Lecture 015 (April 4, 2005)

Groupoids

We first need to know a few things about classifying spaces BG of groupoids G.

1) If G is a groupoid then BG is a Kan complex. In fact, we have

Exercise: Suppose that A is a small category. Then BA is a Kan complex if and only if A is a groupoid.

2) There is a monomorphism

$$\hom(\Delta^n, BG) \subset \hom(\mathrm{sk}_1\,\Delta^n, BG)$$

and an isomorphism

$$\hom(\Delta^n, BG) \cong \hom(\mathrm{sk}_2\,\Delta^n, BG),$$

both defined by restriction. Note that the fact that the first map is an inclusion (and that BG is a Kan complex) implies

Fact: $\pi_n(BG, x) = 0$ for $n \geq 2$ and for all $x \in BG_0 = \mathrm{Ob}(G)$.

3) Say that a functor $f : G \to H$ between groupoids is a weak equivalence if the map $f_* : BG \to BH$ is a weak equivalence.

Fact: $f : G \to H$ is a weak equivalence if and only if $f_* : \pi_0 G \to \pi_0 H$ is surjective and f is fully faithful in the sense that it induces bijections

$$\hom(x,y) \xrightarrow{\cong} \hom(f(x), f(y))$$

for all objects x, y of G.

Proof Suppose that f satisfies the conditions above. Then $[x] = [y] \in \pi_0 G$ if and only if there is a morphism $x \to y$ in G. If $[f(x)] = [f(y)]$ then $\hom(f(x), f(y)) \neq \emptyset$ so that $\hom(x,y) \neq \emptyset$. Thus the induced map

$$\pi_0 BG = \pi_0 G \xrightarrow{f_*} \pi_0 H = \pi_0 BH$$

is injective as well as surjective and is therefore a bijection. There is a natural identification

$$\pi_1(BG, x) = \hom_G(x,x)$$

for all $x \in BG_0 = \mathrm{Ob}(G)$. Then f is fully faithful so that all induced maps

$$\pi_1(BG, x) \to \pi_1(BH, f(x))$$

are isomorphisms. BG and BH have no higher homotopy groups, so that $BG \to BH$ is a weak equivalence.

If $BG \to BH$ is a weak equivalence, then $\pi_0 G \to \pi_0 H$ is a bijection, and is therefore surjective. Suppose that $\alpha : x \to y$ is a morphism of G. Then there is a commutative diagram

$$\begin{array}{ccc} \hom_G(x,x) & \xrightarrow{\cong} & \hom_H(f(x),f(x)) \\ {\scriptstyle \alpha_*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle f(\alpha)_*} \\ \hom_G(x,y) & \longrightarrow & \hom_H(f(x),f(y)) \end{array}$$

It follows that the bottom horizontal map is a bijection if $\hom_G(x,y) \neq \emptyset$. Finally, if $\hom_G(x,y) = \emptyset$ then $[x] \neq [y] \in \pi_0 G$. Then $[f(x)] \neq [f(y)] \in \pi_0 H$, so that $\hom_H(f(x),f(y)) = \emptyset$. $\square$

4) Note that a functor $f : G \to H$ is fully faithful if and only if the induced diagram of functions

$$\begin{array}{ccc} \mathrm{Mor}(G) & \longrightarrow & \mathrm{Mor}(H) \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathrm{Ob}(G) \times \mathrm{Ob}(G) & \longrightarrow & \mathrm{Ob}(H) \times \mathrm{Ob}(H) \end{array}$$

is a pullback. If $f : G \to H$ is also injective on objects in the sense that $\mathrm{Ob}(G) \to \mathrm{Ob}(H)$ is a monomorphism, then the map $\mathrm{Mor}(G) \to \mathrm{Mor}(H)$ is a monomorphism, and so f identifies G with a subcategory of H and the induced simplicial set map $BG \to BH$ is a monomorphism.

5) The fundamental groupoid functor $X \mapsto \pi(X)$ is left adjoint to $G \mapsto BG$ ("Van Kampen theorem"). Explicitly,

$$\pi(X) = GP(X)$$

is the free groupoid on the path category $P(X)$ of X. $\mathrm{Ob}(P(X)) = X_0$, and $P(X)$ is generated as a category by the 1-simplices $x \to y$ of X subject to the relation $d_1(\sigma) = d_0(\sigma)d_2(\sigma)$ for each 2-simplex σ of X.

Note that there is unique lifting in the diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & BG \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

for all $n \geq 2$, so that $\Lambda^n_k \subset \Delta^n$ induces an isomorphism $\pi(\Lambda^n_k) \cong \pi(\Delta^n)$ for $n \geq 2$. Also, the inclusions of vertices $\Delta^0 \to \Delta^1$ induce strong deformation retractions $\pi(\Delta^0) \to \pi(\Delta^1)$.

Lemma: Suppose that $i : G \to G'$ is a strong deformation retraction in the sense that there is a functor $\sigma : G' \to G$ with $\sigma \cdot i = 1$ and $i \cdot \sigma \simeq 1$ relative to G. Suppose given a pushout diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ G' & \longrightarrow & H' \end{array}$$

in groupoids. Then the map i_* is a strong deformation retraction.

Proof The map σ induces $\sigma_* : H' \to H$ such that $\sigma_* \cdot i_* = 1$. The homotopy $i \cdot \sigma \simeq 1$ is a map $h : G' \to (G')^{\mathbf{1}}$ which restricts to a constant homotopy $c : G \to G^{\mathbf{1}}$ on G. It follows that the composites

$$G' \xrightarrow{h} (G')^{\mathbf{1}} \to (H')^{\mathbf{1}} \quad H \xrightarrow{c} H^{\mathbf{1}} \to (H')^{\mathbf{1}}$$

together determine a homotopy $H' \to (H')^{\mathbf{1}}$ from $i_* \cdot \sigma_*$ to 1 which is constant on H. $\square$

Lemma: The functor $X \mapsto \pi(X)$ preserves weak equivalences.

Proof It is enough to show that the fund. groupoid functor takes trivial cofibrations to weak equivalences, by a factorization argument.

The fund. groupoid functor preserves pushouts. Thus, if

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & A \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & B \end{array}$$

is a pushout, then the induced map $\pi(A) \to \pi(B)$ is either an isomorphism or a strong deformation retract, and is therefore a weak equivalence. It follows that π takes anodyne cofibrations to weak equivalences and hence takes all trivial cofibrations to weak equivalences. □

Corollary: Suppose that $i : G \to G'$ is a functor between groupoids which is both a weak equivalence and injective on objects. Suppose that $f : G \to H$ is any groupoid morphism. Then the induced map $i_* : H \to G' \cup_G H$ is a weak equivalence.

Proof i is a strong deformation retraction. □

Presheaves of groupoids

Suppose that $\mathcal{C}$ is a small Grothendieck site.

A **presheaf of groupoids** G on $\mathcal{C}$ is a functor $G : \mathcal{C}^{op} \to \mathbf{Gpd}$ taking values in groupoids. Alternatively, G consists of presheaves $\mathrm{Mor}(G)$ and $\mathrm{Ob}(G)$ of morphisms and objects respectively, together with source and target maps $s, t : \mathrm{Mor}(G) \to \mathrm{Ob}(G)$, an identity $e : \mathrm{Ob}(G) \to \mathrm{Mor}(G)$ which is a section for both s and t, a law of composition

$$\mathrm{Mor}(G) \times_{t,s} \mathrm{Mor}(G) \to \mathrm{Mor}(G)$$

and an inverse

$$\begin{array}{ccc} \mathrm{Mor}(G) & \xrightarrow{\quad\sigma\quad} & \mathrm{Mor}(G) \\ & {\scriptstyle (s,t)}\searrow \qquad \swarrow{\scriptstyle (t,s)} & \\ & \mathrm{Ob}(G) \times \mathrm{Ob}(G) & \end{array}$$

all of which do the obvious things. Here, $\mathrm{Mor}(G) \times_{t,s} \mathrm{Mor}\, G$ is defined by the pullback

$$\begin{array}{ccc} \mathrm{Mor}(G) \times_{t,s} \mathrm{Mor}(G) & \longrightarrow & \mathrm{Mor}(G) \\ \downarrow & & \downarrow{\scriptstyle s} \\ \mathrm{Mor}(G) & \xrightarrow[t]{} & \mathrm{Ob}(G) \end{array}$$

It can (and should) be identified with the presheaf BG_2 of 2-simplices in BG. The object G is said to be a **sheaf of groupoids** if $\mathrm{Mor}\, G$ and $\mathrm{Ob}(G)$ are sheaves.

A functor $f : G \to H$ consists of presheaf maps $\mathrm{Mor}(G) \to \mathrm{Mor}(H)$ and $\mathrm{Ob}(G) \to \mathrm{Ob}(H)$ which respect the groupoid structure in the obvious way. Write $\mathrm{Pre}(\mathbf{Gpd})(\mathcal{C})$ and $\mathrm{Shv}(\mathbf{Gpd})(\mathcal{C})$ for the respective categories of presheaves and sheaves of groupoids on the site $\mathcal{C}$

1) A map $f : G \to H$ of presheaves of groupoids is said to be a **local weak equivalence** if the induced map $BG \to BH$ is a local weak equivalence of simplicial sheaves.

2) A map $p : G \to H$ of presheaves of groupoids is said to be a **global fibration** if $p : BG \to BH$ is a global fibration of simplicial presheaves.

3) **Cofibrations** in presheaves of groupoids are those maps which have the LLP wrt all trivial fibrations.

Facts:

a) Every cofibration $A \to B$ of simplicial presheaves induces a cofibration $\pi(A) \to \pi(B)$ of presheaves of groupoids.

b) Every map $f : G \to H$ of presheaves of groupoids has a factorization

$$\begin{array}{ccc} G & \xrightarrow{j} & K \\ & \searrow^{f} & \downarrow p \\ & & H \end{array}$$

where p is a global fibration and a local weak equivalence, and j is a cofibration which is injective on objects.

c) All cofibrations are injective on objects.

Proof a) is an adjointness argument.

b) is a small object argument in presheaves of groupoids, based on solving lifting problems for all maps $\pi(Y) \to \pi(L_U\Delta^n)$ arising from the inclusions $Y \subset L_U\Delta^n$. Note that π takes cofibrations to groupoid morphisms which are injective on objects, and that the class of groupoid morphisms which are injective on objects is stable under pushout and transfinite composition.

c) Every cofibration is a retract of a map of the form j in b), by a standard argument. □

Lemma: The class of trivial cofibrations of presheaves of groupoids is closed under pushout.

Proof Suppose given a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & G \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ B & \longrightarrow & H \end{array}$$

where i is a trivial cofibration. In order to show that i' is a trivial cofibration, it is enough to show that the induced map $i' : \tilde{G} \to \tilde{H}$ of sheaves of groupoids is a local weak equivalence.

Suppose that $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ is a Boolean localization. Then the diagram

$$\begin{array}{ccc} p^*\tilde{A} & \longrightarrow & p^*\tilde{G} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'_*} \\ p^*\tilde{B} & \longrightarrow & p^*\tilde{H} \end{array}$$

is a pushout of sheaves of groupoids. The map i_* is a weak equivalence and is injective in objects in all sections, and is therefore a strong deformation retract in each section. It follows that i'_* is a local (even sectionwise) weak equivalence. But then $i' : G \to H$ is a local weak equivalence. □

Lemma: The fundamental groupoid functor $X \to \pi(X)$ preserves local weak equivalences.

Proof It is enough to show that the functor $X \mapsto \tilde{\pi}(X)$ preserves local weak equivalences of locally fibrant simplicial sheaves, since there are local weak equivalences

$$\pi(X) \to \pi(\operatorname{Ex}^\infty X) \to \tilde{\pi}(\operatorname{Ex}^\infty X) \cong \tilde{\pi} L^2(\operatorname{Ex}^\infty X)$$

Here $\tilde{\pi} X = L^2(\pi(X))$.

Let $p : \operatorname{Shv}(\mathcal{B}) \to \operatorname{Shv}(\mathcal{C})$ be a Boolean localization. Then the direct image functor p_* preserves fibre products, and hence preserves sheaves of groupoids and their classifying objects. It follows that there is a natural isomorphism

$$p^* \tilde{\pi} X \cong \tilde{\pi} p^* X$$

for all simplicial sheaves X on $\mathcal{C}$. Thus, if $f : X \to Y$ is a local weak equivalence of locally fibrant simplicial sheaves, then the induced map $p^* X \to p^* Y$ is a sectionwise weak equivalence, and so $\pi p^* X \to \pi p^* Y$ is a sectionwise weak equivalence of presheaves of groupoids. It follows that $\tilde{\pi} p^* X \to \tilde{\pi} p^* Y$, or rather $p^* \tilde{\pi} X \to p^* \tilde{\pi} Y$ is a local weak equivalence, so that $\tilde{\pi} X \to \tilde{\pi} Y$ is a local weak equivalence. □

Here's another result that is used repeatedly:

Lemma: Suppose that $f : G \to H$ is a local weak equivalence of sheaves of groupoids. Then all functors $G(U) \to H(U)$ are fully faithful.

Proof For $x \in \mathrm{Ob}(G)(U)$, there is a sheaf isomorphism

$$G|_U(x,x) \cong \tilde{\pi}_1(BG|_U, x),$$

and this isomorphism is natural in G. The $\tilde{\pi}_1$-sheaf isomorphisms induced by j therefore induce sheaf isomorphisms $G|_U(x,x) \to H|_U(f(x), f(x))$ which restrict to isomorphisms

$$G(U)(x,x) \to H(U)(f(x), f(x))$$

in global sections for all $U \in \mathcal{C}$ and $x \in G(U)$. It follows that if $G(U)(x,y) \neq \emptyset$ then the induced maps

$$G(U)(x,y) \to H(U)(f(x), f(y))$$

are bijections. One further concludes that all of these maps are monomorphisms in general.

If $\beta : f(x) \to f(y)$ is some morphism of $H(U)$, then $[x] = [y]$ in $\tilde{\pi}_0 G(U)$, since $\tilde{\pi}_0 G \to \tilde{\pi}_0 H$ is a sheaf isomorphism. But then there is a covering sieve $R \subset \hom(\ , U)$ such that there is a morphism $g_\phi : \phi^*(x) \to \phi^*(y)$ in $G(V)$ for all $\phi : V \to U$ in R. We can choose g_ϕ such that $g_\phi \mapsto \phi^*(\beta)$ in $H(V)$. Then g_ϕ is an R-compatible family of maps, so that there is a morphism $g : x \to y$ in

$G(U)$ which specializes to all g_ϕ. This means that $G(U)(x,y) \neq \emptyset$ if $H(U)(f(x),f(y)) \neq \emptyset$, and so

$$G(U)(x,y) \to H(U)(f(x),f(y))$$

is a bijection for all $x,y \in G(U)$. $\square$

Corollary: A map $f : G \to H$ of sheaves of groupoids is a local weak equivalence if and only if all functors $G(U) \to H(U)$ in sections are fully faithful, and the sheaf map $\tilde{\pi}_0 G \to \tilde{\pi}_0 H$ is a local epimorphism.

Corollary: Suppose that $G \to H$ is a local weak equivalence of sheaves of groupoids which is injective on objects. Then the induced simplicial sheaf map $BG \to BH$ is a cofibration.

Proof The functor $G \to H$ is fully faithful and injective on objects in each section, and therefore induces a monomorphism $BG(U) \to BH(U)$ for each $U \in \mathcal{C}$ by remark **4)** above. $\square$

Theorem: With the definitions given above, the category Pre(**Gpd**)($\mathcal{C}$) of presheaves of groupoids on a small Grothendieck site $\mathcal{C}$ satisfies the axioms for a proper closed simplicial model category.

Proof One of the factorization axioms has already been proved.

Every map $f : G \to H$ has a factorization

$$\begin{array}{ccc} G & \xrightarrow{i} & L \\ & \searrow_{f} & \downarrow q \\ & & H \end{array}$$

where q is a global fibration and i is a cofibration and a local weak equivalence. The proof is a transfinite small object argument based on solving liftings for maps $\pi(A) \to \pi(B)$ associated to a generating set of trivial cofibrations $A \to B$ in the simplicial presheaf category. We need to know that trivial cofibration are closed under pushout to conclude that the map i is a trivial cofibration.

The map i in the factorization above also has the LLP wrt all fibrations. Every trivial cofibration is therefore a retract of a map which has the LLP wrt all fibrations, by the standard argument.

Suppose given pushout diagrams

$$\begin{array}{ccccc} \pi(X) & \longrightarrow & A & \xrightarrow{f} & G \\ {\scriptstyle i_*}\downarrow & & \downarrow & & \downarrow \\ \pi(Y) & \longrightarrow & C & \xrightarrow[f_*]{} & H \end{array}$$

where $i : X \to Y$ is a cofibration of simplicial presheaves and f is a local weak equivalence. Form the adjoint diagram

$$\begin{array}{ccccc} X & \longrightarrow & BA & \longrightarrow & BG \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & BC & \longrightarrow & BH \end{array}$$

and observe that there is an induced diagram

$$\begin{array}{ccc} \pi(Y \cup_X BA) & \longrightarrow & \pi(Y \cup_X BG) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ C & \xrightarrow[f_*]{} & H \end{array}$$

in which the vertical maps are isomorphisms since the fund. groupoid functor π preserves pushouts. The map

$$Y \cup_X BA \to Y \cup_X BG$$

is a weak equivalence by properness for simplicial presheaves, so that $f_* : C \to H$ is a local weak equivalence.

It follows that the class of local weak equivalences is preserved by pushouts along cofibrations.

We have established left properness. Right properness is trivial.

The simplicial model structure is given by

$$G \otimes \Delta^n = G \times \pi(\Delta^n).$$

Note that $\pi(\Delta^n)$ is the trivial groupoid on the set $\{0, 1, \ldots, n\}$. $\square$

We make similar definitions for sheaves of groupoids: a map $f : G \to H$ is a local weak equivalence (respectively global fibration) if $f_* : BG \to BH$ is a local weak equivalence (respectively global fibration) of simplical sheaves, and cofibrations are defined by a LLP wrt all trivial fibrations. Then we have the following, with essentially the same proof:

Theorem: (Joyal-Tierney) With the definitions given above, the category $\mathrm{Shv}(\mathbf{Gpd})(\mathcal{C})$ of sheaves of groupoids on a small Grothendieck site $\mathcal{C}$ satisfies the axioms for a proper closed simplicial model category.

Facts: The following hold for all small Grothendieck sites $\mathcal{C}$:

1) The associated sheaf and forgetful functors induce a Quillen equivalence

$$\mathrm{Ho}(\mathrm{Pre}(\mathbf{Gpd})(\mathcal{C})) \simeq \mathrm{Ho}(\mathrm{Shv}(\mathbf{Gpd})(\mathcal{C}))$$

2) The fundamental groupoid and classifying space functors induce Quillen adjunctions

$$\pi : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})) \leftrightarrows \mathrm{Ho}(\mathrm{Pre}(\mathbf{Gpd})(\mathcal{C})) : B$$

and

$$\tilde{\pi} : \mathrm{Ho}(s\,\mathrm{Shv}(\mathcal{C})) \leftrightarrows \mathrm{Ho}(\mathrm{Shv}(\mathbf{Gpd})(\mathcal{C})) : B$$

Stacks

Suppose that $R \subset \hom(U)$ is a sieve for $U \in \mathcal{C}$, and identify R with the full subcategory of $\mathcal{C}/U$ with objects the morphisms $\phi : V \to U$ of R.

There is a functor $R \to \mathrm{Pre}(\mathcal{C})$ which is defined by sending a morphism $\phi : V \to U$ to the presheaf $V = \hom(\ ,V)$. The corresponding translation category E_R is a presheaf of categories on $\mathcal{C}$: the objects of $E_R(W)$ are the strings of morphisms

$$W \to V \xrightarrow{\phi} U$$

with $\phi \in R$, and the morphisms of $E_R(W)$ are the commutative diagrams

$$\begin{array}{ccccc} & \nearrow & V & \overset{\phi}{\searrow} & \\ W & & \downarrow & & U \\ & \searrow & V' & \underset{\phi'}{\nearrow} & \end{array}$$

where ϕ and ϕ' are morphisms of R. There is a canonical functor $E_R \to \hom(\ ,U)$ which is defined by sending the string $W \to V \to U$ to the composite $W \to U$.

Lemma: Suppose that $R \subset \hom(\ ,U)$ is a covering sieve. The map $BE_R \to K(U, 0)$ is a local weak equivalence.

Proof The fibre of the functor $E_R(W) \to \hom(W,U)$ over the morphism $\alpha : W \to U$ is the nerve of all factorizations

$$\begin{array}{ccc} W & \longrightarrow & V \\ & {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle\phi} \\ & & U \end{array}$$

with $\phi \in R$. If $\alpha \in R$, then this nerve has an initial object, namely

$$\begin{array}{ccc} W & \xrightarrow{1} & W \\ & {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle\alpha} \\ & & U \end{array}$$

It follows that BE_R is sectionwise equivalent to the subpresheaf R of $\hom(\ ,U)$. R is a covering sieve, so that every morphism $\alpha : W \to U$ is locally in R: in effect, $\alpha^{-1}(R)$ covers W. The presheaf map $R \subset \hom(\ ,U)$ therefore induces an isomorphism of associated sheaves. □

Definition: Suppose that G is a sheaf of groupoids on $\mathcal{C}$. An **effective descent datum** on a covering sieve $R \subset \hom(\ ,U)$ is a functor $f : E_R \to G$, or equivalently a simplicial presheaf map $BE_R \to BG$.

An effective descent datum $f : E_R \to G$ is determined by a family of objects

$$x_\phi = f(V \xrightarrow{1} V \xrightarrow{\phi} U) \in G(V), \quad \phi \in R$$

together with morphisms $c_\psi : \psi^* x_\phi \to x_{\phi\psi}$ in $G(W)$ for each composable pair

$$W \xrightarrow{\psi} V \xrightarrow{\phi} U$$

in R such that $c_1 = 1$ and for each sequence

$$W' \xrightarrow{\omega} W \xrightarrow{\psi} V \xrightarrow{\phi} U$$

the diagram

$$\begin{array}{ccc} \omega^*\psi^* x_\phi & \xrightarrow{\omega^* c_\psi} & \omega^* x_{\phi\psi} \\ = \downarrow & & \downarrow c_\omega \\ (\psi\omega)^* x_\phi & \xrightarrow[c_{\psi\omega}]{} & x_{\phi\psi\omega} \end{array}$$

commutes. This is the old definition.

Note that the functors $E_R \to G$ and their natural transformations form a groupoid $\hom(E_R, G)$, and there is a functor $G(U) \to \hom(E_R, G)$.

Definition: A sheaf of groupoids G is a **stack** if all induced maps $\pi_0 G(U) \to \pi_0 \hom(E_R, G)$ are surjective.

Note that that functors $G(U) \to \hom(E_R, G)$ are already fully faithful since G is a sheaf of groupoids, so one could say that G is a stack if and only if all functors $G(U) \to \hom(E_R, G)$ are equivalences of groupoids.

Lemma: Suppose that G is a stack and that $j : G \to H$ is a local weak equivalence with H a globally fibrant sheaf of groupoids. Then j is a sectionwise equivalence.

Proof j is already sectionwise fully faithful since G and H are sheaves of groupoids and j is a local weak equivalence. It suffices to show that j induces an epimorphism

$$\pi_0 G(U) \to \pi_0 H(U),$$

or rather that j induces epimorphisms

$$\pi_0 BG(U) \to \pi_0 BH(U)$$

for all $U \in \mathcal{C}$. There is a commutative diagram

$$\begin{array}{ccc} \pi_0 BG(U) & \longrightarrow & \pi_0 BH(U) \\ \downarrow & & \downarrow \cong \\ \pi_0 H(U, BG) & \xrightarrow[\cong]{} & \pi_0(U, BH) \end{array}$$

It suffices, therefore, to show that $\pi_0 BG(U) \to \pi_0 H(U, BG)$ is surjective.

Suppose given an object

$$\begin{array}{ccc} X & \longrightarrow & BG \\ {\scriptstyle\simeq}\downarrow & & \\ U & & \end{array}$$

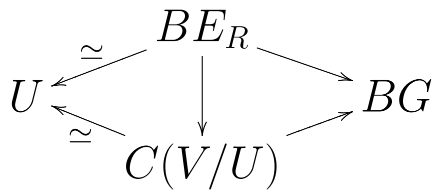
of $H(U, BG)$. Then $X_0 \to U$ is a local epimorphism, and the associated Čech resolution $C(X_0/U) \to U$ is a local weak equivalence. The Čech resolution $C(X_0/U)$ is also equivalent to the nerve of the fundamental groupoid $\tilde{\pi}X$, and there is a commutative diagram

$$\begin{array}{ccccc} & \overset{\simeq}{\swarrow} & X & \searrow & \\ U & & \downarrow & & BG \\ & \underset{\simeq}{\nwarrow} & C(X_0/U) & \nearrow & \end{array}$$

The local epimorphism $X_0 \to U$ can be refined by a covering family $V \to U$ in the sense that all such maps factor through X_0. It follows that there is a diagram

$$\begin{array}{ccccc} & \overset{\simeq}{\swarrow} & C(V/U) & \searrow & \\ U & & \downarrow & & BG \\ & \underset{\simeq}{\nwarrow} & C(X_0/U) & \nearrow & \end{array}$$

where $C(V/U)$ denotes the Čech resolution for the covering. Finally if R is the corresponding covering sieve, $C(V/U)$ is the nerve of the fundamental groupoid for BE_R, so there is a diagram



The corresponding effective descent datum $BE_R \to BG$ factors uniquely up to homotopy through a map $U \to BG$, since G is a stack. $\square$

Corollary: A sheaf of groupoids G is a stack if and only if it satisfies descent.

Proof Suppose that G is a globally fibrant sheaf of groupoids, and let $R \subset \hom(\ ,U)$ be a covering sieve. The local weak equivalence $BE_R \to U$ induces a weak equivalence

$$\mathbf{hom}(U, BG) \to \mathbf{hom}(BE_R, BG)$$

since BG is globally fibrant. In particular, the map

$$\pi_0 G(U) \to \pi_0 \hom(E_R, G) \tag{1}$$

is an isomorphism.

Note that the map (1) is an invariant of sectionwise equivalence. Thus, if G satisfies descent then it is sectionwise equivalent to a globally fibrant model, and so the map (1) is surjective. $\square$

The moral is that any globally fibrant model $j : G \to H$ of a sheaf (or presheaf) of groupoids G is its "associated stack".

There is a stack completion functor $G \mapsto \mathrm{St}(G)$ from days of yore. Specifically, $\mathrm{St}(G)$ is the sheaf associated to the presheaf of groupoids $\mathrm{St}^p(G)$, where

$$\mathrm{St}^p(G)(U) = \varinjlim_{R \subset \hom(\ ,U)} \hom(E_R, G)$$

where the (filtered) colimit is indexed over all covering sieves. Then one shows

Lemma:

1) $\mathrm{St}(G)$ is a stack.

2) The canonical map $G \to \mathrm{St}(G)$ is a local weak equivalence.

Proof (sketch)

For 2) note that the map $G \to \mathrm{St}^p(G)$ is fully faithful in all sections, so it suffices to show that $\pi_0 G \to \pi_0 \mathrm{St}^p(G)$ is a local epimorphism. One finishes by observing that every effective descent datum lifts locally to G.

One proves 1) by showing that every effective descent datum in $\mathrm{St}(G)$ can be refined by an effective descent datum in G. □

More generally, a **stack** can be taken to be a presheaf of groupoids which satisfies descent.

Examples:

1) Suppose that G is a sheaf of groups. A (classicial) G-torsor is a sheaf X with a G-action such that the canonical map $EG \times_G X \to *$ is a local weak equivalence. ($\tilde{\pi}_0 = *$ means that G acts effectively, $\tilde{\pi}_1 = e$ means that G acts principally). If X is a G-torsor on $\mathcal{C}$ then $X|_U$ is a $G|_U$-torsor for $\mathcal{C}/U$. Any G-equivariant morphism $X \to Y$ is an isomorphism (since X is the fibre of $EG \times_G X \to BG$). Write $\mathbf{Tors}_G(U)$ for the groupoid of $G|_U$-torsors on $\mathcal{C}/U$. $\mathbf{Tors}_G$ is the resulting sheaf of groupoids. There is a distinguished element of $\mathbf{Tors}_G(U)$, given by the group $G|_U$, and the collection of all such elements define a groupoid map $G \to \mathbf{Tors}_G$. Then we have the following:

a) $G \to \mathbf{Tors}_G$ is a local weak equivalence.

b) $\mathbf{Tors}_G$ is a stack.

2) Suppose that a group G acts on a (pre)sheaf N. Then the Borel construction $EG \times_G N$ is the nerve of a groupoid, namely the translation groupoid $E_G N$ for the G-action. The corr. **quotient stack** is the stack completion of $E_G N$.

3) A **gerbe** is a locally connected stack ...