# RESEARCH STATEMENT

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## 1. Introduction

My current work is part of an effort by many researchers, including John Baez, Louis Crane, Laurent Freidel, Dan Freed, and Ruth Lawrence, to apply ideas from higher dimensional category theory to mathematical physics, and especially quantum gravity and the foundations of quantum mechanics.

In particular, I have studied applications of the program of "groupoidification" initiated by Baez and Dolan, and have developed an extension of this program which I call "2-linearization". The basis of this program is a map $|\cdot| : Span(Gpd) \rightarrow Vect$ from the category of groupoids and spans into that of vector spaces and linear maps. I have shown a similar functor $\Lambda : Span(Gpd) \rightarrow 2Vect$ 2-vector spaces in the sense of Kapranov and Voevodsky. These are "higher categorical" analogs of vector spaces.

Physical applications of this program arise from the fact that groupoids can be used to describe spaces (of classical *states*) and their local *symmetries*. In this approach, processes are described by *spans* of groupoids: that is, pairs of maps $A \leftarrow X \rightarrow B$, where the objects of $X$ are the space of histories starting at a state in $A$ and ending at a state in $B$. Both groupoidification and 2-linearization are related to constructions that can be interpreted in terms of a "sum-over-histories". Taking spans of groupoids into $Vect$, the category of vector spaces and linear maps, gives models of quantum mechanical systems, which are described in terms of Hilbert spaces and linear operators. The 2-linearization extends this picture in ways which have produced results of possible physical significance.

One application I have studied is the groupoidification of the quantum harmonic oscillator, which turns out to provide a direct link between the combinatorics of Feynman diagrams and an algebraic representation of this system. Another application involves using 2-linearization to construct Extended Topological Quantum Field Theories (which, moreover, are related to 3 dimensional quantum gravity).

### 1.1. Categorifying the Quantum Harmonic Oscillator.

The quantum harmonic oscillator is a fundamental entity in quantum mechanics, and indeed quantum field theory, since any field can be treated as a system of (possibly interacting) oscillators. I described a combinatorial interpretation of the Heisenberg representation of the oscillator, using a generalization of the *stuff types* of Baez and Dolan, which in turn generalize the *structure types* of Joyal. These have a natural interpretation in terms of spans of groupoids, and the groupoidification program. In a new result, my paper showed that there is a well-defined notion of $U(1)$-stuff type, where objects of the groupoids involved are labeled by phases in $U(1)$. The category of all such $U(1)$-stuff types is the categorification of the Fock space for the oscillator.

All the operations of the Hilbert space in the usual quantum system, such as the inner product which gives probabilities, have corresponding structures in this category. So do the "raising" and "lowering" operators which are the generators of the algebra of observables.

These create or annihilate "quanta" of energy in the oscillator. The combinatorial equivalent of these operators correspond to "Feynman diagrams", and reproduce exactly the usual Feynman rules for the harmonic oscillator.

1.2. **Extended Topological Quantum Field Theory.** One physically interesting example of a concept which has been framed in categorical language is that of Topological Quantum Field Theories (TQFT's). Atiyah showed that the axioms for a TQFT are exactly those for a monoidal functor from a category of spaces and cobordisms into the category *Hilb* of Hilbert spaces and bounded linear operators. This assigns a Hilbert space to each manifold, and an operator to each *cobordism* interpolating between manifolds.

TQFT's are interesting to physicists because they are quantum field theories with no local degrees of freedom. They are interesting to mathematicians, particularly topologists, as a tool for so-called *quantum topology*, which involves assigning algebraic data to topological objects as in quantum field theory, and is related to the classification of knots, for example, since Reshitikhin and Turaev exhibited a TQFT which produces the Jones polynomial for knots.

Extended Topological Quantum Field Theories (ETQFT's) were proposed by Ruth Lawrence and Dan Freed, among others, as a "categorification" of Atiyah's axiomatization of a TQFT. These would map from $n$-categories of cobordisms into $n$-vector spaces: in particular, into Kapranov-Voevodsky 2-vector spaces in the case $n = 2$. In my thesis I gave a construction using 2-linearization to provide an analogous construction.

An Extended TQFT is then a weak 2-functor assigning 2-Hilbert spaces to boundaries of manifolds, and other higher algebraic data to other parts of the manifolds. I gave a construction which gives such an ETQFT for each finite group $G$, using the spans of groupoids of flat $G$-connections on $S$ (and conjectured the same construction should work for $G$ a compact Lie group). In the case of finite groups, this turns out to be the same as the (untwisted) Dijkgraaf-Witten model, a known topological gauge theory. The "states" of such a theory denote the geometric structure on a "slice" of space, and processes between states are described in terms of the geometric structure on the spacetime between them, giving a span of groupoids as before.

As part of the ETQFT program, I developed general category-theoretic tools for representing spacetimes, and the evolution of spacetimes with boundaries, in terms of spans. In particular, I described how "double spans" give a broad class of examples of structures called double bicategories. These are used in the ETQFT construction to represent portions of spacetime which can be "glued" along boundaries in time (so that one portion follows another), or boundaries in space (so that the portions of spacetime are adjacent in a space direction).

1.3. **ETQFT's and Quantum Gravity.** Work on TQFT's is of interest in topology and geometry - particularly knot theory - and also in mathematical physics. One key example is the fact that quantum gravity in 3D can be represented as a TQFT. In particular, in 3D, the circle appears as a component of a *boundary* of a 2-dimensional slice of space. The ETQFT I construct assigns to the circle a 2-vector space with a basis labeled by a choice of isomorphism class of $G$ and representation of its automorphism group. In the case of 3D gravity, where $G = SU(2)$, these can be interpreted as giving the total mass and spin of some matter. So categorification produces a theory containing matter from one which described only a vacuum theory.

This realizes - at least in a toy model - a program which goes back at least to Wheeler, and has recently been taken up by Freidel, among other researchers. Specifically, the view of "matter as holes", in which holes in spacetime can be treated as matter. The ETQFT gives exactly this picture in 3 dimensions.

## 2. Future Research

2.1. **Mathematical Questions.** One major area in which further work needs to be made to make these physical applications more broadly applicable is in the intersection of categorical methods with analysis. To handle the "2-vector spaces" which would arise from groupoids of connections with infinite gauge groups such as Lie groups, one can appeal to a notion based the 2-category $\mathbf{Meas}(X)$ introduced by Louis Crane and David Yetter, for a measurable space $X$. This is the 2-category of measurable fields of Hilbert spaces on $X$ - a notion which goes back to von Neumann and the original development of Hilbert spaces, and which was developed by Dixmier. In my thesis I outline a closely related concept of 2-Hilbert spaces for measure space $X$, and call this $\mathbf{2L^2}(X)$, which captures more of the structure of Hilbert spaces, such as adjoint operators. A similar construction for groupoids with measure should be canonical examples of infinite-dimensional 2-Hilbert spaces.

In the next stage of my program, I aim to make this more precise by giving a rigorous treatment of infinite-dimensional 2-Hilbert spaces, and rigorously extend the work in my thesis from the case of finite groups to compact Lie groups. This will build upon the work of Crane and Yetter mentioned above, as well as of Baez on finite-dimensional 2-Hilbert spaces, Michael Wendt on categories of measurable Hilbert sheaves, and others. These fields of spaces give an analog of the space of measurable functions on $X$.

2.2. **Physical Applications.** With this background, it should be possible to extend the results about 2-linear maps from spans of groupoids to the smooth case, at least for compact Lie groups, where there is a measure to build a 2-Hilbert space. This would give an Extended TQFT, described in terms of a categorification of the "sum over histories" picture of field theories in physics. In particular, I will show that the case in dimension 3 with $G = SU(2)$ does indeed recover the Ponzano-Regge model of gravity—or for any dimension and compact $G$, a BF theory—with particle insertions, consistent with the results for finite $G$. By not removing the gauge transformations from the groupoid—by taking the moduli *stack* rather than the moduli *space*—we leave room for the theory to contain matter.

Field theories related to "quantum" invariants of manifolds are of particular interest to both topologists and physicists, and this setup should then give "categorified" versions of these invariants, which should again give invariants for manifolds with boundary. This relies on the fact that groupoids of connections are invariants for manifolds. Other structures, such as Gromov-Witten invariants, or Conformal Field Theories, can be formulated in terms of groupoids of geometric structures on manifolds, and I expect that these may be "Extended", in the same fashion as TQFT'-, to the setting of manifolds with boundary.

Finally, a fuller picture of the topology of a manifold is given by its fundamental $n$-groupoid, which suggests a role for an $n$-groupoid of connections valued in an $n$-group. This is the program of *higher gauge theory*, as suggested by Baez, and touches on research into topological invariants by Kauffman, Yetter, and others. In particular, these invariants provide tools for studying the geometry and topology of manifolds, directly connected to physically interesting field theories, and should have generalizations analogous to ETQFT's.

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