Lecture 014 (April 4, 2005)

Localization: some details

As before, $\mathcal{A}$ is a small category, and $\mathcal{A}-\mathbf{Set}$ is the category of $\mathcal{A}$-sets, or functors $\mathcal{A}^{op} \to \mathbf{Set}$.

The class of anodyne cofibrations (or anodyne extensions) is the saturation of the set of inclusions $\Lambda(S)$ specified by

$$(Y \otimes \Box^n) \cup (\Delta^a \otimes \sqcap^n_{(i,\epsilon)}) \subset \Delta^a \otimes \Box^n \qquad (1)$$

$$(A \otimes \Box^n) \cup (B \otimes \partial\Box^n) \subset B \otimes \Box^n. \qquad (2)$$

Recall that we are tacitly assuming that $\mathcal{A}-\mathbf{Set}$ is equipped with a pairing

$$\otimes : (\mathcal{A}-\mathbf{Set}) \times (\Box-\mathbf{Set}) \to \mathcal{A}-\mathbf{Set}$$

which behaves well with respect to inclusions of both $\mathcal{A}$-sets and inclusions of cubical sets. Most often, the pairing $\otimes$ is specified by

$$X \otimes \Box^n = X \times I^{\times n}$$

where I is an $\mathcal{A}$-set equipped with two distinct rational points $* \to I$.

In all that follows, the definitions and proofs depend on the generating set S and the pairing $\otimes$.

An injective morphism is a map $p : X \to Y$ which has the RLP wrt all anodyne extensions, and an object X is injective if $X \to *$ is an injective morphism.

A weak equivalence is a map $f : X \to Y$ which induces a bijection $\pi(Y, Z) \to \pi(X, Z)$ for all injective Z. A cofibration is just a monomorphism, and a fibration is a map which has the RLP wrt all trivial cofibrations.

We shall sketch the proof of the following:

Theorem A: (Cisinski, “Swiss army knife theorem”) With the definitions given above, the category of $\mathcal{A}$-sets has the structure of cubical model category.

There is a properness assertion as well:

Theorem B: Suppose that the interval theory $\otimes$ on $\mathcal{A}$-sets is defined by an interval I. Suppose that all cofibrations in the set S pull back to weak equivalences along all fibrations $p : X \to Y$ with Y fibrant. Then the corresponding model structure on $\mathcal{A}$-sets is proper.

The proof of Theorem B can be found in “Categorical homotopy theory”.

Proof of Theorem A

1) Cardinality tricks

Suppose that T is some set of cofibrations of $\mathcal{A}$-sets, and choose a cardinal α such that $\alpha > \zeta$ where ζ is an infinite cardinal chosen as above (ie. so that $|X \otimes \square^n| < \gamma$ if $|X| < \gamma$, for all $\gamma > \zeta$). Suppose also that $\alpha > |T|$ and that $\alpha > |D|$ for all $C \to D$ in T.

Suppose that $\lambda > 2^\alpha$

Every $f : X \to Y$ has a functorial system of factorizations

$$\begin{tikzcd} X \arrow[r, "i_s"] \arrow[dr, "f"'] & E_s(f) \arrow[d, "f_s"] \\ & Y \end{tikzcd}$$

for $s < \lambda$ defined by the lifting property for maps in T, and which form the stages of a transfinite small object argument.

Specifically, given the factorization $f = f_s i_s$ form the pushout diagram

$$\begin{tikzcd} \sqcup_{\mathcal{D}} C \arrow[r] \arrow[d] & E_s(f) \arrow[d] \\ \sqcup_{\mathcal{D}} D \arrow[r] & E_{s+1}(f) \end{tikzcd}$$

where $\mathcal{D}$ runs through all diagrams

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "i"'] & E_s(f) \arrow[d] \\
D \arrow[r] & Y
\end{tikzcd}$$

with i in T. Then $f_{s+1} : E_{s+1}(f) \to Y$ is the obvious induced map. Set $E_t(f) = \varinjlim_{s<t} E_s(f)$ at limit ordinals $t < \lambda$.

Then there is a functorial factorization

$$\begin{tikzcd}
X \arrow[r, "i_\lambda"] \arrow[dr, "f"'] & E_\lambda(f) \arrow[d, "f_\lambda"] \\
& Y
\end{tikzcd}$$

with $E_\lambda(f) = \varinjlim_{s<\lambda} E_s(f)$. Also f_λ has the right lifting property with respect to all $C \to D$ in T, and i_λ is in the saturation of T.

Write $\mathcal{L}(X) = E_\lambda(X \to *)$.

Lemma:

1) Suppose that $t \mapsto X_t$ is a diagram of cofibrations indexed by $\omega > 2^\alpha$. Then the map

$$\varinjlim_{t<\omega} \mathcal{L}(X_t) \to \mathcal{L}(\varinjlim_{t<\omega} X_t)$$

is an isomorphism.

2) $X \mapsto \mathcal{L}(X)$ preserves cofibrations.

3) Suppose that γ is a cardinal with $\gamma > \alpha$, and let $\mathcal{F}_\gamma(X) =$ the subobjects of X having cardinality less than γ. Then the map

$$\varinjlim_{Y \in \mathcal{F}_\gamma(X)} \mathcal{L}(Y) \to \mathcal{L}(X)$$

is an isomorphism.

4) If $|X| < 2^\omega$ where $\omega \geq \lambda$ then $|\mathcal{L}(X)| < 2^\omega$.

5) Suppose that U, V are subobjects of X. Then the natural map

$$\mathcal{L}(U \cap V) \to \mathcal{L}(U) \cap \mathcal{L}(V)$$

is an isomorphism.

Proof It suffices to prove all statements with $\mathcal{L}(X)$ replaced by $E_1(X)$. There is a pushout diagram

$$\begin{tikzcd}
\sqcup_T(C \times \hom(C, X)) \arrow[r] \arrow[d] & X \arrow[d] \\
\sqcup_T(D \times \hom(C, X)) \arrow[r] & E_1X
\end{tikzcd}$$

Then, in sections,

$$E_1X = \bigsqcup_T((D(a) - C(a)) \times \hom(C, X)) \sqcup X(a).$$

so 5) follows. The remaining statements are exercises. $\square$

Corollary: Every $\mathcal{A}$-set map $f : X \to Y$ has a functorial factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where j is anodyne and p is injective.

Remark: The Lemma also gives us a set of good properties for the injective model construction $X \mapsto \mathcal{L}(X)$, including cardinality bounds.

Suppose that $\alpha > \zeta$, $\alpha > |\Lambda(S)|$ and that $\alpha > |D|$ for all $C \to D$ in $\Lambda(S)$. Suppose that $\lambda > 2^{\alpha}$

Here is the bounded cofibration condition:

Lemma: Suppose given a diagram

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] & Y \end{tikzcd}$$

of cofibrations such that i is a weak equivalence and $|A| < 2^{\lambda}$. Then there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| < 2^{\lambda}$ and $B \cap X \to B$ is an equivalence.

Proof The proof is due to Cisinski. It is innovative in the sense that it uses nothing but naive homotopy.

The map $i_* : \mathcal{L}X \to \mathcal{L}Y$ is a cofibration (by the previous lemma) and is a naive homotopy equivalence of injective objects. There is a map $\sigma : \mathcal{L}Y \to \mathcal{L}Y$ such that $\sigma \cdot i_* \simeq 1$ via a naive homotopy $h : \mathcal{L}X \otimes \square^1 \to \mathcal{L}X$. Form the diagram

$$\begin{tikzcd}
(\mathcal{L}Y \otimes \square^0) \cup (\mathcal{L}X \otimes \square^1) \arrow[r, "{(\sigma,h)}"] \arrow[d] & \mathcal{L}X \\
\mathcal{L}Y \otimes \square^1 \arrow[ur, dotted, "H"'] &
\end{tikzcd}$$

The other end of the homotopy H gives a map σ' such that $\sigma' \cdot i_* = 1$, and $i_*\sigma' \simeq i_*\sigma \simeq 1$. We can therefore assume that $\sigma \cdot i_* = 1$.

Suppose that $A_s \subset Y$ and $|A_s| < 2^\lambda$. Then $|\mathcal{L}A_s \otimes \square^1| < 2^\lambda$. Also, there is a 2^λ-bounded subobject A_{s+1} such that $A_s \subset A_{s+1}$ and there is a diagram

$$\begin{tikzcd}
\mathcal{L}A_s \otimes \square^1 \arrow[r] \arrow[d] & \mathcal{L}A_{s+1} \arrow[d] \\
\mathcal{L}Y \otimes \square^1 \arrow[r, "K"'] & \mathcal{L}Y
\end{tikzcd}$$

where $K : i_*\sigma \simeq 1$.

This is the successor ordinal step in the construction of a system $s \mapsto A_s$ with $s < \lambda$ (recall that $\lambda > 2^\alpha$) and $A = A_0$. Let $B = \varinjlim_s A_s$. Then, by construction, B is 2^λ-bounded and the restriction of the homotopy K to $\mathcal{L}B \otimes \square^1$ factors through the inclusion $j_* : \mathcal{L}B \to \mathcal{L}Y$.

There is a pullback

$$\begin{tikzcd}
\mathcal{L}(B \cap X) \arrow[r, "\tilde{j}"] \arrow[d, "\tilde{i}"'] & \mathcal{L}X \arrow[d, "i_*"] \\
\mathcal{L}B \arrow[r, "j_*"'] & \mathcal{L}Y
\end{tikzcd}$$

and $i_*\sigma(\mathcal{L}B) \subset \mathcal{L}B$. It follows that there is a map $\sigma' : \mathcal{L}B \to \mathcal{L}(B \cap X)$ such that $\sigma' \cdot \tilde{i} = 1$. K restricts to a homotopy $\mathcal{L}B \otimes \square^1 \to \mathcal{L}B$ (by construction), and this is a homotopy $\tilde{i}\sigma' \simeq 1$. $\square$

2) Trivial cofibrations are preserved by pushout

Note first that anodyne extensions are closed under pushout.

Lemma: Suppose given a diagram

$$\begin{tikzcd}
C \arrow[r, "f{,}g"] \arrow[d, swap, "i"] & E \\
D &
\end{tikzcd}$$

where i is a cofibration, and suppose that there is a naive homotopy $h : C \otimes \Box^1 \to E$ from f to g. Then $g_* : D \to D \cup_g E$ is a weak equivalence if and only if $f_* : D \to D \cup_f E$ is a weak equivalence.

Proof There are pushout diagrams

$$\begin{tikzcd}
C \arrow[r, "d_0"] \arrow[d, swap, "i"] & C \otimes \Box^1 \arrow[r, "h"] \arrow[d, "i_*"] & E \arrow[d, "i_*"] \\
D \arrow[r, swap, "d_{0*}"] & D \cup_C (C \otimes \Box^1) \arrow[r, swap, "h'"] \arrow[d, swap, "j"] & D \cup_f E \arrow[d, "j_*"] \\
& D \otimes \Box^1 \arrow[r, swap, "h_*"] & (D \otimes \Box^1) \cup_h E
\end{tikzcd}$$

where the top composite is f. d_{0*}, j and j_* are anodyne cofibrations. Thus $f_* = h' \cdot d_{0*}$ is equivalent to h' and h' is equivalent to h_*, so f_* is a weak equivalence if and only if h_* is a weak equivalence.

Similarly, g_* is a weak equivalence if and only if h_* is a weak equivalence. $\square$

Lemma: Suppose that $i : C \to D$ is a trivial cofibration. Then the cofibration

$$(C \otimes \square^1) \cup (D \otimes \partial\square^1) \to D \otimes \square^1$$

is a weak equivalence.

Proof The diagram

$$\begin{tikzcd}
C \otimes \partial\square^1 \arrow[r] \arrow[d] & D \otimes \partial\square^1 \arrow[r] \arrow[d] & \mathcal{L}D \otimes \partial\square^1 \arrow[d] \\
C \otimes \square^1 \arrow[r] & D \otimes \square^1 \arrow[r] & \mathcal{L}D \otimes \square^1
\end{tikzcd}$$

induces a diagram

$$\begin{tikzcd}
(C \otimes \square^1) \cup (D \otimes \partial\square^1) \arrow[r] \arrow[d] & (C \otimes \square^1) \cup (\mathcal{L}D \otimes \partial\square^1) \arrow[d] \\
D \otimes \square^1 \arrow[r] & \mathcal{L}D \otimes \square^1
\end{tikzcd}$$

in which the horizontal maps are anodyne extensions, and hence weak equivalences.

There is a factorization

$$\begin{tikzcd}
C \arrow[r, "i'"] \arrow[dr, "i"'] & D' \arrow[d, "p"] \\
 & D
\end{tikzcd}$$

where i' is anodyne and p is both injective and a weak equivalence. In the induced diagram

$$\begin{tikzcd}
(C \otimes \square^1) \cup (\mathcal{L}D' \otimes \partial\square^1) \arrow[r] \arrow[d] & (C \otimes \square^1) \cup (\mathcal{L}D \otimes \partial\square^1) \arrow[d] \\
\mathcal{L}D' \otimes \square^1 \arrow[r] & \mathcal{L}D \otimes \square^1
\end{tikzcd}$$

the top horizontal map is induced by the homotopy equivalence

$$\mathcal{L}D' \otimes \partial\square^1 \to \mathcal{L}D \otimes \partial\square^1,$$

and is therefore an equivalence by the previous Lemma. The bottom horizontal map is also a homotopy equivalence. The left hand vertical map is an equivalence by comparison with the map

$$(C \otimes \square^1) \cup (D' \otimes \partial\square^1) \to D' \otimes \square^1$$

which is an anodyne extension. $\square$

Lemma: The class of trivial cofibrations is closed under pushout.

Proof If $j : C \to D$ is a cofibration and a weak equivalence, then every map $\alpha : C \to Z$ with Z injective extends to a map $D \to Z$.

In effect, there is a homotopy $h : C \otimes \square^1 \to Z$ from α to a map $\beta \cdot j$ for some map $\beta : D \to Z$, and then the homotopy extends: diagram

$$\begin{tikzcd}
(C \otimes \square^1) \cup (D \otimes \{1\}) \arrow[r, "{(h,\beta)}"] \arrow[d] & Z \\
D \otimes \square^1 \arrow[ur, dotted, "H"'] &
\end{tikzcd}$$

Note that the vertical map is an anodyne extension.

The diagram

$$\begin{tikzcd}
(C \otimes \square^1) \cup (D \otimes \partial\square^1) \arrow[r] \arrow[d] & (C' \otimes \square^1) \cup (D' \otimes \partial\square^1) \arrow[d] \\
D \otimes \square^1 \arrow[r] & D' \otimes \square^1
\end{tikzcd}$$

is a pushout. The left vertical map is a trivial cofibration by the previous Lemma, and therefore has the left lifting property with respect to the map $Z \to *$. Thus, if two maps $f, g : D' \to Z$ restrict to homotopic maps on C', then $f \simeq g$. $\square$

3) Many injective maps are fibrations

Lemma: Suppose that $p : X \to Y$ is injective and that Y is injective. Then p is a fibration.

Proof Suppose given a diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "\beta"'] & Y
\end{tikzcd} \tag{3}$$

where i is a trivial cofibration. Then there is a map $\theta : B \to X$ such that $\theta \cdot i = \alpha$ since X is injective.

The constant homotopy $A \otimes \square^1 \xrightarrow{pr} A \xrightarrow{\alpha} X$ extends to a homotopy $h : B \otimes \square^1 \to Y$ as in the diagram

$$\begin{tikzcd}
(A \otimes \square^1) \cup (B \otimes \partial\square^1) \arrow[r, "{(p\alpha pr_A,(\beta,p\theta))}"] \arrow[d] & Y \\
B \otimes \square^1 \arrow[ur, dotted, "h"'] &
\end{tikzcd}$$

since the vertical map is a trivial cofibration and Y is injective. It follows that there is a homotopy

$$\begin{tikzcd}
A \otimes \square^1 \arrow[r, "\alpha pr_A"] \arrow[d, "i \times i"'] & X \arrow[d, "p"] \\
B \otimes \square^1 \arrow[r, "h"'] & Y
\end{tikzcd}$$

from the original diagram to a diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "p\theta"'] \arrow[ur, dotted, "\theta"] & Y
\end{tikzcd}$$

Form the diagram

$$\begin{tikzcd}
(A \otimes \square^1) \cup B \arrow[r, "(\alpha pr_A{,}\theta)"] \arrow[d] & X \arrow[d, "p"] \\
B \otimes \square^1 \arrow[r, "h"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$

to show that the required lifting exists for the original diagram. $\square$

Corollary: Every injective object is fibrant.

4) Final approach

Lemma: (**CM4**) Suppose that $p : X \to Y$ is a fibration and a weak equivalence. Then p has the right lifting property with respect to all cofibrations.

Proof Suppose first that Y is injective. Then p is a naive homotopy equivalence, and has a section $\sigma : Y \to X$. The map σ is a trivial cofibration so the lift exists in the diagram

$$\begin{tikzcd}
(Y \otimes \square^1) \cup (X \otimes \partial\square^1) \arrow[rr, "{(\sigma\cdot pr,(1_X,\sigma\cdot p))}"] \arrow[d] & & X \arrow[d, "p"] \\
X \otimes \square^1 \arrow[r, "p\otimes 1"'] \arrow[urr, dotted, "H"] & Y \otimes \square^1 \arrow[r, "pr"'] & Y
\end{tikzcd}$$

since the left vertical map is a weak equivalence by one of the Lemmas above. It follows that the identity diagram on $p : X \to Y$ is homotopic to the diagram

$$\begin{tikzcd}
X \arrow[r, "\sigma\cdot p"] \arrow[d, "p"'] & X \arrow[d, "p"] \\
Y \arrow[r, "1"'] \arrow[ur, dotted, "\sigma"] & Y
\end{tikzcd}$$

Thus, any diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "p"] \\
B \arrow[r] & Y
\end{tikzcd}$$

is homotopic to a diagram which admits a lifting.

It follows that p has the right lifting property with respect to all cofibrations.

If Y is not injective, form the diagram

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "p"'] & Z \arrow[d, "q"] \\
Y \arrow[r, "j_Y"'] & \mathcal{L}(Y)
\end{tikzcd}$$

where j is an anodyne cofibration, q is injective, and j is an injective model for X. Then q is a fibration (previous Lemma) and is a weak equivalence, so that q has the RLP wrt all cofibrations, by the previous paragraph.

Factorize the map $X \to Y \times_{\mathcal{L}(Y)} Z$ as

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr] & W \arrow[d, "\pi"] \\
& Y \times_{\mathcal{L}(Y)} Z
\end{tikzcd}$$

where π has the right lifting property with respect to all cofibrations and i is a cofibration. Write q_* for the induced map $Y \times_{\mathcal{L}(Y)} Z \to Y$. Then the composite $q_*\pi$ has the RLP wrt all cofibrations and is therefore a homotopy equivalence. The cofibration i is also a weak equivalence, and it follows that the lifting exists in the diagram

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
Z \arrow[r, "q_*\pi"'] \arrow[ur] & Y
\end{tikzcd}$$

so that p is a retract of a map which has the RLP wrt all cofibrations. $\square$

Corollary: A map $p : X \to Y$ is a fibration and a weak equivalence if and only if it has the RLP wrt all cofibrations.

Proof of Theorem A: One part of **CM5** is also a consequence of the last Lemma: every map $f : X \to Y$ has a factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& W \arrow[ur, "p"'] &
\end{tikzcd}$$

where i is a cofibration and p is a fibration and a weak equivalence.

The other part of **CM5** follows from the bounded cofibration condition: every $f : X \to Y$ has a factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\
& Z \arrow[ur, "q"'] &
\end{tikzcd}$$

where j is a cofibration and a weak equivalence and q is a fibration. In order to conclude that j is a weak equivalence, we need to know that trivial cofibrations are closed under pushout, but this was proved above. $\square$