Lecture 009 (April 4, 2005)

Local weak equivalences

Suppose that $\mathcal{C}$ is a small Grothendieck site. Let $s\,\mathrm{Pre}(\mathcal{C}$ and $s\,\mathrm{Shv}(\mathcal{C})$ denote the categories of simplicial presheaves and simplicial sheaves on $\mathcal{C}$, respectively.

Recall that a simplicial set map $f : X \to Y$ is a weak equivalence if all maps

a) $\pi_0 X \to \pi_0 Y$, and

b) $\pi_i(X, x) \to \pi_i(Y, fx)$, $x \in X_0, i \geq 1$

are bijections. Here $\pi_i(X, x) = \pi_i(|X|, x)$ in general, but

$$\pi_i(X, x) = [(S^i, *), (X, x)] = \pi((S^i, *), (X, x))$$

if X is a Kan complex (recall that $S^i = \Delta^i / \partial\Delta^i$ is the simplicial i-sphere).

There is a different way to organize this: $f : X \to Y$ is a weak equivalence if

a) $\pi_0 X \to \pi_0 Y$ is a bijection, and

b) all diagrams

$$\begin{array}{ccc} \pi_i X & \longrightarrow & \pi_i Y \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

are pullbacks for $i \geq 1$.

Here,

$$\pi_i X = \bigsqcup_{x \in X_0} \pi_i(X, x).$$

The basic idea behind the homotopy theory of simplicial presheaves is that the topology of the underlying site $\mathcal{C}$ should create the weak equivalences.

It's easy to see how to do this in cases where there are enough points:

Example: A map $f : X \to Y$ of simplicial presheaves on $op|_T$ for some topological space T should be a local weak equivalence if and only if it induces a weak equivalence in stalks $X_x \to Y_x$ for all $x \in T$. In particular f should induce isomorphisms

$$\pi_i(X_x, y) \to \pi_i(Y_X, f(y))$$

for all $i \geq 1$ *and all choices of base point* $y \in X_x$.

$$X_x = \varinjlim_{x \in U} X(U)$$

is a filtered colimit, and so each base point y comes from somewhere, namely some $z \in X(U)$ for some U. The point z determines a global section of $X|_U$, which is the composite

$$((op|_T)/U)^{op} \to (op|_T)^{op} \xrightarrow{X} s\mathbf{Set}$$

and f restricts to a simplicial presheaf map $f|_U : X|_U \to Y|_U$. The one can show that f is a local weak equivalence if and only if all induced sheaf maps

a) $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$, and

b) $\tilde{\pi}_i(X|_U, z) \to \tilde{\pi}_i(Y|_U, f(z))$, $i \geq 1$, $U \in \mathcal{C}$, $z \in X_0(U)$

are isomorphisms.

This is equivalent to the following: the map $f : X \to Y$ is a local weak equivalence if and only if

a) $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism

b) all presheaf diagrams

$$\begin{array}{ccc} \pi_i X & \longrightarrow & \pi_i Y \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

induce pullback diagrams of associated sheaves.

Both descriptions generalize to equivalent conditions for maps of simplicial presheaves on an arbitrary site $\mathcal{C}$:

Definition A: A map $f : X \to Y$ of $s\,\mathrm{Pre}(\mathcal{C})$ is a *local weak equivalence* if and only if

a) the map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism of sheaves, and

b) all maps $\tilde{\pi}_i(X|_U, x) \to \tilde{\pi}_i(Y|_U, f(x))$ are isomorphisms of sheaves on $\mathcal{C}/U$ for all $i \geq 1$, $U \in \mathcal{C}$, $x \in X_0(U)$.

Here, $X|_U$ is the composite

$$(\mathcal{C}/U)^{op} \to \mathcal{C}^{op} \xrightarrow{X} s\mathbf{Set}.$$

Definition B: A map $f : X \to Y$ of $s\,\mathrm{Pre}(\mathcal{C})$ is a *local weak equivalence* if and only if

a) the map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism of sheaves, and

b) all diagrams

$$\begin{array}{ccc} \pi_i X & \longrightarrow & \pi_i Y \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

induce pullback diagrams of associated sheaves.

Exercise: Show that Definition A is equivalent to Definition B.

Definition: Suppose that $i : K \subset L$ is a cofibration of finite simplicial sets and that $f : X \to Y$ is a map of simplicial presheaves. We say that f has the local right lifting property with respect to i if for every diagram

$$\begin{array}{ccc} K & \longrightarrow & X(U) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ L & \longrightarrow & Y(U) \end{array}$$

there is a covering $R \subset \hom(\ ,U)$ such that the lift exists in the diagram

$$\begin{array}{ccccc} K & \longrightarrow & X(U) & \xrightarrow{\phi^*} & X(V) \\ {\scriptstyle i}\downarrow & & & \nearrow & \downarrow{\scriptstyle f} \\ L & \longrightarrow & Y(U) & \xrightarrow[\phi^*]{} & Y(V) \end{array}$$

for every $\phi : V \to U$ in R.

Write X^K for the presheaf defined by

$$X^K(U) = \mathbf{hom}(K, X(U))$$

Exercise: A map $f : X \to Y$ has the local RLP with respect to $i : K \to L$ if and only if the simplicial presheaf map

$$X^L \xrightarrow{(i^*, f_*)} X^K \times_{Y^K} Y^L$$

is a local epimorphism in degree 0.

Definition: A local fibration is a map which has the local RLP wrt all $\Lambda^n_k \subset \Delta^n$. A simplicial presheaf X is locally fibrant if the map $X \to *$ is a local fibration.

Remark: A local fibration has the local RLP wrt all anodyne extensions $A \subset B$ with B finite.

Lemma: Suppose that X and Y are presheaves of Kan complexes. Then a map $p : X \to Y$ is a local fibration and a local weak equivalence if and only if it has the RLP wrt all $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.

Proof (sketch — the argument is given in detail on pp. 51–53 of "Simplicial presheaves")

Suppose that p is a local fibration and a local weak equivalence, and that we have a diagram

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X(U) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y(U) \end{array}$$

The idea is to show that this diagram is locally homotopic to diagrams

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X(V) \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y(V) \end{array}$$

for which the lift exists. This means that there are homotopies

$$\begin{array}{ccc} \partial\Delta^n \times \Delta^1 & \longrightarrow & X(V) \\ \downarrow & & \downarrow p \\ \Delta^n \times \Delta^1 & \longrightarrow & Y(V) \end{array}$$

from the diagrams

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & X(U) & \xrightarrow{\phi^*} & X(V) \\ \downarrow & & & & \downarrow p \\ \Delta^n & \longrightarrow & Y(U) & \xrightarrow[\phi^*]{} & Y(V) \end{array}$$

to the corresponding diagrams above for all $\phi : V \to U$ in a covering for U. If such local homotopies exist, then solutions to the lifting problems

$$\begin{array}{ccc} (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) & \longrightarrow & X(V) \\ \downarrow & & \downarrow p \\ \Delta^n \times \Delta^1 & \longrightarrow & Y(V) \end{array}$$

have local solutions for each V, and so the original lifting problem is solved on the refined covering of U. The local homotopies are created by arguments similar to the proof of the corresponding result in the simplicial set case.

For the converse show that the presheaf maps $\pi_0 X \to \pi_0 Y$ and $\pi_i(X|_U, x) \to \pi_i(Y|_U, p(x))$ are local epis and monics. $\square$

Lemma: Suppose that $f : X \to Y$ is a section-wise weak equivalence in the sense that all $X(U) \to Y(U)$ are weak equivalences of simplicial sets. Then f is a local weak equivalence.

Proof: The map $\pi_0 X \to \pi_0 Y$ is an isomorphism of presheaves and all diagrams

$$\begin{array}{ccc} \pi_i X & \longrightarrow & \pi_i Y \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

are pullbacks of presheaves. Sheafify. $\square$

The Ex^∞ functor

The functor $\mathrm{Ex} : \mathbf{S} \to \mathbf{S}$ is defined by

$$\mathrm{Ex}(X)_n = \hom(\mathrm{sd}\,\Delta^n, X).$$

$\mathrm{sd}\,\Delta^n = BN\Delta^n$, where $N\Delta^n$ is the poset of non-degenerate simplices of Δ^n (subsets of $\{0, 1, \dots, n\}$). Any ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ induces a functor $N\Delta^m \to N\Delta^n$, and hence induces a simplicial set map $\mathrm{sd}\,\Delta^m \to \mathrm{sd}\,\Delta^n$. Precomposition with this map gives the simplicial structure of $\mathrm{Ex}(X)$. There is a last vertex functor $N\Delta^n \to \mathbf{n}$, which is natural in $\mathbf{n}$; the collection of such functors determines a natural simplicial set map

$$\eta : X \to \mathrm{Ex}(X).$$

Iterating gives $\mathrm{Ex}^\infty(X) = \varinjlim \mathrm{Ex}^n(X)$. The salient features of the construction are the following:

1) the map $\eta : X \to \mathrm{Ex}(X)$ is a weak equivalence,

2) the functor $X \mapsto \mathrm{Ex}(X)$ preserves Kan fibrations

3) $\mathrm{Ex}^\infty(X)$ is a Kan complex, and the natural map $j : X \to \mathrm{Ex}^\infty(X)$ is a weak equivalence.

Lemma: Suppose that a simplicial presheaf map $f : X \to Y$ has the local RLP with respect to all $\partial\Delta^n \subset \Delta^n$. Then f is a local fibration and a local weak equivalence.

Proof The local fibration part is trivial: f has the RLP wrt all inclusions of finite simplicial sets.

The induced map $f : \mathrm{Ex}(X) \to \mathrm{Ex}(Y)$ has the RLP wrt all $\partial\Delta^n \subset \Delta^n$, since f has the local RLP wrt all $\mathrm{sd}\,\partial\Delta^n \to \mathrm{sd}\,\Delta^n$. Thus, $f : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$ has the local RLP wrt all $\partial\Delta^n \subset \Delta^n$ and is a map of presheaves of Kan complexes. Use the Lemma above. $\square$

Corollary: The maps $\eta : X \to LX$ and $\eta : X \to L^2X$ are local fibrations and local weak equivalences.

Proof Show that $\eta : X \to LX$ has the local RLP wrt all $\partial\Delta^n \subset \Delta^n$: the map

$$X^{\Delta^n} \to X^{\partial\Delta^n} \times_{LX^{\partial\Delta^n}} LX^{\Delta^n}$$

is a local epi in degree 0 if and only if the map of associated sheaves is a sheaf epi. But the map of associated sheaves is an isomorphism. $\square$

Say that a map $p : X \to Y$ which has the local RLP wrt all $\partial\Delta^n \subset \Delta^n$ is a local trivial fibration

Suppose that $\mathcal{B}$ is a complete Boolean algebra. Here is the meaning of some of these concepts for simplicial sheaves on $\mathcal{B}$:

Lemma: Suppose that $\mathcal{B}$ is a complete Boolean algebra.

1) A map $p : X \to Y$ of simplicial sheaves on $\mathcal{B}$ is a local (resp. local trivial) fibration if and only if all maps $p : X(b) \to Y(b)$ are Kan fibrations (resp. trivial Kan fibrations).
2) A map $f : X \to Y$ of locally fibrant simplicial sheaves on $\mathcal{B}$ is a local weak equivalence if and only if all maps $f : X(b) \to Y(b)$ are weak equivalences of simplicial sets.

Proof An induced map

$$X^{\Delta^n} \to Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}$$

is a sheaf epi in degree 0 if and only if it is a sectionwise epi in degree 0, since Shv($\mathcal{B}$) satisfies the axiom of choice. The local fibration statement is similar.

Suppose that f is a local weak equivalence. The map f has a factorization

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \times_Y Y^{\Delta^1} \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle p} \\ & & Y \end{array}$$

where p is a sectionwise Kan fibration and j is right inverse to a sectionwise trivial Kan fibration (all objects are simplicial sheaves). The map p is a local weak equivalence and a local fibration, and is therefore a sectionwise weak equivalence by the Lemma above. But then f is a sectionwise weak equivalence. □

Example: Suppose that $\mathcal{B} = \mathcal{P}(I)$ for some set I. A presheaf F on $\mathcal{P}(I)$ is a sheaf if and only if the canonical presheaf map

$$F(U) \xrightarrow{\eta} \tilde{F}(U) = \prod_{x \in U} F(\{x\})$$

This map is, in general, the associated sheaf map.

A simplicial presheaf map $f : X \to Y$ on $\mathcal{P}(I)$ is a local fibration (resp. local trivial fibration, local weak equivalence) if and only if all simplicial set maps $f : X(\{x\}) \to Y(\{x\})$, $x \in I$ are Kan fibrations (resp. triv. Kan fibrations, weak equivalences). A local weak equivalence $f : X \to Y$ of simplicial sheaves is not necessarily a sectionwise weak equivalence, unless all $X(\{x\})$ and $Y(\{x\})$ are Kan complexes, ie. X and Y are locally fibrant.

Lemma: Suppose that $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ is a Boolean localization and that $f : X \to Y$ is a map of simplicial sheaves on $\mathcal{C}$. Then f is a local fibration (respectively local trivial fibration) if and only if the induced map $p^*X \to p^*Y$ is a sectionwise Kan fibration (respectively sectionwise trivial Kan fibration) of simplicial sheaves on $\mathcal{B}$.

Proof The simplicial sheaf map

$$X^{\Delta^n} \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$$

is a sheaf epi in degree zero if and only if the induced map

$$p^*X^{\Delta^n} \to p^*X^{\partial\Delta^n} \times_{p^*Y^{\partial\Delta^n}} p^*Y^{\Delta^n}$$

is a sheaf epi in degree 0 (note: $p^*(Y^K) \cong (p^*Y)^K$ if K is a finite simplicial set). Now use the previous Lemma. □

Proposition: Suppose that $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ is a Boolean localization, and that $f : X \to Y$ is a map of simplicial presheaves on $\mathcal{C}$. Then f is a local weak equivalence if and only if $f : p^*L^2X \to$

p^*L^2Y is a local weak equivalence of simplicial sheaves on $\mathcal{B}$.

Proof f is a local weak equivalence if and only the induced map $f_* : L^2 \operatorname{Ex}^\infty X \to L^2 \operatorname{Ex}^\infty Y$ is a weak equivalence of locally fibrant simplicial sheaves.

The map $f_* : \operatorname{Ex}^\infty X \to \operatorname{Ex}^\infty Y$ of presheaves of Kan complexes has a factorization

$$\begin{array}{ccc} \operatorname{Ex}^\infty X & \xrightarrow{j} & Z \\ & \searrow^{f_*} & \downarrow{q} \\ & & \operatorname{Ex}^\infty Y \end{array}$$

where q is a sectionwise Kan fibration and j is a section of a sectionwise trivial Kan fibration $\pi : Z \to \operatorname{Ex}^\infty X$. Then $j_* : L^2 \operatorname{Ex}^\infty X \to L^2 Z$ is a section of a local trivial fibration $\pi_* : L^2 Z \to L^2 \operatorname{Ex}^\infty X$, and the induced map $q_* : L^2 Z \to L^2 \operatorname{Ex}^\infty Y$ is a local fibration between locally fibrant simplicial sheaves. It follows that $f : X \to Y$ is a local weak equivalence if and only if q_* is a local trivial fibration. But this is so if and only if p^*q_* is a sectionwise trivial fibration, by the previous Lemma. Thus, $f : X \to Y$ is a weak equivalence if and only if the induced map $f_* : p^*L^2 \operatorname{Ex}^\infty X \to p^*L^2 \operatorname{Ex}^\infty Y$ is a sectionwise weak equivalence of simplicial sheaves on $\mathcal{B}$.

Finally, by exactness of p^* and L^2, there is a natural isomorphism

$$p^* L^2 \operatorname{Ex}^\infty X \cong L^2 \operatorname{Ex}^\infty p^* L^2 X.$$

Thus $f : X \to Y$ is a local weak equivalence of simplicial presheaves on $\mathcal{C}$ if and only if $f_* : p^* L^2 X \to p^* L^2 Y$ is a local weak equivalence of simplicial sheaves on $\mathcal{B}$. $\square$

Corollary: (of proof) Suppose that $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ is a Boolean localization. Then a simplicial presheaf map $f : X \to Y$ is a local weak equivalence if and only if the induced map $f_* : p^* L^2 \operatorname{Ex}^\infty X \to p^* L^2 \operatorname{Ex}^\infty Y$ is a sectionwise weak equivalence of simplicial sheaves on $\mathcal{B}$.

Now for some applications:

Lemma: Suppose given a commutative diagram of simplicial presheaf maps

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{h} & \downarrow{g} \\ & & Z \end{array}$$

on a Grothendieck site $\mathcal{C}$. If any two of f, g or h are local weak equivalences then so is the third.

Proof Apply $p^* L^2 \operatorname{Ex}^\infty$. $\square$

Say that a simplicial presheaf map $i : A \to B$ is a cofibration if it is a monomorphism in all sections and in all simplicial degrees.

Lemma: Suppose given a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ B & \longrightarrow & D \end{array}$$

in the category $s\,\mathrm{Shv}(\mathcal{B})$ such that i is a cofibration and a local equivalence. Then i_* is a cofibration and a local equivalence.

Proof Form the diagram of simplicial presheaf maps

$$\begin{array}{ccc} \operatorname{Ex}^\infty A & \longrightarrow & \operatorname{Ex}^\infty C \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ \operatorname{Ex}^\infty B & \longrightarrow & E \end{array}$$

where i_* is a cofibration. Then the induced map $D \to E$ is a sectionwise weak equivalence. Sheafifying gives a pushout diagram of simplicial sheaves

$$\begin{array}{ccc} L^2 \operatorname{Ex}^\infty A & \longrightarrow & L^2 \operatorname{Ex}^\infty C \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ L^2 \operatorname{Ex}^\infty B & \longrightarrow & L^2 E \end{array}$$

which is locally equivalent to the original. We can therefore assume that the simplicial sheaves A, B and C are locally fibrant.

The map $i : A \to B$ is a local weak equivalence of locally fibrant simplicial sheaves on $\mathcal{B}$ and is therefore a sectionwise weak equivalence. Sectiowise trivial cofibrations are closed under pushout in the simplicial presheaf category, and since $D = L^2(B \cup_A C)$ is the associated sheaf of the presheaf pushout, the map $C \to D$ must then be a local weak equivalence. $\square$

Lemma: Suppose given a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ B & \longrightarrow & D \end{array}$$

of simplicial presheaves on a Grothendieck site $\mathcal{C}$, and suppose that i is a cofibration and a local weak equivalence. Then i_* is a local weak equivalence.

Proof Suppose that $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ is a Boolean localization. The functor p^*L^2 preserves cofibrations and pushouts, and preserves and reflects local weak equivalences. The map $i_* : p^*L^2A \to p^*L^2B$ is a local weak equivalence and a cofibration, so the map $p^*L^2C \to p^*L^2D$ induced by i_* is a local weak equivalence. But then $i_* : C \to D$ must be a local weak equivalence. $\square$

Lemma: Suppose that $p : X \to Y$ is a map of $s\,\mathrm{Shv}(\mathcal{B})$ such that p is a sectionwise Kan fibration and is a local weak equivalence. Then p is a sectionwise trivial fibration.

Proof The functor $X \mapsto L^2 \operatorname{Ex}^\infty X$ preserves sectionwise Kan fibrations and preserves pullbacks. Also the sectionwise fibration $p : X \to Y$ is local weak equivalence if and only if the induced map $p_* : L^2 \operatorname{Ex}^\infty X \to L^2 \operatorname{Ex}^\infty Y$ is a sectionwise weak equivalence. It follows that the family of all maps which are simultaneously sectionwise Kan fibrations and local weak equivalences is closed under base change.

Suppose given a diagram

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha} & X(b) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[\beta]{} & Y(b) \end{array}$$

The simplex Δ^n contracts onto the vertex 0; write $h : \Delta^n \times \Delta^1 \to \Delta^n$ for the contracting homotopy. Let $h' : \partial\Delta^n \times \Delta^1 \to X(b)$ be a choice of lifting

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha} & X(b) \\ \downarrow & \overset{h'}{\nearrow} & \downarrow{\scriptstyle p} \\ \partial\Delta^n \times \Delta^1 & \xrightarrow[\beta\cdot h\cdot(i\times 1)]{} & Y(b) \end{array}$$

Then the original diagram is homotopic to a diagram of the form

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha'} & X(b) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[x]{} & Y(b) \end{array}$$

where $x : \Delta^n \to X(b)$ factors through a vertex $x \in Y(b)$. Consider the induced diagram of sheaf maps

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & (L_b\Delta^0 \times_Y X)(b) \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p_*} \\ \Delta^n & \longrightarrow & L_b\Delta^0(b) \end{array}$$

Then $L_b\Delta^0$ is a diagram of points as a simplicial presheaf and hence is locally fibrant; the associated sheaf is therefore a diagram of Kan complexes. The pulled back map $p_* : L_b\Delta^0 \times_Y X \to L_b\Delta^0$ is a local fibration and a local weak equivalence between sheaves of Kan complexes and is therefore a sectionwise trivial fibration, so the indicated lift exists. □

Corollary: Suppose that $q : X \to Y$ is a local weak equivalence and a local fibration in $s\,\mathrm{Pre}(\mathcal{C})$. Then q has the local RLP wrt all $\partial\Delta^n \subset \Delta^n, n \geq 0$.

Proof Suppose that $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ is a Boolean localization. p^*L^2q is a local weak equivalence and a local fibration, and is therefore a sectionwise trivial fibration. The functor p^*L^2 reflects local epimorphisms, so that the map

$$X^{\Delta^n} \to Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}$$

is a local epi in degree 0.