Lecture 010 (March 3, 2005)

"Injective" model structure

I want to review the classical fibration replacement construction from simplicial homotopy theory, because it is so important.

1) Suppose that $f : X \to Y$ is a map of Kan complexes, and form the diagram

$$\begin{array}{ccccc} X \times_Y Y^I & \xrightarrow{f_*} & Y^I & \xrightarrow{d_1} & Y \\ {\scriptstyle d_{0*}} \downarrow & \nearrow^{sf} & \downarrow {\scriptstyle d_0} & & \\ X & \xrightarrow[f]{} & Y & & \end{array}$$

Then d_0 is a trivial fibration since Y is a Kan complex, so d_{0*} is a trivial fibration. The section s of d_0 (and d_1) induces a section s_* of d_{0*}. Then

$$(d_1 f_*)s_* = d_1(sf) = f$$

Finally, there is a pullback diagram

$$\begin{array}{ccc} X \times_Y Y^I & \xrightarrow{f_*} & Y^I \\ {\scriptstyle (d_{0*}, d_1 f_*)} \downarrow & & \downarrow {\scriptstyle (d_0, d_1)} \\ X \times Y & \xrightarrow[f \times 1]{} & Y \times Y \end{array}$$

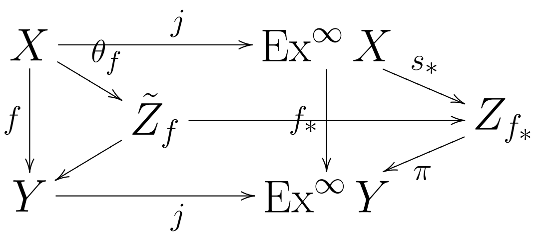
and the map $pr_R : X \times Y \to Y$ is a fibration since X is fibrant, so that $pr_R(d_{0*}, d_1 f_*) = d_1 f_*$ is a fibration.

Write $Z_f = X \times_Y Y^I$ and $\pi_f = d_1 f_*$. Then we have functorial replacement

$$\begin{array}{ccccc} X & \xrightarrow{s_*} & Z_f & \xrightarrow{d_{0*}} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} & & \\ & & Y & & \end{array}$$

of f by a fibration π, where d_{0*} is a trivial fibration such that $d_{0*} s_* = 1$.

2) Suppose that $f : X \to Y$ is a simplicial set map, and form the diagram



where the diagram

$$\begin{array}{ccc} \tilde{Z}_f & \longrightarrow & Z_{f_*} \\ {\scriptstyle \tilde{\pi}_f}\downarrow & & \downarrow{\scriptstyle \pi_{f_*}} \\ Y & \xrightarrow[j]{} & \mathrm{Ex}^\infty Y \end{array}$$

is a pullback. Then $\tilde{\pi}_f$ is a fibration, and θ_f is a weak equivalence. Furthermore, the construction taking a map f to the factorization

$$\begin{array}{ccc} X & \xrightarrow{\theta_f} & \tilde{Z}_f \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi_f} \\ & & Y \end{array}$$

has the following properties:

a) it is natural in f

b) it preserves filtered colimits in f

c) if X and Y are α-bounded where α is some infinite cardinal, then so is $\tilde{Z}_f$

This construction therefore carries over to simplicial presheaves, giving a natural factorization

$$\begin{array}{ccc} X & \xrightarrow{\theta_f} & \tilde{Z}_f \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi_f} \\ & & Y \end{array}$$

of a simplicial presheaf map $f : X \to Y$ such that θ_f is a sectionwise weak equivalence and π_f is a sectionwise fibration. Here are some further properties of this factorization:

a) it preserves filtered colimits in f

b) if X and Y are α-bounded where α is some infinite cardinal, then so is $\tilde{Z}_f$

c) f is a local weak equivalence if and only if π_f has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$.

To fix notation, suppose that $\mathcal{C}$ is a small Grothendieck site.

Suppose that α is an infinite cardinal such that $\alpha > |\operatorname{Mor}(\mathcal{C})|$. Choose another infinite cardinal $\lambda > 2^{\alpha}$.

Lemma: Suppose that $i : X \to Y$ is a cofibration and a local weak equivalence of $s\operatorname{Pre}(\mathcal{C})$. Suppose further that $A \subset Y$ is an α-bounded subobject of Y. Then there is an α-bounded subobject of Y such that $A \subset B$ and the map $B \cap X \to B$ is a local weak equivalence.

Proof Write $\pi_B : Z_B \to B$ for the natural pointwise Kan fibration replacement for the cofibration $B \cap X \to B$. The map $\pi_Y : Z_Y \to Y$ has the local RLP wrt all $\partial\Delta^n \subset \Delta^n$.

Supppose given a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Z_A(U) \\ \downarrow & \nearrow & \downarrow \pi_A \\ \Delta^n & \longrightarrow & A(U) \end{array}$$

where A is α-bounded. The lifting problem can be solved locally over Y along some covering sieve for U having at most α elements. $Z_Y = \varinjlim_{|B|<\alpha} Z_B$ since Y is a filtered colimit of its α-bounded subobjects. It follows that there is an α-bounded subobject $A' \subset Y$ with $A \subset A'$ such that the original lifting problem can be solved over A'. The list of all such lifting problems is α-bounded, so there is an α-bounded subobject $B_1 \subset Y$ with $A \subset B_1$ so that all lifting problems as above over A can be solved locally over B_1. Repeat this procedure countably many times to produce an ascending family

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of α-bounded subobjects of Y such that all lifting local lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Z_{B_i}(U) \\ \downarrow & \nearrow & \downarrow \pi_{B_i} \\ \Delta^n & \longrightarrow & B_i(U) \end{array}$$

over B_i can be solved over B_{i+1}. Set $B = \cup_i B_i$ $\square$

Say that a map $p : X \to Y$ of $s\,\mathrm{Pre}(\mathcal{C})$ is a **global fibration** if p has the right lifting property with respect to all maps $A \to B$ which are cofibrations and local weak equivalences.

Corollary: $p : X \to Y$ is a global fibration if and only if it has the RLP wrt all cofibrations $A \to B$ which are α-bounded and local weak equivalence.

Lemma: Suppose that $q : Z \to W$ has the RLP wrt all cofibrations. Then q is a global fibration and a local weak equivalence.

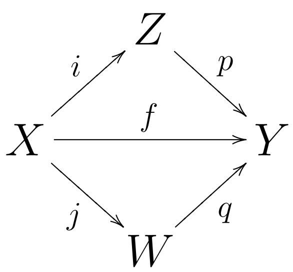
Proof q is obviously a global fibration. q has the RLP wrt all $L_U \partial \Delta^n \to L_U \Delta^n$, so that all maps $q : Z(U) \to W(U)$ are trivial Kan fibrations. But then q is a local weak equivalence. $\square$

Exercise: Alternatively, show that a map which has the RLP wrt all cofibrations is a simplicial homotopy equivalence.

Lemma: $q : Z \to W$ has the RLP wrt all cofibrations if and only if it has the RLP wrt all α-bounded cofibrations.

Proof exercise.

Lemma: Any simplicial presheaf map $f : X \to Y$ has factorizations



where

1) i is a cofibration and a local weak equivalence, and p is a global fibration,

2) j is a cofibration and p has the RLP wrt all cofibrations (and is therefore a global fibration and a local weak equivalence)

Proof For the first factorization, choose a cardinal $\lambda > 2^\alpha$ and do a transfinite small object argument of size λ to solve all lifting problems

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

arising from locally trivial cofibrations i which are α-bounded. We need to know that locally trivial cofibrations are closed under pushout, but we proved that before with a Boolean localization argument. The small object argument stops on account of the condition on the size of the cardinal λ.

The second factorization is similar. $\square$

Theorem: Suppose that $\mathcal{C}$ is a small Grothendieck site. The category $s\,\mathrm{Pre}(\mathcal{C})$ with local weak equivalences, cofibrations and global fibrations, satisfies the axioms for a proper closed simplicial model category.

Proof $s\,\mathrm{Pre}(\mathcal{C})$ has all small limits and colimits, giving **CM1**. The weak equivalence axiom **CM2**

was proved with a Boolean localization argument. The weak equivalence part of the retract axiom **CM3** can also be proved with Boolean localization — the fibration and cofibration parts are trivial. The factorization axiom **CM5** has just been proved.

Suppose that $\pi : X \to Y$ is a global fibration and a local weak equivalence. Then π has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & \searrow^{\pi} & \downarrow p \\ & & Y \end{array}$$

where p has the RLP wrt all cofibrations and is therefore a local weak equivalence. Then j is a local weak equivalence, and so π is a retract of p (exercise). Thus π has the RLP wrt all cofibrations, giving **CM4**.

The simplicial model structure comes from the function complex

$$\mathbf{hom}(X, Y)_n = \hom_{s\,\mathrm{Pre}(\mathcal{C})}(X \times \Delta^n, Y).$$

Properness is proved with a Boolean localization argument (exercise). $\square$

Simplicial sheaves

Write $s\,\mathrm{Shv}(\mathcal{C})$ for the category of simplicial sheaves on $\mathcal{C}$. Say that a map $f : X \to Y$ is a **local weak equivalence** of simplicial sheaves if it is a local weak equivalence of simplicial presheaves. A **cofibration** of simplicial sheaves is a monomorphism, and a **global fibration** is a map which has the RLP wrt all trivial cofibrations.

Theorem: Suppose that $\mathcal{C}$ is a small Grothendieck site.

1) (Joyal) The category $s\,\mathrm{Shv}(\mathcal{C})$ with local weak equivalences, cofibrations and global fibrations, satisfies the axioms for a proper closed simplicial model category.

2) The geometric morphism $\mathrm{Shv}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C})$ defined by the inclusion i of sheaves in presheaves and the associated sheaf functor L^2 induces a Quillen equivalence of homotopy categories

$$i : \mathrm{Ho}(s\,\mathrm{Shv}(\mathcal{C})) \simeq \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})) : L^2.$$

Proof L^2 preserves and reflects local weak equivalences. The inclusion functor i preserves global fibrations and L^2 preserves cofibrations. The associated sheaf map $\eta : X \to L^2X$ is a local weak equivalence, while the counit of the adjunction is an iso. Thus, we get 2) if we can prove 1).

CM1 follows from completeness and cocompleteness for $s\,\mathrm{Shv}(\mathcal{C})$. **CM2** follows from the corr. statement for simplicial presheaves. **CM3** is trivial. **CM4** follows from the corr. statement for simplicial presheaves.

A map $p : X \to Y$ is a global fibration (resp. trivial global fibration) of $s\,\mathrm{Shv}(\mathcal{C})$ if and only if it is a global fibration (resp. trivial global fibration) of $s\,\mathrm{Pre}(\mathcal{C})$ (exercise).

Thus, p is a global fibration if and only if it has the RLP wrt all inclusions $A \subset B$ of α-bounded subobjects of $s\,\mathrm{Shv}(\mathcal{C})$ (α is a cardinal bigger than $|\,\mathrm{Mor}(\mathcal{C})|$), and it is a trivial global fibration iff it has the RLP wrt all inclusions $Y \subset L^2 L_U \Delta^n$ of subobjects in $s\,\mathrm{Shv}(\mathcal{C})$.

The factorization axiom **CM5** is then proved by transfinite small object arguments of size λ where $\lambda > 2^\alpha$ (there is an alternative argument — see "Simplicial presheaves").

The simplicial model structure (aka. function complexes) is inherited from simplicial presheaves, as is properness. $\square$

Example: Suppose that A is a sheaf, and let $K(A,0)$ be the constant simplicial object associated to A.There is a bijection

$$\hom(X, K(A,0)) \cong \hom(\tilde{\pi}_0(X), A)$$

It follows that $K(A,0)$ is globally fibrant.

Suppose that X is a simplicial presheaf such that all higher local homotopy groups vanish in the sense that $\tilde{\pi}_n(X) \to \tilde{X}_0$ is an isomorphism for $n \geq 1$. Then the map $X \to K(\pi_0(X),0)$ is a local weak equivalence. It follows that the composite

$$X \to K(\pi_0(X),0) \to K(\tilde{\pi}_0(X),0)$$

is a local weak equivalence, and therefore gives a globally fibrant model for X. Note that all higher homotopy groups

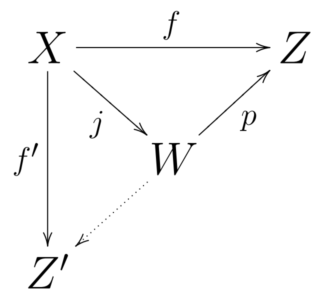
$$\pi_n(K(\tilde{\pi}_0(X),0)(U),x),\ n \geq 1,$$

vanish in all sections.

This observation is a special case of (and starting point for) a result which asserts that if X is a simplicial presheaf such that $\tilde{\pi}_n(X) \to \tilde{X}_0$ is an isomorphism for $n \geq k$, then any globally fibrant model $X \to Y$ has the same property in all sections: $\pi_n(Y(U),x) = 0$ for $n \geq k$, for all $x \in Y(U)$ and all $U \in \mathcal{C}$. (GECT: Prop. 6.11, p.201). This result is particular to simplicial presheaves — it does not hold in motivic homotopy theory, where every motivic homotopy type is representable by a presheaf ("Motivic symmetric spectra", appendix).

Definition: A **globally fibrant model** for a simplicial presheaf X is a local weak equivalence $f : X \to Z$ such that Z is globally fibrant.

Any two globally fibrant models for a simplicial presheaf X are equivalent in a rather strong sense: given models $f : X \to Z$ and $f' : X \to Z'$, f has a factorization $f = p \cdot j$ where p is a global fibration and j is a cofibration and both are local weak equivalences, and there is a commutative diagram



where the dotted arrow exists since j is a trivial cofibration and Z' is globally fibrant. Note that all morphisms in the picture are local weak equivalences, and we have the following:

Lemma: Suppose that $f : Z \to W$ is a weak equivalence of globally fibrant objects. Then all maps $f : Z(U) \to W(U)$ are weak equivalences of simplicial sets.

Proof The map f is a simplicial homotopy equivalence since Z and W are cofibrant and globally fibrant. In other words, there is a map $g : W \to Z$ and homotopies $Z \times \Delta^1 \to Z$ from gf to 1_Z and $W \times \Delta^1 \to W$ from fg to 1_W. The map g restricts to $g : W(U) \to Z(U)$ in each section, and the homotopies restrict to simplicial set maps $Z(U) \times \Delta^1 \to Z(U)$ and $W(U) \times \Delta^1 \to W(U)$. In particular $f : Z(U) \to W(U)$ is a simplicial homotopy equivalence with homotopy inverse $g : W(U) \to Z(U)$, for each $U \in \mathcal{C}$. $\square$

Corollary: Any two globally fibrant models for a simplicial presheaf X are sectionwise homotopy equivalent.

Example: As a category, $s\,\mathrm{Pre}(\mathcal{C})$ is just contravariant simplicial set-valued functors on $\mathcal{C}$ with natural transformations for morphisms. It is also the category of simplicial sheaves for the Grothendieck topology on $\mathcal{C}$ whose covering sieves are the representable functors $\hom(\ ,U)$, $U \in \mathcal{C}$. The theorems of this section, for simplicial presheaves or simplicial sheaves, specialize to the inductive model structure for diagrams of simplicial sets given in Lecture 002.

Other model structures

Recall that there is also a projective model structure on $s\,\mathrm{Pre}(\mathcal{C})$, for which the fibrations are sectionwise Kan fibrations and the weak equivalences are sectionwise weak equivalences. The cofibrations for this theory are the projective cofibrations, and this class of maps has a generating set S_0 consisting of all maps $L_U(\partial\Delta^n) \to L_U(\Delta^n)$. Write $\mathbf{C}_P$ for the class of projective cofibrations, and write $\mathbf{C}$ for the full class of cofibrations. Obviously $\mathbf{C}_P \subset \mathbf{C}$.

Let S be any **set** of cofibrations which contains S_0. Let $\mathbf{C}_S$ be the "saturation" of all cofibrations of the form

$$(B \times \partial\Delta^n) \cup_{(A\times\partial\Delta^n)} (A \times \Delta^n) \subset B \times \Delta^n$$

which are induced by members $A \to B$ of the set S. "Saturation" means the smallest class of cofibration containing the list above which is contains all isomorphisms, and is closed under pushout and all transfinite compositions. $\mathbf{C}_S$ is called the class of S**-cofibrations**.

An S**-fibration** is a map which has the RLP wrt all S-cofibrations which are local weak equivalences.

Theorem: The category $s\,\mathrm{Pre}(\mathcal{C})$ and the classes of S-cofibrations, local weak equivalences, and S-fibrations, satisfies the axioms for a proper closed simplicial model category.

Proof CM1 – **CM3** are trivialities.

Any $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {}_{f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where $j \in \mathbf{C}_S$ and p has the RLP with respect to all members of $\mathbf{C}_S$. Then p is an S-fibration and is a sectionwise hence local weak equivalence.

The map f also has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & {}_{f}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

where q is a global fibration and i is a cofibration and local weak equivalence. Then q is an S-fibration. Factorize i as $i = p \cdot j$ where $j \in \mathbf{C}_S$ and p is an S-fibration and a local weak equivalence (as above). Then j is a local weak equivalence, so $f = (qp) \cdot j$ factorizes f as an S-fibration following a map which is an S-cofibration and a local weak equivalence.

Exercise: prove **CM4**.

The simplicial model structure is the usual one.

Exercise: prove that the structure is proper — use Boolean localization. □

Remarks:

1) The case $S = S_0$ gives the local projective structure of Blander.

2) The model structure of the Theorem is cofibrantly generated. This was originally proved by Beke, but see also "Intermediate model structures for simplicial presheaves".