Lecture 011 (March 31, 2005)

Cocycles

Let $\mathcal{M}$ be any of the model structure on $s\,\mathrm{Pre}(\mathcal{C})$ of the last section, where the weak equivalences are always local weak equivalences. All that we require for the following is that

1) $\mathcal{M}$ is right proper in the sense that weak equivalences pull back to weak equivalences along fibrations, and

2) the class of weak equivalences is closed under products: if $f : X \to Y$ is a weak equivalence, so is any map $f \times 1 : X \times Z \to Y \times Z$

We've already seen 1), and 2) holds by a Boolean localization argument. The following results hold for any model category $\mathcal{M}$ which satisfies 1) and 2).

Suppose that X, Y are simplicial presheaves.

Write $H(X,Y)$ for the category whose objects are all pairs of maps (f,g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism $\alpha : (f,g) \to (f',g')$ of $H(X,Y)$ is a commutative diagram

$$\begin{array}{ccccc} & \overset{f}{\swarrow} & Z & \overset{g}{\searrow} & \\ X & & \downarrow \alpha & & Y \\ & \underset{f'}{\nwarrow} & Z' & \underset{g'}{\nearrow} & \end{array}$$

We shall sometimes refer to $H(X,Y)$ as the **category of cocycles** from X to Y.

Write $\pi_0 H(X,Y)$ for the class of path components of $H(X,Y)$. There is a function

$$\phi : \pi_0 H(X,Y) \to [X,Y]$$

defined by $(f,g) \mapsto g \cdot f^{-1}$.

Lemma: Suppose that $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences. Then the function

$$(\alpha,\beta)_* : \pi_0 H(X,Y) \to \pi_0 H(X',Y')$$

is a bijection.

Proof An object (f, g) of $H(X', Y')$ is a map $(f, g) : Z \to X' \times Y'$ such that f is a weak equivalence. There is a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & W \\ & \searrow^{(f,g)} & \downarrow{\scriptstyle (p_{X'},p_{Y'})} \\ & & X' \times Y' \end{array}$$

such that j is a trivial cofibration and $(p_{X'}, p_{Y'})$ is a fibration. The map $p_{X'}$ is a weak equivalence. Form the pullback

$$\begin{array}{ccc} W_* & \xrightarrow{(\alpha\times\beta)_*} & W \\ {\scriptstyle (p_X^*,p_Y^*)}\downarrow & & \downarrow{\scriptstyle (p_{X'},p_{Y'})} \\ X \times Y & \xrightarrow[\alpha\times\beta]{} & X' \times Y' \end{array}$$

Then the map (p_X^*, p_Y^*) is a fibration and $(\alpha \times \beta)_*$ is a local weak equivalence (since $\alpha \times \beta$ is a weak equivalence, and by right properness). The map p_X^* is also a weak equivalence.

The assignment $(f, g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X', Y') \to \pi_0 H(X, Y)$$

which is inverse to $(\alpha, \beta)_*$. $\square$

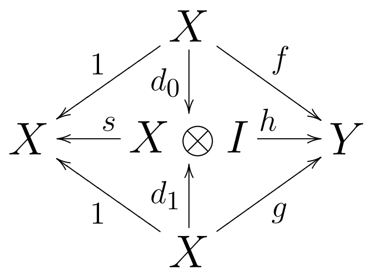
Lemma: Suppose that Y is fibrant and X is cofibrant. Then the canonical map

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

is a bijection.

Proof The function $\pi(X, Y) \to [X, Y]$ relating naive homotopy classes to morphisms in the homotopy category is a bijection since X is cofibrant and Y is fibrant.

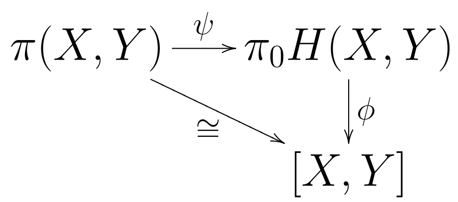
If $f, g : X \to Y$ are homotopic, there is a diagram



where h is the homotopy. Thus, sending $f : X \to Y$ to the class of $(1_X, f)$ defines a function

$$\psi : \pi(X, Y) \to \pi_0 H(X, Y)$$

and there is a diagram



It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.

The weak equivalence f has a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & V \\ & \searrow^{f} & \downarrow p \\ & & X \end{array}$$

where j is a trivial cofibration and p is a trivial fibration. The object Y is fibrant, so the dotted arrow θ exists in the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & \downarrow j & \searrow^{g} & \\ X & & & & Y \\ & \nwarrow_{p} & V & \nearrow_{\theta} & \end{array}$$

Since X is cofibrant, the trivial fibration p has a section σ, and so there is a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{1} & \downarrow \sigma & \searrow^{\theta\sigma} & \\ X & & & & Y \\ & \nwarrow_{p} & V & \nearrow_{\theta} & \end{array}$$

Then the composite $\theta\sigma$ is the required map k. $\square$

Theorem: The canonical map $\phi : \pi_0 H(X,Y) \to [X,Y]$ is a bijection for all X and Y.

Proof There are weak equivalences $\pi : X' \to X$ and $j : Y \to Y'$ such that X' and Y' are cofibrant and fibrant, respectively, and there is a commutative diagram

$$\begin{array}{ccc} \pi_0 H(X,Y) & \xrightarrow{\phi} & [X,Y] \\ {\scriptstyle (1,j)_*}\downarrow\cong & & \cong\downarrow{\scriptstyle j_*} \\ \pi_0 H(X,Y') & \xrightarrow{\phi} & [X,Y'] \\ {\scriptstyle (\pi,1)_*}\uparrow\cong & & \cong\downarrow{\scriptstyle \pi^*} \\ \pi_0 H(X',Y') & \xrightarrow[\phi]{\cong} & [X',Y'] \end{array}$$

The functions $(1,j)_*$ and $(\pi,1)_*$ are bijections by the first Lemma, and the bottom map ϕ is a bijection by the second Lemma. $\square$

Sheaf cohomology

Lemma: (van Osdol) Suppose that $f : X \to Y$ is a local weak equivalence of simplicial presheaves. Then the induced map $f_* : \mathbb{Z}X \to \mathbb{Z}Y$ of simplicial abelian presheaves is also a local weak equivalence.

Proof It's enough to show that if $f : X \to Y$ is a local equivalence of locally fibrant simplicial sheaves, then $f_* : \tilde{\mathbb{Z}}X \to \tilde{\mathbb{Z}}Y$ is a local equivalence of simplicial abelian sheaves.

It's also enough to assume that the map $f : X \to Y$ is a morphism of sheaves on a complete Boolean algebra $\mathcal{B}$, since the inverse image functor p^* for a Boolean localization $p : \mathbf{Shv}(\mathcal{B}) \to \mathbf{Shv}(\mathcal{C})$ commutes with the free abelian sheaf functor (p_* preserves abelian group structures).

But then $f : X \to Y$ is a sectionwise weak equivalence, so $f_* : \mathbb{Z}X \to \mathbb{Z}Y$ is a sectionwise weak equivalence of associated free abelian presheaves, so that $f_* : \tilde{\mathbb{Z}}X \to \tilde{\mathbb{Z}}Y$ is a local weak equivalence. $\square$

Remark: Of course, $f_* : \tilde{\mathbb{Z}}X \to \tilde{\mathbb{Z}}Y$ is a sectionwise equivalence of simplicial sheaves on $\mathcal{B}$, since $\tilde{\mathbb{Z}}X$ and $\tilde{\mathbb{Z}}Y$ are locally fibrant.

Remark: Once upon a time, the Lemma above was called the Illusie conjecture.

Suppose that A is a simplicial abelian group. Then A is a Kan complex, and we know that there is a natural isomorphism

$$\pi_n(A, 0) \cong H_n(NA).$$

There is a canonical isomorphism

$$\pi_n(A, 0) \xrightarrow{\cong} \pi_n(A, a)$$

which is defined for any $a \in A_0$ by $[\alpha] \mapsto [\alpha + a]$ where we have written a for the composite

$$\Delta^n \to \Delta^0 \xrightarrow{a} A$$

The collection of these isomorphisms, taken together, define isomorphisms

$$\begin{array}{ccc} \pi_n(A,0) \times A_0 & \xrightarrow{\cong} & \pi_n A \\ & {}_{pr}\searrow \quad \swarrow & \\ & A_0 & \end{array}$$

of abelian groups fibred over A_0, and these isomorphisms are natural in simplicial abelian group homomorphisms.

Lemma: A map $A \to B$ of presheaves of simplicial abelian groups is a local weak equivalence if and only if the presheaf of chain complex maps $NA \to NB$ induces an isomorphism in all homology sheaves.

Proof If $NA \to NB$ induces an isomorphism in all homology sheaves, then the map $\tilde{\pi}_0(A) \to \tilde{\pi}_0(B)$ and all maps $\tilde{\pi}_n(A,0) \to \tilde{\pi}_n(B,0)$ are isomorphisms of sheaves. The diagram of sheaves associated to

$$\begin{array}{ccc} \pi_n(A,0) \times A_0 & \longrightarrow & \pi_n(B,0) \times B_0 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & B_0 \end{array}$$

coincides with the diagram of sheaves associated to the picture

$$\begin{array}{ccc} \tilde{\pi}_n(A,0) \times A_0 & \longrightarrow & \tilde{\pi}_n(B,0) \times B_0 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & B_0 \end{array}$$

which is a pullback. □

Suppose that A is a sheaf of abelian groups, and let $A \to J$ be an injective resolution of A, thought of as a $\mathbb{Z}$-graded chain complex, concentrated in negative degrees. Write $A[-n]$ for the chain complex consisting of A concentrated in degree n, and consider the chain map $A[-n] \to J[-n]$. Recall that $K(A,n) = \Gamma A[-n]$ defines the Eilenberg-Mac Lane sheaf associated to A. Let $K(J,n) = \Gamma T(J[-n])$ where $T(J[-n])$ is the good truncation of $J[-n]$ in non-negative degrees.

Lemma: Every local weak equivalence $f : X \to Y$ induces an isomorphism

$$\pi_{ch}(N\tilde{\mathbb{Z}}Y, J[-n]) \xrightarrow{\cong} \pi_{ch}(N\tilde{\mathbb{Z}}X, J[-n])$$

in chain homotopy classes for all $n \geq 0$.

Proof The map f induces a homology sheaf isomorphism $N\tilde{\mathbb{Z}}X \to N\tilde{\mathbb{Z}}Y$, and then a comparison of spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(X), A) \Rightarrow \pi_{ch}(N\tilde{\mathbb{Z}}X, J[-p-q])$$

gives the desired result. □

If two chain maps $f, g : N\tilde{\mathbb{Z}}X \to J[-n]$ are chain homotopic, then there is a right homotopy

$$\begin{array}{ccc} & & Z \\ & \nearrow & \downarrow p \\ X & \xrightarrow[(f_*, g_*)]{} & K(J,n) \times K(J,n) \end{array}$$

for some path object Z over $K(J,n)$ in the projective model structure for $\mathcal{C}^{op}$-diagrams. Choose a sectionwise trivial fibration $\pi : W \to X$ such that W is projective cofibrant. Then $f_*\pi$ is left homotopic to $g_*\pi$ for some choice of cylinder object $W \otimes I$ for W, again in the projective structure. This means that there is a diagram

$$\begin{array}{ccccc} & & W & \xrightarrow{\pi} & X & & \\ & \swarrow^{1} & \downarrow i_0 & & & \searrow^{f_*} & \\ W & \xleftarrow{s} & W \otimes I & & \xrightarrow{h} & & K(J,n) \\ & \nwarrow_{1} & \uparrow i_1 & & & \nearrow_{g_*} & \\ & & W & \xrightarrow[\pi]{} & X & & \end{array}$$

where the maps s, i_0, i_1 are all part of the cylinder object structure for $W \otimes I$, and are sectionwise weak equivalences. It follows that

$$(1, f_*) \sim (\pi, f_*\pi) \sim (\pi s, h) \sim (\pi, g_*\pi) \sim (1, g_*)$$

in $\pi_0 H(X, K(J,n))$. It follows that there is a well defined abelian group homomorphism

$$\phi : \pi_{ch}(N\tilde{\mathbb{Z}}X, J[-n]) \to \pi_0 H(X, K(J,n)).$$

This map is natural in X.

Lemma: ϕ is an isomorphism.

Proof Suppose that $X \xleftarrow{f} Z \xrightarrow{g} K(J,n)$ is an object of $H(X, K(J,n))$. Then there is a unique chain homotopy class $[v] : N\tilde{\mathbb{Z}}X \to J[-n]$ such that $[v_* f] = [g]$ since f is a local weak equivalence. This chain homotopy class $[v]$ is also independent of choice of representative for the component of (f,g). We therefore have a well defined function

$$\psi : \pi_0 H(X, K(J,n)) \to \pi_{ch}(N\tilde{\mathbb{Z}}X, J[-n]).$$

Then the composites $\psi \cdot \phi$ and $\phi \cdot \psi$ are identity morphisms. □

Corollary: Suppose that A is a sheaf of abelian groups on $\mathcal{C}$, and let $A \to J$ be an injective resolution of A in the category of abelian sheaves. Let X be a simplicial presheaf on $\mathcal{C}$. Then there is an isomorphism

$$\pi_{ch}(N\tilde{\mathbb{Z}}X, J[-n]) \cong [X, K(A,n)].$$

This isomorphism is natural in X.

Suppose that A is an abelian (pre)sheaf on $\mathcal{C}$ and that X is a simplicial presheaf. Write

$$H^n(X, A) = [X, K(A, n)],$$

and say that this group is the n^{th} **cohomology group** of X with coeffients in A.

Remarks:

1) The associated sheaf map

$$K(A, n) \to K(\tilde{A}, n)$$

is a local weak equivalence, so that

$$H^n(X, A) \cong H^n(X, \tilde{A}).$$

2) One can (and does) define sheaf cohomology $H^n(\mathcal{C}, B)$ for an abelian sheaf B by

$$H^n(\mathcal{C}, B) = H_{-n}(\Gamma_* J)$$

where $B \to J$ is an injective resolution of B concentrated in negative degrees and Γ_* is global sections (ie. inverse limit). But $\Gamma_* Y = \hom(*, Y)$ for any Y, and so

$$H^n(\mathcal{C}, B) \cong \pi_{ch}(\tilde{\mathbb{Z}}*, J[-n]) \cong [*, K(B, n)].$$

3) There is a **universal coefficients spectral sequence**

$$E_2^{p,q} = \operatorname{Ext}^q(\tilde{H}_p(X), \tilde{A}) \Rightarrow H^{p+q}(X, A)$$

4) Suppose that

$$X \xleftarrow{\simeq} X' \to K(A,n), \quad Y \xleftarrow{\simeq} Y' \to K(B,m)$$

are cocycles. Then the adjoint simplicial abelian presheaf maps

$$\mathbb{Z}X' \to K(A,n), \quad \mathbb{Z}Y' \to K(B,n)$$

have a (simplicial abelian group) tensor product

$$\mathbb{Z}(X' \times Y') \cong \mathbb{Z}X' \otimes \mathbb{Z}Y' \to K(A,n) \otimes K(B,n)$$

and there is a natural weak equivalence

$$K(A,n) \otimes K(B,m) \simeq K(A \otimes B, n+m).$$

in simplicial abelian groups, hence in simplicial abelian presheaves (Exercise: suppose first that $A = B = \mathbb{Z}$). The composite

$$X \times Y \xleftarrow{\simeq} X' \times Y' \to K(A \otimes B, n+m)$$

represents the external cup product of the classes represented by the two cocycles. We have defined an **external cup product**

$$H^n(X,A) \times H^m(Y,B) \to H^{n+m}(X \times Y, A \otimes B).$$

If A happens to be a presheaf of rings this construction specializes to the cup product pairing

$$H^n(X,A) \times H^m(X,A) \to H^{n+m}(X \times X, A) \xrightarrow{\Delta^*} H^{n+m}(X,A).$$

where $\Delta : X \to X \times X$ is the diagonal map.

Up the the early 1980s, cup products in sheaf cohomology could only be constructed in the presence of enough points on the underlying topos. It was not even known, for example, how to construct cup products for flat cohomology.

Major principle: Suppose that $f : X \to Y$ induces an isomorphism $\tilde{H}_*(X) \cong \tilde{H}_*(Y)$ in all homology sheaves. Then the induced map in cohomology

$$H^*(Y, A) \to H^*(X, A)$$

is an isomorphism for all coefficient presheaves A.

Proof Compare universal coefficients spectral sequences. □

There is a torsion coeffients version:

Fact: If $f : X \to Y$ induces a homology sheaf isomorphism $\tilde{H}_*(X, \mathbb{Z}/n) \cong \tilde{H}_*(Y, \mathbb{Z}/n)$ then f induces an isomorphism

$$H^*(Y, A) \to H^*(X, A)$$

for all n-torsion presheaves A.

Proof Construct the universal coefficients spectral sequence in the category of n-torsion sheaves. □

Example: Suppose that k is an alg. closed field and ℓ is a prime $\neq char(k)$. Let $\mathcal{C} = (Sm|_k)_{et} =$ smooth schemes over k with the étale topology. The **Gabber rigidity** theorem asserts, in this case, that the map $\epsilon : \Gamma^* BGl(k) \to BGl$ of simplicial presheaves on $(Sm|_k)_{et}$ induces an isomorphism

$$\tilde{H}_*(\Gamma^* BGl(k), \mathbb{Z}/\ell) \cong \tilde{H}_*(BGl, \mathbb{Z}/\ell)$$

on mod ℓ homology sheaves. It follows that the induced map

$$H^*(BGl, \mathbb{Z}/\ell) \to H^*(\Gamma^* BGl(k)\mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell)$$

is an isomorphism.

There are several consequences:

a) $H^*(BGl(k), \mathbb{Z}/\ell) \cong H^*(BU, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, c_2, \dots]$ is a polynomial ring in Chern classes.

b) Any inclusion $k \subset L$ of algebraically closed fields induces isomorphisms

$$H^*(BGl(L), \mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell)$$
$$K_*(k, \mathbb{Z}/\ell) \cong K_*(L, \mathbb{Z}/\ell)$$

c) $K_*(k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta]$ is a polynomial ring on $\beta \in K_2(k, \mathbb{Z}/\ell)$. Here, the Bott element $\beta \mapsto \zeta_\ell$ where ζ_l is a primitive ℓ^{th} root of unity under the isomorphism

$$K_2(k,\mathbb{Z}/\ell) \cong \mathrm{Tor}(\mathbb{Z}/\ell, K_1(k)) \cong \mathrm{Tor}(\mathbb{Z}/\ell, k^*).$$

d) The simplicial presheaf map

$$\epsilon : \Gamma^* K(k,\mathbb{Z}/\ell) \to K(\ ,\mathbb{Z}/\ell)$$

is a local weak equivalence on $(Sm|_k)_{et}$. In other words, the mod ℓ étale K-theory sheaves $\tilde{\pi}_i K(\ ,\mathbb{Z}/\ell)$ are constant.

5) A **cohomology operation** is a map

$$K(A,n) \to K(B,m)$$

in the homotopy category.

The **Steenrod operation** Sq^i is a morphism $K(\mathbb{Z}/2,n) \to K(\mathbb{Z}/2,n+i)$ in the ordinary homotopy category. The constant presheaf functor preserves weak equivalences, and so Sq^i induces a morphism $K(\mathbb{Z}/2,n) \to K(\mathbb{Z}/2,n+i)$ in the homotopy category of simplicial presheaves on an arbitrary small site $\mathcal{C}$. It therefore induces a homomorphism

$$\mathrm{Sq}^i : H^n(X,\mathbb{Z}/2) \to H^{n+i}(X,\mathbb{Z}/2)$$

which is natural in simplicial presheaves X. The collection of Steenrod operations $\{\mathrm{Sq}^i\}$ for simplicial presheaves has the same basic list of properties as the Steenrod operations for ordinary spaces.

Steenrod operations for mod 2 étale cohomology were first introduced by Breen, and the definition given here for mod 2 cohomology of arbitrary simplicial presheaves is a vast generalization. The first calculational application was in questions concerning Hasse-Witt classes for non-degenerate symmetric bilinear forms in the mod 2 Galois cohomology of fields — see "Universal Hasse-Witt classes" and "Higher spinor classes".

Suppose that k is a field such that $char(k) \neq 2$. A non-degenerate symmetric bilinear form α on k represents an element of

$$H^1_{et}(k, O_n) \cong [*, BO_n]$$

where the homotopy classes of maps are in the homotopy category of simplicial presheaves on $(Sch|_k)_{et}$ (this remains to be explained — see the next section of this file).

There are isomorphisms

$$\begin{aligned} H^*_{et}(BO_n, \mathbb{Z}/2) &\cong H^*(BO_n, \mathbb{Z}/2) \\ &\cong H^*_{Gal}(k, \mathbb{Z}/2)[HW_1, \dots, HW_n] \end{aligned}$$

where the polynomial generator HW_i has degree i. In fact HW_i is characterized by mapping to the i^{th} elementary symmetric polynomial $\sigma_i(x_1, \dots, x_n)$

under the isomorphism

$$H^*(BO_n, \mathbb{Z}/2) \cong H^*(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2)^{\Sigma_n}$$
$$\cong H^*_{Gal}(k, \mathbb{Z}/2)[x_1, \ldots, x_n]^{\Sigma_n}.$$

where $(\ )^{\Sigma_n}$ denotes invariants for the symmetric group Σ_n

Every symmtric bilinear form α determines a map $\alpha : * \to BO_n$ in the simplicial presheaf homotopy category, and therefore induces a map

$$\alpha^* : H^*_{et}(BO_n, \mathbb{Z}/2) \to H^*_{Gal}(k, \mathbb{Z}/2),$$

and $HW_i(\alpha) = \alpha^*(HW_i)$ is the i^{th} Hasse-Witt class of α. $HW_1(\alpha)$ is the pullback of the determinant $BO_n \to B\mathbb{Z}/2$, and $HW_2(\alpha)$ is the classical Hasse-Witt invariant of α.

The application of the Steenrod algebra is about calculating the relation between Hasse-Witt and Stiefel-Whitney classes for Galois representations, and depends on knowing the Wu formulas for the action of the Steenrod algebra on elementary symmetric polynomials.

Non-abelian cohomology

Suppose that G is a sheaf of groups. A G-torsor is traditionally defined to be a sheaf X with a free G-action such that $X/G \cong *$ in the sheaf category.

The requirement that the action $G \times X \to X$ is free means that the isotropy subgroups of G for the action are trivial in all sections, which is equivalent to requiring that all sheaves of fundamental groups for the Borel construction $EG \times_G X$ are trivial. There is an isomorphism of sheaves

$$\tilde{\pi}_0(EG \times_G X) \cong X/G.$$

Also the simplicial sheaf $EG \times_G X$ is the nerve of a sheaf of groupoids, which is given in each section by the translation category for the action of $G(U)$ on $X(U)$; this means, in particular, that all sheaves of higher homotopy groups for $EG \times_G X$ vanish.

It follows that a G-sheaf X is a G-torsor if and only if the map $EG \times_G X \to *$ is a local weak equivalence.

The category $G - \mathbf{Tors}$ is the category whose objects are all G-torsors and whose maps are all G-equivariant maps between them.

If $f : X \to Y$ is a map of G-torsors, then f is induced as a map of fibres by the comparison of local fibrations

$$\begin{array}{ccc} EG \times_G X & \longrightarrow & EG \times_G Y \\ & \searrow \quad \swarrow & \\ & BG & \end{array}$$

It follows that $f : X \to Y$ is a weak equivalence of constant simplicial sheaves, and is therefore an isomorphism. The category of G-torsors is therefore a groupoid.

Suppose that the picture

$$* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$$

is an object of the cocycle category $H(*, BG)$, and form the pullback

$$\begin{array}{ccc} \mathrm{pb}(Y) & \longrightarrow & Y \\ \downarrow & & \downarrow \alpha \\ EG & \xrightarrow[\pi]{} & BG \end{array}$$

where $EG = B(G/*) = EG \times_G G$ and $\pi : EG \to BG$ is the canonical map. Then $\mathrm{pb}(Y)$ inherits a G-action from the G-action on EG, and the map $EG \times_G \mathrm{pb}(Y) \to Y$ is a sectionwise weak equivalence. Also, the square is homotopy cartesian in sections where $Y(U) \neq \emptyset$, so that $Y(U) \simeq G(U)$

in those sections. It follows that the canonical map $\mathrm{pb}(Y) \to \tilde{\pi}_0\,\mathrm{pb}(Y)$ is a G-equivariant local weak equivalence, and hence that the maps

$$EG \times_G \tilde{\pi}_0\,\mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \to Y \simeq *$$

are natural local weak equivalences. In particular, the G-sheaf $\tilde{\pi}_0\,\mathrm{pb}(Y)$ is a G-torsor.

We therefore have a functor

$$H(*, BG) \to G - \mathbf{Tors}$$

defined by sending $* \xleftarrow{\simeq} Y \to BG$ to the object $\tilde{\pi}_0\,\mathrm{pb}(Y)$. The Borel construction defines a functor

$$G - \mathbf{Tors} \to H(*, BG):$$

the G-torsor X is sent to the cocycle

$$* \xleftarrow{\simeq} EG \times_G X \to BG.$$

It is elementary to check that these two functors, together, induce a bijection

$$\pi_0 H(*, BG) \cong \pi_0(G - \mathbf{Tors}).$$

In view of the fact that $\pi_0(G - \mathbf{Tors})$ is isomorphism classes of G-torsors, and we know that

$$\pi_0 H(*, BG) \cong [*, BG],$$

we have proved

Theorem: Suppose that G is a sheaf of groups on a small Grothendieck site $\mathcal{C}$. Then there is a bijection

$$[*, BG] \cong \{\text{isomorphism classes of } G\text{-torsors}\}$$

Remarks:

1) The theorem was first proved, by a different method, in "Universal Hasse-Witt classes".

2) The non-abelian invariant $H^1(\mathcal{C}, G)$ is traditionally defined to be the collection of isomorphism classes of G-torsors. The theorem therefore gives an identification

$$H^1(\mathcal{C}, G) \cong [*, BG]$$

Example: Suppose that k is a field. Let $\mathcal{C}$ be the étale site $et|_k$ for k, and identify the orthogonal group O_n with a sheaf of groups on this site. The non-abelian cohomology object $H^1_{et}(k, O_n)$ is well known to coincide with the set of isomorphism classes of non-degenerate symmetric bilinear forms over k of rank n. Thus, every such form q determines a morphism $* \to BO_n$ in the simplicial (pre)sheaf homotopy category, and this morphism determines the form q up to isomorphism.

Descent

Proposition: Suppose that A is a presheaf of abelian groups and let $GK(A,n)$ be a globally fibrant model of $K(A,n)$. Then there are isomorphisms

$$\pi_j GK(A,n)(U) \cong \begin{cases} H^{n-j}(\mathcal{C}/U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathcal{C}$.

Exercise: Suppose given a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & Y & \end{array}$$

where p and p' are local fibrations and f is a local weak equivalence. Suppose that $Z \to Y$ is a map of simplicial presheaves. Show that the induced map

$$Z \times_Y X \xrightarrow{f_*} Z \times_Y X'$$

is a local weak equivalence — use Boolean localization.

Lemma: Suppose that $U \in \mathcal{C}$ and write $X|_U$ for restriction of X along the functor $\mathcal{C}/U \to \mathcal{C}$. Then the restriction functor $X \mapsto X|_U$ preserves global fibrations and local weak equivalences.

Proof The restriction functor $X \mapsto X|_U$ has a left adjoint j_U^* where

$$j_U^*(Y)(V) = \bigsqcup_{V \to U} Y(V).$$

j_U^* clearly preserves cofibrations and sectionwise weak equivalences. j_U^* also preserves local trivial fibrations (exercise) and therefore preserves local weak equivalences.

Restriction preserves sectionwise equivalences and local trivial fibrations, and therefore preserves local weak equivalences. $\square$

Proof of Proposition There is a sectionwise fibre sequence

$$\begin{aligned} K(A, n-1) \to & WK(A, n-1) \\ & \to \overline{W}K(A, n-1) = K(A, n) \end{aligned}$$

where $WK(A, n-1)$ is sectionwise contractible. Take a globally fibrant model

$$\begin{array}{ccc} WK(A, n-1) & \xrightarrow{j} & GWK(A, n-1) \\ \downarrow & & \downarrow p \\ K(A, n) & \xrightarrow[j]{} & GK(A, n) \end{array}$$

where the maps labelled j are local weak equivalences, $GK(A, n)$ is globally fibrant and p is a global fibration. Let $F = p^{-1}(0)$. Then F is globally fibrant and the induced map

$$K(A, n-1) \to F$$

is a local weak equivalence, by the Exercise. Write $GK(A, n-1)$ for F.

We have sectionwise fibre sequences

$$\begin{aligned} GK(A, n-1)(U) \to & GWK(A, n-1)(U) \\ & \to GK(A, n)(U) \end{aligned}$$

for all $U \in \mathcal{C}$. The map

$$GWK(A, n-1) \to *$$

is a trivial global fibration, and is therefore a sectionwise trivial fibration. It follows that

$$\pi_j GK(A, n)(U) \cong \pi_{j-1} GK(A, n-1)(U)$$

for $1 \leq j \leq n$, so that

$$\pi_j GK(A, n)(U) \cong H^{n-j}(\mathcal{C}/U, \tilde{A}|_U)$$

for $1 \leq j \leq n$ by induction on n. Finally

$$\begin{aligned} \pi_0 GK(A, n)(U) &\cong [*, GK(A, n)(U)]_{\mathbf{S}} \\ &\cong [*, GK(A, n)|_U]_U \\ &\cong [*, GK(A|_U, n)]_U. \end{aligned}$$

Note that $GK(A,n)|_U$ globally fibrant by the Lemma, giving the second isomorphism; the other isomorphisms are formal. $\square$

Example: Suppose $\mathcal{C}$ is the big site $(Sch|_S)_{et}$ for a scheme S with the étale topology. Then $\mathcal{C}/U$ is isomorphic to the site $(Sch|_U)_{et}$. If A is a sheaf on the big étale site for S, and if $K(A,n) \to GK(A,n)$ is a globally fibrant model for $K(A,n)$, then the presheaves of homotopy groups for $GK(A,n)$ have the form

$$\pi_j GK(A,n)(U) \cong \begin{cases} H^{n-j}_{et}(U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathcal{C}$.

Similar statements obtain for all other geometric topologies on categories of S-schemes.

Say that a simplicial presheaf X **satisfies descent** if some (hence any) globally fibrant model $j : X \to Z$ induces weak equivalences $X(U) \to Z(U)$ of simplicial sets in all sections.

Simplicial presheaves which satisfy descent are not common: Eilenberg-Mac Lane objects $K(A,n)$, for example, almost never satisfy descent.

This concept is, however, at the heart of the applications of the homotopy theory of simplicial presheaves, and was the primary reason for the introduction of that theory. The basic idea is that if a simplicial presheaf satisfies descent for some topology, then its homotopy groups can (or should) be computed with a cohomological **descent spectral sequence** for that same topology.

Suppose that X is a presheaf of pointed Kan complexes, and form the Postnikov tower

$$
\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow \\
 & & P_2X & \xrightarrow{j} & GP_2X \\
 & \nearrow & \downarrow & & \downarrow p \\
 & & P_1X & \xrightarrow{j} & GP_1X \\
 & \nearrow & \downarrow & & \downarrow p \\
X & \longrightarrow & P_0X & \xrightarrow[j]{} & GP_0X
\end{array}
$$

where all maps labelled j are globally fibrant models and the maps p are global fibrations.

The fibre of $GP_nX \to GP_{n-1}X$ is sectionwise equivalent to $GK(\tilde{\pi}_nX, n)$, where

$$\tilde{\pi}_nX = \tilde{\pi}_n(X, *)$$

is the n^{th} homotopy group sheaf, based at the global base point.

Now take $U \in \mathcal{C}$ and consider the tower of fibrations

$$GP_0X(U) \leftarrow GP_1X(U) \leftarrow GP_2X(U) \leftarrow \dots$$

The fibre $GK(\tilde{\pi}_nX, n)(U)$ of the map

$$GP_nX(U) \to GP_{n-1}X(U)$$

has homotopy groups

$$\pi_j GK(\tilde{\pi}_nX, n)(U) \cong \begin{cases} H^{n-j}(\mathcal{C}/U, \tilde{\pi}_nX|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

and so the tower of fibrations spectral sequence (with the Thomason re-indexing trick) determines a spectral sequence with

$$E_2^{s,t}(U) = H^s(\mathcal{C}/U, \tilde{\pi}_sX|_U)$$

This is "the" descent spectral sequence — it is actually a presheaf of spectral sequences.

There are two issues:

1) the spectral sequence might converge to

$$\pi_{t-s} \varprojlim GP_nX(U)$$

2) it can be a bit of work to show that the map $X \to \varprojlim_n GP_nX$ is a local weak equivalence.

Both issues can be resolved (ie. the spectral sequence converges and the map of 2) is a local weak equivalence) if X is locally connected in the sense that $\tilde{\pi}_0 X \cong *$ and there is a uniform bound on cohomological dimension for all sheaves $\tilde{\pi} X|_U$. See "Simplicial presheaves".