## **Lecture 013** (April 4, 2005)

## Localization

Suppose that $\mathcal{A}$ is a small category, and say that an **$\mathcal{A}$-set** is a functor $X : \mathcal{A}^{op} \to \mathbf{Set}$, ie. a contravariant set-valued functor on $\mathcal{A}$. The $\mathcal{A}$-sets and natural transformations form a category, called the category of $\mathcal{A}$-sets and denoted by $\mathcal{A} - \mathbf{Set}$.

**Examples:**

1) $\mathcal{A} = \mathbf{\Delta}$: $\mathbf{\Delta} - \mathbf{Set}$ = simplicial sets.

2) $\mathcal{A} = \Box$: $\Box - \mathbf{Set}$ = cubical sets.

3) $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ models presheaves of $\mathcal{B}$-sets (provided $\mathcal{B}$ and $\mathcal{C}$ are small).

$a \in \mathcal{A}$: the **standard $a$-cell** $\Delta^a$ is the functor represented by $a$.

$$\Delta^a = \hom_{\mathcal{A}}(\ ,a).$$

The **cell category** $i_{\mathcal{A}}X$ for an $\mathcal{A}$-set $X$ has as objects all morphisms $\Delta^a \to X$ and as morphisms all commutative diagrams

$$\begin{tikzcd}
\Delta^a \arrow[rr] \arrow[dr] & & \Delta^b \arrow[dl] \\
& X &
\end{tikzcd}$$

A map $f : X \to Y$ of $\mathcal{A}$-sets is said to be a "simplicial" weak equivalence ($\infty$-equivalence) if $Bi_{\mathcal{A}}X \to Bi_{\mathcal{A}}Y$ is a weak equivalence of simplicial sets.

**Examples:**

1) $X =$ simplicial set. $i_{\mathbf{\Delta}}X = \mathbf{\Delta}/X$, ie. the cell category for $X$ is its simplex category. There are canonical weak equivalences

$$\begin{tikzcd} \underrightarrow{\mathrm{holim}}_{\,\Delta^n \to X}\ \Delta^n \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & X \\ Bi_{\mathbf{\Delta}}X & \end{tikzcd}$$

so that $X \to Y$ is a weak equiv. of simplicial sets if and only if $Bi_{\mathbf{\Delta}}X \to Bi_{\mathbf{\Delta}}Y$ is a weak equiv.

2) $Y =$ cubical set. There are natural weak equivalences

$$\begin{tikzcd} \underrightarrow{\mathrm{holim}}_{\,\square^n \to Y}\ |\square^n| \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & {|Y|} \\ Bi_{\square}Y & \end{tikzcd}$$

The horizontal equivalence is a bit subtle — it's the "regularity" property of cubical sets, in Cisinski's language. It results from the skeletal decomposition for cubical sets, and the fact that $X \mapsto Bi_{\mathcal{A}}X$ takes pushout squares to homotopy cocartesian diagrams (which, in itself, was a surprise).

The pairing

$$(\Box^n, \Box^m) \mapsto \Box^n \otimes \Box^m = \Box^{n+m}$$

defines a monoidal structure $\otimes : \Box \times \Box \to \Box$ on the box category.

An **interval theory** on the category of $\mathcal{A}$-sets is a coherent action

$$\otimes : \mathcal{A} - \mathbf{Set} \times \Box \to \mathcal{A} - \mathbf{Set}$$

of $\Box$ on the category of $\mathcal{A}$-sets, written as

$$(X, \mathbf{1}^n) \mapsto X \otimes \Box^n,$$

subject to the following conditions:

**DH1** $X \mapsto X \otimes \Box^1$ preserves filtered colimits and monomorphisms.

**DH2** Given a monomorphism $i : X \to Y$ and a $d^{(i,\epsilon)} : \Box^{n-1} \to \Box^n$ the following is a pullback

$$\begin{tikzcd}
X \otimes \Box^{n-1} \arrow[r, "i\otimes 1"] \arrow[d, "1\otimes d^{(i,\epsilon)}"'] & Y \otimes \Box^{n-1} \arrow[d, "1\otimes d^{(i,\epsilon)}"] \\
X \otimes \Box^n \arrow[r, "i\otimes 1"'] & Y \otimes \Box^n
\end{tikzcd}$$

**DH3** The following is a pullback for $1 \leq i \leq n$:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \otimes \Box^{n-1} \arrow[d, "d^{(i,0)}"] \\
X \otimes \Box^{n-1} \arrow[r, "d^{(i,1)}"'] & X \otimes \Box^n
\end{tikzcd}$$

**Examples:**

1) If $I$ is any $\mathcal{A}$-set with a monomorphism $(d_0, d_1) : * \sqcup * \longrightarrow I$ (ie. $d_0$ and $d_1$ are disjoint "rational points" of the interval $I$) then $(X, \mathbf{1}^n) \mapsto X \times I^{\times n}$ defines an interval theory

$$I : \mathcal{A} - \mathbf{Set} \times \Box \to \mathcal{A} - \mathbf{Set}$$

2) $(X, Y) \mapsto X \otimes Y$ defines a monoidal structure on the category of cubical sets, and this structure induces a coherent action

$$\otimes : \Box - \mathbf{Set} \times \Box \to \Box - \mathbf{Set}$$

of the box category on the category of cubical sets.

We shall denote interval theories by $\otimes$.

$X \in \mathcal{A} - \mathbf{Set}$, $K$ cubical set:

$$X \otimes K = \varinjlim_{\Box^n \to K} X \otimes \Box^n$$

For $X, Y \in \mathcal{A}-\mathbf{Set}$, define a cubical set $\mathbf{hom}_{\Box}(X, Y)$ by

$$\mathbf{hom}_{\Box}(X, Y)_n = \hom(X \otimes \Box^n, Y).$$

There is a natural bijection

$$\hom(X \otimes K, Y) \cong \hom(K, \mathbf{hom}_{\Box}(X, Y)).$$

**Some basics:**

1) $\partial\square^n \subset \square^n$ induces a monomorphism

$$X \otimes \partial\square^n \to X \otimes \square^n.$$

Consequence: any cubical set inclusion $K \to L$ induces a monic $X \otimes K \to X \otimes L$.

2) If $X \to Y$ is a monomorphism of $\mathcal{A}$-sets and $K \to L$ is a monomorphism of cubical sets, then the map

$$(Y \otimes K) \cup_{(X \otimes K)} (X \otimes L) \to Y \otimes L$$

is a monomorphism. The map $X \otimes L \to Y \otimes L$ is a monomorphism for all $L$.

3) There is a cardinal $\zeta$ such that $|X \otimes \square^n| < \lambda$ if $|X| < \lambda$ for all $\lambda > \zeta$.

How to think of 3): choose $\alpha$ such that $|\operatorname{Mor}(\mathcal{A})| < \alpha$. Choose $\zeta$ such that $|A \otimes \square^n| < \zeta$ for all $\alpha$-bounded objects $A$.

4) About cubical sets: the map

$$(\sqcap^n_{(i,\epsilon)} \otimes \square^k) \cup (\square^n \otimes \partial\square^k) \subset \square^n \otimes \square^k$$

is isomorphic to the inclusion $\sqcap^{n+k}_{(i,\epsilon)} \subset \square^{+k}$. Similarly, the map

$$(\partial\square^n \otimes \square^k) \cup (\square^n \otimes \partial\square^k) \subset \square^n \otimes \square^k$$

is isomorphic to $\partial\square^{n+k} \subset \square^{n+k}$.

$S =$ a set of monomorphisms of $\mathcal{A}$-sets

The class of **anodyne cofibrations** is the saturation of the set of inclusions

$$(Y \otimes \square^n) \cup (\Delta^a \otimes \sqcap^n_{(i,\epsilon)}) \subset \Delta^a \otimes \square^n \qquad (1)$$

arising from the set of all inclusions of subobjects $Y \subset \Delta^a$, together with the set of inclusions

$$(A \otimes \square^n) \cup (B \otimes \partial\square^n) \subset B \otimes \square^n \qquad (2)$$

induced by the maps $A \to B$ of the set $S$.

Write $\Lambda(S)$ for the set of all maps appearing in (1) and (2).

**Fact:** Any inclusion $C \to D$ of $\mathcal{A}$-sets induces an anodyne cofibration

$$(C \otimes \square^n) \cup (D \otimes \sqcap^n_{(i,\epsilon)}) \subset D \otimes \square^n.$$

**Lemma:** If $C \to D$ is an anodyne cofibration, then so is

$$(C \otimes \square^1) \cup (D \otimes \partial\square^1) \subset D \otimes \square^1.$$

**Proof** It's enough to prove this for $C \to D$ of the form (1) or (2) above, but this is just fun with the identifications of cubical set morphisms given in 4) above. $\square$

**Definition:** Say that a map $p : X \to Y$ of $\mathcal{A}$-sets is **injective** if it has the RLP wrt all anodyne cofibrations. An $\mathcal{A}$-set $X$ is injective if $X \to *$ is injective.

**Definition:** A **naive homotopy** between maps $f, g : X \to Y$ is a map $h : X \otimes \square^1 \to Y$ which makes the obvious diagram commute:

$$\begin{tikzcd}
X \arrow[d, "d_0"'] \arrow[dr, "f"] & \\
X \otimes \square^1 \arrow[r, "h"] & Y \\
X \arrow[u, "d_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

**Fact:** Naive homotopy of maps $X \to Z$ is an equivalence relation if $Z$ is injective.

**Proof** Suppose that $f_0 \simeq f_1$ and $f_1 \simeq f_2$ via homotopies $h_1, h_2 : X \otimes \square^1 \to Z$, respectively. Then $h_1, h_2$ and the constant homotopy $c$ at $f_2$ together determine a map

$$H : X \otimes \sqcap^2_{(2,0)} \to Z$$

which can be represented by the picture

$$\begin{tikzcd}
f_1 \arrow[r, "h_2"] & f_2 \\
f_0 \arrow[u, "h_1"] \arrow[r, dotted] & f_2 \arrow[u, "c"']
\end{tikzcd}$$

The map $H$ extends to a map $H' : X \otimes \Box^2 \to Z$ since the cofibration $X \otimes \sqcap^2_{(2,0)} \to X \otimes \Box^2$ is anodyne. Restriction to the $(2,0)$ face gives a homotopy $f_0 \simeq f_2$. Symmetry has a similar proof:

$$\begin{tikzcd}
f_1 \arrow[r, "c"] & f_1 \\
f_1 \arrow[u, "c"] \arrow[r, dotted] & f_0 \arrow[u, "h"']
\end{tikzcd}$$

Reflexity is trivial: the map $\Box^1 \to \Box^0$ induces a map $X \otimes \Box^1 \to X$ $\Box$

$\pi(X, Y)$ = set of naive homotopy classes of maps from $X$ to $Y$, meaning collapse $\hom(X, Y)$ by the equiv. relation generated by naive homotopy.

A map $f : X \to Y$ is said to be a **weak equivalence** if it induces a bijection

$$\pi(Y, Z) \xrightarrow{\cong} \pi(X, Z)$$

for all injective $Z$.

A **cofibration** is a monomorphism.

A **fibration** is a map which has the RLP wrt all trivial cofibrations.

We'll see later on that all injective objects are fibrant. Every fibrant object is obviously injective, so that the classes of fibrant and of injective objects coincide.

**Fact:** Every $X$ has an injective model $j : X \to \mathcal{L}X$ by a standard transfinite small object argument, meaning that $j$ is anodyne and $\mathcal{L}X$ is injective.

**Lemma:** All anodyne cofibrations are weak equivalences.

**Proof** Suppose that $i : A \to B$ is anodyne and $Z$ is injective. The lifting exists in any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Z \\ B \arrow[ur, dotted] & \end{tikzcd}$$

so that $\pi(B, Z) \to \pi(A, Z)$ is surjective.

Suppose that $f, g : B \to Z$ become homotopic on restriction to $A$, via a homotopy $h : A \otimes \square^1 \to Z$. Then the lifting exists in the diagram

$$\begin{tikzcd} (B \otimes \partial\square^1) \cup (A \otimes \square^1) \arrow[r, "((f,g),h)"] \arrow[d] & Z \\ B \otimes \square^1 \arrow[ur, dotted] & \end{tikzcd}$$

and so $f \simeq g$. Thus $\pi(B, Z) \to \pi(A, Z)$ is a monomorphism. $\square$

**Lemma:** Suppose that $X$ and $Y$ are injective objects. Then $f : X \to Y$ is a weak equivalence if and only if it is a naive homotopy equivalence.

**Proof:** $f_* : \pi(Y,X) \to \pi(Y,Y)$ is a bijection, so there is a unique naive homotopy class $g : Y \to X$ such that $fg \simeq 1$. $f_* : \pi(X,X) \to \pi(X,Y)$ is a bijection and $fgf \simeq f$. Thus $gf \simeq 1$. $\Box$

**Corollary:** $f : X \to Y$ is a weak equivalence if and only if $\mathcal{L}X \to \mathcal{L}Y$ is a naive homotopy equivalence.

**Theorem A:** (Cisinski, "Swiss army knife theorem") With the definitions given above, the category of $\mathcal{A}$-sets has the structure of cubical model category.

The model structure given by this theorem is called the $(S, \otimes)$**-model structure** on the category $\mathcal{A}$-sets, reflecting the fact that it depends only on the interval theory $\otimes$ and the generating set of cofibrations $S$. If $\otimes$ is specified by an interval $I$, one calls this the $(S, I)$**-model structure**.

**Theorem B:** Suppose that the interval theory $\otimes$ on $\mathcal{A}$-sets is defined by an interval $I$ in the sense that

$$Z \otimes \square^n = Z \times I^{\times n}$$

Suppose that all cofibrations in the set $S$ pull back to weak equivalences along all fibrations $p : X \to Y$ with $Y$ fibrant. Then the corresponding model structure on $\mathcal{A}$-sets is proper.

The proofs will come later. Here are some consequences:

**Example 1:** $\mathcal{A} = \mathcal{C} \times \mathbf{\Delta}$: $\mathcal{A}$-sets are simplicial presheaves on $\mathcal{C}$, $S =$ generating set of local trivial cofibrations for the standard (injective) model structure on $s\,\mathrm{Pre}(\mathcal{C})$, $I = \Delta^1$.

The $(S, \Delta^1)$-model structure given by Theorem A is the standard model structure for $s\,\mathrm{Pre}(\mathcal{C})$: every injective object is globally fibrant and the map $X \to \mathcal{L}X$ is a local weak equivalence, so $f : X \to Y$ is a local weak equivalence iff $f$ is a weak equivalence for the $(S, \Delta^1)$-structure.

**NB**: The case $\mathcal{C} = *$ gives the standard model structure for simplicial sets. In that case $S$ is the set of all inclusions $\Lambda^n_k \subset \Delta^n$, $n \geq 0$.

**Example 2:** $S$ can be empty: the interval theory $X \times (\Delta^1)^{\times n}$ alone gives a model structure for simplicial sets which is a priori weaker (has fewer weak equivalences) than the standard model structure.

Say that the case $S = \emptyset$ is a **primitive model structure**.

**Example 3:** Back to $\mathcal{A} = \mathcal{C} \times \mathbf{\Delta}$: suppose that $f : A \to B$ is a monomorphism (or a set of monomorphisms) of simplicial presheaves on $\mathcal{C}$. Take the set $S$ of generating cofibrations from Example 1, and add the set of all cofibrations

$$(Y \times B) \cup (L_U \Delta^n \times A) \to L_U \Delta^n \times B$$

induced by all subobjects $Y \subset L_U \Delta^n$. Denote the enlarged set of cofibrations by $S_f$. Let $I = \Delta^1$, as before. The resulting $(S_f, \Delta^1)$-model structure on $s\,\mathrm{Pre}(\mathcal{C})$ is the $f$-local model structure on $s\,\mathrm{Pre}(\mathcal{C})$.
**NB:** The $f$-local model structure is proper if $f$ is a map $* \to J$ for some simplicial presheaf $J$.

**Example 4:** Suppose that $\mathcal{C} = (Sm|_S)_{Nis}$ where $S$ is a scheme of finite dimension, and let $f : * \to \mathbb{A}^1$ be the rational point $0$ (or any other). The the $f$-local structure of the previous example, in this case is the **motivic model structure** on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$. This model structure is proper.

**Different construction:** Use the interval theory $\mathbb{A}^1$ given by the presheaf $\mathbb{A}^1$ and the rational points $0, 1 : * \to \mathbb{A}^1$. Let $S$ be the generating set of trivial cofibrations for the standard model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$. Then the $(\mathbb{A}^1, S)-$model structure on $s\,\mathrm{Pre}(Sm|_S)_{Nis}$ is the motivic model structure.

**Example 5:** Recall that a map $f : X \to Y$ of simplicial presheaves on $\mathcal{C}$ is a homology sheaf isomorphism if $\tilde{H}_*(X) \to \tilde{H}_*(Y)$ is an isomorphism of sheaves. Suppose that $\alpha > |\,\mathrm{Mor}(\mathcal{C})|$.

**Exercise:** Suppose that the cofibration $i : X \to Y$ is a homology sheaf isomorphism, and that $A$ is an $\alpha$-bounded subobject of $Y$. Then there is an $\alpha$-bounded subobject $B \subset Y$ with $A \subset B$ such that $B \cap X \to B$ is a homology sheaf isomorphism.

Let $S$ be the set of $\alpha$-bounded cofibrations which are homology sheaf isomorphisms, and let $I = \Delta^1$. Then the $(S, I)$-model structure on $s\,\mathrm{Pre}(\mathcal{C})$ is integral homology localization structure, and the fibrant models $X \to \mathcal{L}X = L_{\mathbb{Z}}(X)$ are homology (sheaf) localizations.

a) If $\mathcal{C}$ has no topology, this construction specializes to sectionwise integral homology localization on $\mathcal{C}^{op}$-diagrams.

b) If $\mathcal{C} = *$, this construction specializes further to Bousfields integral homology localization theory for simplicial sets.

**Remark:** This construction generalizes to a localization construction for any homology theory, sheaf theoretic or not.

**Example:** The $H_*(\ ,\mathbb{Q})$-local theory is rational homotopy theory.