\uparrow & & \uparrow \\ \varinjlim \pi^* A & \xrightarrow[\eta]{} & L^2(\varinjlim \pi^* A) \end{array}$$

Then $\pi^*(\eta)$ is a local epi and a local monic by the previous lemma. It follows that the map

$$L^2(\varinjlim \pi^* A) \to \pi^*(L^2(\varinjlim A))$$

is a local epi and a local monomorphism of sheaves, and is therefore an isomorphism. □

The following result asserts that any Grothendieck topos has a localic cover (see also Mac Lane-Moerdijk, p.511). The topos $\mathrm{Shv}(\mathbf{St}(\mathcal{C}))$ is also called the Diaconescu cover of $\mathrm{Shv}(\mathcal{C})$.

Theorem: (Diaconescu) The right adjoint $\pi_* : \mathrm{Pre}(\mathbf{St}(\mathcal{C})) \to \mathrm{Pre}(\mathcal{C})$ of precomposition with π restricts to a functor

$$\pi_* : \mathrm{Shv}(\mathbf{St}(\mathcal{C})) \to \mathrm{Shv}(\mathcal{C})$$

which is right adjoint to π^*. The functors π^* and π_* determine a geometric morphism

$$\pi : \mathrm{Shv}(\mathbf{St}(\mathcal{C})) \to \mathrm{Shv}(\mathcal{C}).$$

The functor π^* is faithful.

Proof A covering sieve $R \subset \hom(\ , x)$ in $\mathcal{C}$ determines an isomorphism of sheaves

$$\varinjlim_{y \to x} L^2 \hom(\ , y) \cong L^2 \hom(\ , x).$$

The functor π^* preserves colimits of sheaves, and so $\pi_* G$ is a sheaf if G is a sheaf.

The functor π^* plainly preserves finite limits. $\square$

We have assembled a proof of the following:

Theorem: (Barr's theorem; Boolean localization) Suppose that $\mathcal{C}$ is a small Grothendieck site. Then there are geometric morphisms

$$\mathrm{Shv}(B) \xrightarrow{f} \mathrm{Shv}(\mathbf{St}(\mathcal{C}) \xrightarrow{\pi} \mathrm{Shv}(\mathcal{C})$$

such that the inverse image functors f^* and π^* are faithful, and such that B is a complete Boolean algebra.