ANALOGUE OF THE BRUHAT DECOMPOSITION FOR ALGEBRAIC MONOIDS II.

THE LENGTH FUNCTION AND THE TRICHOTOMY

Lex E. Renner

0. Introduction.

The purpose of this paper is to continue the study of $B \times B$ orbits on a reductive monoid M. This was first taken up systematically by the author in [9], where several basic properties analogous to the case of groups were established. The $B \times B$ orbits were identified with a finite monoid $\mathcal{R}$, defined in terms of the normalizer of a maximal torus, and a direct analysis of these orbits led to an extension of Tits' axiom "$\rho Br \subseteq BrB \cup B\rho rB$" to the case of reductive monoids. However, there is more to this axiom when dealing with monoids. Obviously,

$$B\rho BrB = \begin{cases} BrB, \\ B\rho rB, \text{ or,} \\ BrB \cup B\rho rB. \end{cases}$$

But in the case of groups, the first possibility does not occur, while for monoids it is ubiquitous.

Putcha [5] has discussed a truly semigroup theoretic approach to "monoids with BN pair". However, it is not yet clear how to define a suitable length function at that generality, nor is it clear that one could obtain the analogue of Tits' axiom.

At the other extreme, Solomon [10] has obtained a length function $l : \mathcal{R}_n \to \mathbb{N}$ for the $B \times B$ orbits $\mathcal{R}_n$ of $M_n(\mathbb{F}_q)$, guided by the requirement that it provides the correct relationship between Iwahori–Hecke algebra constructions and ennumerative formulae. For example, he obtains the formula,

$$|M_n(\mathbb{F}_q)^r| = (q-1)^r q^{r(r-1)/2} \sum_{x \in \mathcal{R}_n^r} q^{l(x)}$$

where $M_n(\mathbb{F}_q)^r = \{A \in M_n(\mathbb{F}_q) | rk(A) = r\}$, and $\mathcal{R}_n^r = \{x \in \mathcal{R}_n | rk(x) = r\}$. A key point in Solomon's analysis in the observation that, for any reasonable length function, there is a unique element of minimal length in $\mathcal{R}^r$.

The main results of this paper provide a detailed analysis of the length function for any reductive monoid M. Our more abstract approach also leads us to the set of *order preserving* elements O of $\mathcal{R}$. This is an entirely new notion which is not suggested by any results from group theory. The main structure theorem of section 3 states that any $r \in \mathcal{R}$ can be written uniquely as where $r\mathcal{J}r_+\mathcal{J}r_0\mathcal{J}r_-$, $r_+, r_- \in O$ and $BHB = H$, where H is the $\mathcal{H}$-class of r_0 in M. Furthermore, $BrB = Br_+Br_0Br_-B$. This leads to a varied assortment of results about the $B \times B$ orbits on M (see 3.3, 3.4, 3.10, 3.11, 4.2, 4.3).

In a sequel to this paper we intend to obtain cell decompositions for compactifications (even the singular ones) of algebraic groups. While this can be done in the nonsingular case using G_m-actions [2], we expect to obtain more detailed information, even in this case, about the cells using the result of the present paper.

1. The Length Function. Let M be a reductive monoid with unit group G and let $B \subseteq G$ be a Borel subgroup with maximal torus T. Let $R = \overline{N_G(T)} \subseteq M$ (Zariski closure) and define

$$\mathcal{R} = R/T = T\backslash R.$$

In [9] we obtained the following results:

1.1. Theorem.

(a) *$\mathcal{R}$ is a finite inverse monoid with unit group W, the Weyl group of T.*
(b) *$E(\mathcal{R}) = \{e \in \mathcal{R} | e^2 = e\} = E(\overline{T})$.*
(c) *$M = \bigsqcup_{r\in\mathcal{R}} BrB$, disjoint union.*
(d) *If $S \subseteq W$ is the set of simple involutions relative to B and T, then $\rho Br \subseteq BrB \cup B\rho rB$ if $\rho \in S$ and $r \in \mathcal{R}$.*

It is our intention in this section to introduce a length function on $\mathcal{R}$ (following [10]) and to establish its most important properties. First we find the length zero elements.

By [8; Theorem 8.2] there exists an antiinvolution $\tau : M \to M$ so that

$$\begin{aligned} \tau^2(x) &= x, \\ \tau(xy) &= \tau(y)\tau(x), \\ \tau|T &= \text{id, and} \\ \tau(B) &= B^-, \text{ the opposite Borel subgroup.} \end{aligned}$$

Let $w \in N_G(T)$ represent the longest element (so $wBw^{-1} = B^-$) and define

$$\theta = int(w) \circ \tau.$$

1.2 Proposition. *For each $W \times W$ orbit J of $\mathcal{R}$, there exists a unique $\nu \in J$ such that $B\nu = \nu B$.*

Proof. First notice that $\theta(B) = B$, and $\theta(e) = wew^{-1}$ for all $e \in E(\mathcal{R})$, since $\tau(e) = e$ for all $e \in E(\mathcal{R})$.

So let $\Lambda = \{e \in E(\mathcal{R}) | Be \subseteq eB\}$ be the cross section lattice. By the results of [3], $|\Lambda \cap J| = 1$, for each $W \times W$ orbit J of $\mathcal{R}$. So let $\Lambda \cap J = \{e\}$. By [9; Theorem 9.6] there exists a unique $\nu \in \mathcal{R}$ such that $B\nu \subseteq \nu B$ and $\nu \in W \cdot f$, where $f = wew^{-1}$. But if we write $\nu = e'\sigma$ for some $e' \in J$ we obtain $Be'\sigma \subseteq e'\sigma B$, so that $Be' \subseteq e'B$, since $e'Be' \subseteq e'B$. Thus, $e' = e$ since (as noted above) $\Lambda \cap J = \{e\}$. To summarize, we can write $\nu = e\sigma = \sigma f$ for some $\sigma \in W$ such that

$$\sigma^{-1}e\sigma = \theta(e) = wew^{-1} = f$$

But also

$$\begin{aligned} fB &= \theta(e)\theta(B) \\ &= \theta(Be) \\ &\subseteq \theta(eB) \\ &= \theta(B)\theta(e) \\ &= Bf. \end{aligned}$$

So $fB = fBf$, and thus $\nu B = \sigma fBf$. But also, $Be = eBe$ and so $B\nu = eBe\sigma$. On the other hand $B\nu \subseteq \nu B$ and so $B\nu = B\nu f \subseteq \nu Bf$. Hence $eBe\sigma \subseteq \nu Bf$. But then $eBe \subseteq \sigma fBf\sigma^{-1}$. But these are both Borel subgroups of H_e, and thus $eBe = \sigma fBf\sigma^{-1}$. Finally, $B\nu = eBe\sigma = \sigma fBf = \nu B$. □

1.3 Definition. Define $l : \mathcal{R} \to \mathbb{N}$ by $l(r) = \dim(BrB) - \dim(B\nu B)$, where $\nu \in WrW$ is the unique element such that $B\nu = \nu B$.

Remark. This agrees in spirit with Solomon's definition in [10] for $M = M_n(\mathbb{F}_q)$ and $\mathcal{R}$ =the rook monoid. His approach however, is entirely different.

1.4 Theorem. *Let $S \subseteq W$ be the set of simple reflections relative to B and T. If $\rho \in S$ and $r \in \mathcal{R}$ then*

$$B\rho BrB = \begin{cases} BrB \text{ if } l(\rho r) = l(r) \\ B\rho rB \text{ if } l(\rho r) = l(r) + 1 \\ B\rho rB \cup BrB \text{ if } l(\rho r) = l(r) - 1. \end{cases}$$

Proof. By 1.1 (d), $B\rho BrB = BrB$, $B\rho rB$ or $B\rho rB \cup BrB$.

Suppose $B\rho BrB = BrB$. Then $\rho r \in BrB$ and so by 1.1 (c) $\rho r = r$. Thus, $l(\rho r) = l(r)$.

Suppose $B\rho BrB = B\rho rB$. So, in particular, $\rho(BrB) \subseteq B\rho rB$, and it follows that $l(\rho r) \geq l(r)$. But also $B\rho B = U_\alpha \rho B$ for some unique $\alpha \in \triangle \subseteq \Phi^+$. So $B\rho BrB = U_\alpha \rho BrB$, and it follows that $l(\rho r) \leq l(r) + 1$. Hence, either $l(\rho r) = l(r)$ or $l(\rho r) = l(r) + 1$. If $l(\rho r) = l(r)$ then $\rho BrB = B\rho rB$. So $u\rho BrB = v\rho BrB$ for any $u, v \in U_\alpha$. But also $aBrB = bBrB$ for any $a, b \in U_{-\alpha}$ since $\rho = \sigma_\alpha$, the simple reflection associated with α. So $BrB = U_{-\alpha}BrB$, and consequently, $P_\alpha rB = BrB$, where $P_\alpha = P \cup B\rho B$. But $\sigma_\alpha = \rho \in P_\alpha$ and so $\rho r \in BrB$. Then by 1.1 (c) $\rho r = r$. It follows also from this arguement that if $B\rho BrB = B\rho rB$ and $\rho r \neq r$, then $l(\rho r) = l(r) + 1$.

Suppose $B\rho BrB = B\rho rB \cup BrB$, and without loss of generality $\rho r \neq r$. Then $B\rho BrB = (B\rho B \cup B)rB = P_\alpha rB$, which is a subvariety of M. Furthermore, $B\rho rB \subsetneqq P_\alpha rB = P_\alpha \rho rB$. Let $X = \{b \in B | \rho br \in B\rho rB\} \subsetneqq B$. Write $B = VU_\alpha = U_\alpha V$, where $V \subseteq B$ is the closed, normal, codimension one subgroup of B such that $\rho V = V\rho$. So if $b \in X$ and $v \in V$,

$$\rho vbr = v'\rho br \in B\rho rB.$$

Hence $VX \subseteq X$. But also $V \subseteq X$. Now let $u \in U_\alpha$ and suppose that $\rho ur \in B\rho rB$. But if $t \in T$ then $\rho tut^{-1}r = t'\rho urt'' \in B\rho rB$. It is well known that U_α consists of two conjugacy classes under T: $\{1\}$ and U $\setminus \{1\}$. We conclude that $u = 1$

since otherwise $U_\alpha \subseteq X$ which is impossible. Hence, $X = V$, and furthermore $\rho(B \setminus V)r \subseteq BrB$. Thus,

$$BrB = B\rho(B \setminus V)rB.$$

So $BrB \subseteq B\rho BrB = P_\alpha rB$ is open and dense, with $B\rho rB \subseteq \overline{BrB}$. So $l(\rho r) < l(r)$. But $\overline{P_\alpha rB} \subseteq M$ is an irreducible affine variety with $B \times B$ acting with finitely many orbits. Thus, by [9; Theorem 8], the set of $B \times B$ orbits forms a ranked poset under the closure operation. The rank function is $\zeta(r) = \dim(BrB)$. Since $P_\alpha rB \subseteq \overline{P_\alpha rB}$ is open, it follows that $\dim(B\rho rB) = \dim(BrB) - 1$. Thus, $l(\rho r) = l(r) - 1$. This establishes the theorem completely. $\square$

We now establish a few useful facts related to this length function.

1.5 Proposition. *Let* $r = e\sigma \in \mathcal{R}$. *Then* $\dim(BrB) = \dim(Br) + \dim(rB) - \dim(Br \cap rB)$.

Proof. Define $\zeta : Be \times rB \to BrB$ by $\zeta(b_1, erb_2) = b_1, erb_2 = b_1, rb_2$. Now $\dim(Be) = \dim(Br)$. Furthermore, all fibres of ζ have the same dimension since ζ is equivariant for the $B \times B$ action. So it suffices to show that $\zeta^{-1}(r) \cong Br \cap rB$. Define $f : \zeta^{-1}(r) \to Br \cap rB$ by $f(x, y) = xr$. This makes sense since if $x = b_1 e$ and $y = rb_2$, with $b_1 erb_2 = r$, then $b_1 r = rb_2^{-1}$. So indeed, $xr = b_1 er = b_1 r \in Br \cap rB$. ζ is surjective since, if $z = b_1 r = rb_2 \in Br \cap rB$ then $z = f(zr^*, z)$, where $r^* = \sigma^{-1}e \in R$ is the inverse of r. But f is also injective since if $x_1, x_2 \in Be$ and $x_1 r = x_2 r$ then $x_1 rr^* = x_2 rr^*$, while $x_1 rr^* = x_1 e = x_1$ and $x_2 rr^* = x_2$. $\square$

1.6 Proposition. *If* $r = e\sigma = f\sigma \in R$ *then* $Br \cap rB = eBr \cap rBf$.

Proof. Let $V = Br \cap rB$. Then $V\sigma^{-1} = Be \cap eB^\sigma$. So if $x \in V\sigma^{-1}$ then $x = eb^\sigma$ for some $b \in B$. Thus, $x = ex$. But also $x = b'e$ for some $b'B$. Hence, $x = ex = eb'e \in eBe$. Therefore, $v\sigma^{-1} \subseteq eBe$; or $V \subseteq eBr$. Similarly, $V \subseteq rBf$.

Conversely, if $x \in eBr \cap rBf$ then recall that $eBr = C_B(e)r$ and $rBf = rC_B(f)$. So $x \in C_B(e)r \cap rC_B(f) \subseteq Br \cap rB$. $\square$

2. Order Preserving.

In this section we consider the relationship of elements of $\mathcal{R}$ to B, and obtain the subset of *order preserving* elements. The standard example here is $M = M_n(k)$, where $\mathcal{R}$ can be identified with the set of 01 matrices with at most one nonzero entry in each row or column. An element $r \in \mathcal{R}$ is order preserving if the matrix obtained from r by deleting all the zero rows and all the zero columns is an identity matrix. The general definition is as follows.

2.1 Definition. Let $O \subseteq \mathcal{R}$ be the subset of elements $r \in \mathcal{R}$ with the property

$$rBr^* \subset Brr^*$$

where $r^* \in \mathcal{R}$ is the unique element satisfying $rr^*r = r$ and $r^*rr^* = r^*$. Any element with this property is called *order preserving.*

2.2 Lemma.

(a) *If* $r, s \in O$ *then* $rs \in O$.
(b) $W \cap O = \{1\}$.
(c) $E(\mathcal{R}) \subseteq O$.
(d) *If* $r \in O$ *then* $rBr^* \subseteq rr^*Brr^*$

Proof.

(a) Suppose $rBr^* \subseteq Brr^*$ and $sBs^* \subseteq Bss^*$. Consider $ss^*r^* \in \mathcal{R}$. Write $ss^* = e$ and $r^* = f\sigma^{-1}$. Then $ss^*r^* = ef\sigma^{-1} = fe\sigma^{-1} = f\sigma^{-1}\sigma e\sigma^{-1} = r^*(ss^*)^\sigma$. Now compute:

$$\begin{aligned} rsB(rs)^* &= rsBs^*r^* \subseteq rBss^*r^* \\ &= rBr^*(ss^*)^\sigma \subseteq Brr^*(ss^*)^\sigma \\ &= Brss^*r^* = B(rs)(rs)^*. \end{aligned}$$

(b) Suppose $r \in W \cap O$. Then $r^* = r^{-1}$, and $rBr^* \subseteq Brr^*$ implies $rBr^{-1} = B$.

(c) If $e \in E(\mathcal{R})$ then by [4; Theorems 6.16 and 6.30] $eBe^* = eBe = eC_B(e)e \subseteq Be = Bee^*$.

(d) Write $r = e\sigma$ so that $r^* = \sigma^{-1}e$ and $rr^* = e$. Then $rBr^* \subseteq eM \cap Be$. But if $be \in eM \cap Be$ then $be = ex$, and so $ebe = ex = be$. So $be \in eBe$. Hence, $rBr^* \subseteq eM \cap Be \subseteq eBe = rr^*Brr^*$. ☐

2.3 Corollary. *The following are equivalent.*

(a) $rBr^* \subseteq Brr^*$.

(b) $rBr^* \subseteq rr^*B$.

(c) $rBr^* \subseteq rr^*Brr^*$.

Proof. $rr^*Brr^* \subseteq Brr^* \cap rr^*B$. ☐

Recall now the involution $\tau : M \to M$. By [8; Theorem 8.2], it has the properties

$$\begin{aligned} \tau|T &= id, \\ \tau(B) &= B^-, \\ \tau^2 &= id \\ \tau(xy) &= \tau(y)\tau(x), \text{ and} \\ \tau(r) &= r^*, \text{ for all } r \in \mathcal{R}. \end{aligned}$$

Recall also $w \in W$, the longest element. It satisfies $wBw^{-1} = B^-$.

2.4 Proposition. *Let* $O^- = \{r \in \mathcal{R} | rB^-r^* \subseteq B^-rr^*\}$ *and let* $O^w = \{wrw^{-1} \in \mathcal{R} | r \in O\}$. *Then* $O = O^- = O^w$.

Proof. Let $r \in \mathcal{R}$. Then $r \in O$ iff $rBr^* \subseteq Brr^*$ iff $\tau(rBr^*) \subset \tau(Brr^*)$ iff $rB^-r^* \subseteq rr^*B^-$ iff $rB^-r^* \subseteq B^-rr^*$ (by 2.3) iff $r \in O^-$. So $O = O^-$.

Again if $r \in \mathcal{R}$, $r \in O^w$ iff $wrwB(wrw)^* \subseteq Bwrw(wrw)^*$ iff $wrB^-r^*w \subseteq Bwrr^*w$
iff $rB^-r^* \subseteq B^-rr^*$ iff $r \in O^-$. So $O^w = O^-$. ☐

2.5 Proposition. *Let* $O^* = \{r^* | r \in O\}$. *Then* $O = O^*$. *In particular,* O *is an inverse monoid.*

Proof. $r \in O$ iff $rBr^* = rr^*Brr^*$ iff $r^*rBr^*r = r^*rr^*Brr^*r = r^*Br$ iff $r \in O^*$. So $O = O^*$. ☐

2.6 Lemma. *Let $e\sigma \in \mathcal{R}$. Then there exists $c \in C_W(e)$ such that $ce\sigma \in O$. In particular, $ce\sigma \mathcal{H} e\sigma$.*

Proof. $eB^{\sigma}e$ and eBe are Borel subgroups of H_e containing eT. Thus, there exists $c \in C_W(e)$ such that $c(eB^{\sigma}e)c^{-1} = eBe$. But, $c(eB^{\sigma}e)c^{-1} = eB^{c\sigma}e$ and so $ce\sigma = ec\sigma \in O$. ☐

2.7 Proposition. *Let e, $f \in E(\mathcal{R})$, $e\mathcal{J}f$. Choose $\sigma \in W$ of minimal length such that $\sigma^{-1}e\sigma = f$, and let $r = e\sigma$. Then $r \in O$.*

Proof. We proceed by induction on this minimal length l. If $l = 0$ then $e = f$ and the result follows from 2.2(c). So assume $l = n + 1$. We can write $e\sigma = e\tau\rho$ where $l(\tau) = n = l(\sigma) - 1$. Now $e\tau \in O$ since if $\zeta^{-1}e\zeta = \tau^{-1}e\tau$ with $l(\zeta) < l(\tau)$ then $(\zeta\rho)^{-1}e(\zeta\rho) = \sigma^{-1}e\sigma$ with $l(\zeta\rho) < l(\sigma)$, a contradiction. Also $\tau^{-1}e\tau\rho \neq \rho\tau^{-1}e\tau$ since otherwise $\tau^{-1}e\tau = \sigma^{-1}e\sigma$ which contradicts the minimality of $l(\sigma)$. So we show that $\tau^{-1}e\tau \cdot \rho \in O$. Then by 2.2(a) $e\tau \cdot \tau^{-1}e\tau\rho = e\sigma \in O$. So let $f = \tau^{-1}e\tau$. Then we have $f\rho \neq \rho f$. Now $\rho = \sigma_{\alpha}$ for some unique $\alpha \in \triangle$ and there exists a closed subgroup $V \subseteq B$ such that $B = VU_{\alpha} = U_{\alpha}V$ and $B^{\rho} = VU_{-\alpha} = U_{-\alpha}V$. By [9; Lemma 5.1] we have that either $fU_{\alpha} = \{f\}$ and $U_{-\alpha}f = \{f\}$ or $U_{\alpha}f = \{f\}$ and $fU_{-\alpha} = \{f\}$. In either case $fBf = fVf = fB^{\rho}f$. Hence, $f\rho \in O$. ☐

2.8 Theorem. *Suppose $r, s \in \mathcal{R}$, $r\mathcal{H}s$, and $r, s \in O$. Then $r = s$.*

Proof. Let $r = e\sigma$ and $s = ce\sigma$ where $ce = ec$. Then $eB^{\sigma}e = eBe = eB^{c\sigma}e = c(eB^{\sigma}e)c^{-1}$. But then $ce \in N_{H_e}(eT) \cap eB^{\sigma}e$ and hence $ce = e$. ☐

There are other ways to characterize the elements of O.

2.9 Proposition.

Let $r = e\sigma = \sigma f \in \mathcal{R}$. The following are equivalent:

(a) $r \in O$
(b) $Br \cap rB = eBr$
(c) $Br \cap rB = rBf$
(d) $eBr = rBf$

Proof. $r \in O$ iff $eBr \subseteq rB$ iff $eBr \subseteq rB \cap Br$ (since always, $eBr \subseteq Br$) iff $eBr = rB \cap Br$ (by 1.6). Similarily, (a) iff (c). But from 1.6, $Br \cap rB = eBr \cap rBf$, for any $r \in \mathcal{R}$. So (b) iff (c). ☐

2.10 Corollary. *Let $r \in O$. Then $l(r) = \dim(Br) + \dim(rB) - 2\dim(eBr)$.*

Proof. Recall that $l(r) = \dim(BrB) - \dim(B\nu B)$. But $\dim(BrB) = \dim(Br) + \dim(rB) - \dim(Br \cap rB)$, while, by 2.9, $\dim(Br \cap rB) = \dim(B\nu B)$. ☐

2.11 Remark. Write $r = e\sigma = \sigma f \in \mathcal{R}$. Then

$$\begin{aligned} Br \cap rB &= eBr \cap rBf \\ &= eBe\sigma \cap \sigma fBf \qquad (1.6) \\ &= (eBe \cap eB^{\sigma}e)\sigma. \end{aligned}$$

This pictures the elements of O as those for which $\dim(Br \cap rB)$ is maximal.

3. The Trichotomy. As usual we fix $B \subseteq G$ a Borel subgroup and $T \subseteq B$ a maximal torus. Recall from 1.2 that for each $W \times W$ orbit J of $\mathcal{R}$ there exists a unique $\nu \in J$ such that $B\nu = \nu B$. We recall the *Green's relations* on $\mathcal{R}$. Let $r, s \in \mathcal{R}$.

($\mathcal{H}$) $r\mathcal{H}s$ if for some $\sigma, \tau \in W$ and $e, f \in E(\mathcal{R})$, $r = e\sigma = \sigma f$ and $s = e\tau = \tau f$.

($\mathcal{J}$) $r\mathcal{J}s$ if for some $\sigma, \tau \in W, r = \sigma s \tau$.

($\mathcal{R}$) $r\mathcal{R}s$ if for some $\sigma \in W, r = s\sigma$.

($\mathcal{L}$) $r\mathcal{L}s$ is for some $\sigma \in W, r = \sigma s$.

See [1; Chapter 2] for more details on these relations.

3.1 Theorem. *Let $r \in \mathcal{R}$. Then there exist unique elements $r_+, r_-, r_0 \in \mathcal{R}$ such that*

(1) $r = r_+ r_0 r_-$

(2) $r_0 \mathcal{H} \nu$, *where* $\nu \mathcal{J} r$ *and* $B\nu = \nu B$

(3) $r_+ \mathcal{R} r$ *and* $r_- \mathcal{L} r$

(4) $r_+,\ r_- \in O$.

Proof. First suppose that for $r \in \mathcal{R}$, we have $r = r_+ r_0 r_-$ staifying (1)-(4). If also, $r = s_+ s_0 s_-$ then (1), (2) and (3) imply that $r_+ \mathcal{H} s_+$ and $r_- \mathcal{H} s_-$. But then by 2.8, $r_+ = s_+$ and $r_- = s_-$. So r_+ and r_- are unique. But then $r_0 = r_+^* r r_-^*$ and so r_0 is also unique.

To establish existence, we count. Let $\nu = e_0\sigma = \sigma f_0 \in J = WrW$ and let

$$\begin{aligned} A &= \{x \in O | x \mathcal{L} e_0\} \\ B &= \{x \in O | x \mathcal{R} f_0\}, \text{ and} \\ C &= \{x \in \mathcal{R} | x \mathcal{H} \nu\}. \end{aligned}$$

By the above, the product map,

$$A \times C \times B \to WrW = W\nu W$$

is injective.

By 2.7 and 2.8 $|A| = |E(J)|$ and $|B| = |E(J)|$, while $|C| = |H_\nu|$, where $H_\nu = C$ is the $\mathcal{H}$-class of ν. Thus, $|A \times C \times B| = |E(J)|^2 |H_\nu|$. To count up J we consider the map

$$\zeta : W\nu W \to E(J) \times E(J)$$

$\zeta(\sigma\nu\tau^{-1}) = (\sigma e_0 \sigma^{-1},\ \tau f_0 \tau^{-1})$. It is easy to check that ζ is well defined and surjective, and all fibres have the same cardinality. But $\zeta^{-1}(e_0,\ f_0) = H_\nu$. Thus, $|J| = |E(J)|^2 |H_\nu|$. □

3.2 Proposition. *Let $r \in \mathcal{R}$, and write $r = r_+ r_0 r_-$ as in 3.1. Then*

$$BrB = Br_+Br_0r_-B = Br_+r_0Br_-B = Br_+Br_0Br_-B.$$

Proof. It suffices to show that $Br_+Br_0Br_-B \subseteq BrB$, since $BrB \subseteq Br_+Br_0r_-B \cup Br_+r_0Br_-B \subseteq Br_+Br_0Br_-B$. First notice that we can write $r_+ = \sigma_+ e_0$, $r_0 = e_0\sigma_0 = \sigma_0 f_0$ and $r_- = f_0 \sigma_-$, where $e_0, f_0 \in E(R)$ and $\sigma_+, \sigma_0, \sigma_- \in W$. Thus,

$$\begin{aligned} Br_+Br_0Br_-B &= Br_+Be_0r_0f_0Br_-B \\ &\subseteq Br_+r_0f_0Br_-B, \text{ by 2.9 (c)}, \\ &\subseteq Br_+r_0r_-B, \text{ by 2.9 (b)}. \end{aligned}$$ □

3.3 Proposition. *Let* $r = r_+ r_0 r_-$ *as in 3.1. Then for some* $\sigma, \tau \in W$, $Br \cap rB = \sigma(Br_0 \cap r_0 B)\tau$. *Furthermore,* $r \in O$ *iff* $r_0 = \nu$.

Proof. Write $r_+ = \sigma e_0 = e\sigma$ and $r_- = f_0 \tau = \tau f$. Then

$$\begin{aligned} Br \cap rB &= eBr \cap rBf, \text{ by 1.6} \\ &= eBr_+ r_0 r_- \cap r_+ r_0 r_- Bf, \\ &\subseteq r_+ Br_0 r_- \cap r_+ r_0 Br_-, \text{ since } r_+, r_- \in O \\ &= \sigma(e_0 Br_0 \cap r_0 Bf_0)\tau, \\ &= \sigma(Br_0 \cap r_0 B)\tau, \text{ by 1.6.} \end{aligned}$$

The third step amounts to the inclusions $eBr_+ r_0 r_- \subseteq r_+ Br_0 r_-$ and $r_+ r_0 r_- Bf \subseteq r_+ r_0 Br_-$. But in each case these are equalities. Thus the inclusion is also an equality. Hence, $Br \cap rB = \sigma(Br_0 \cap r_0 B)\tau$.

Now if $r_0 = \nu$ then $r_+, \nu, r_- \in O$ and so $r = r_+ \nu r_- \in O$, by 2.2 (a). Conversely, if $r \in O$, then $r_0 = r_+^* r r_-^* \in O$. Thus, by 2.8, $r_0 = \nu$. ☐

We now study the relationship of the length function $l : \mathcal{R} \to \mathbb{N}$ (1.3) to the decomposition $r = r_+ r_0 r_-$ of 3.1.

3.4 Theorem. *Let* $r = r_+ r_0 r_-$ *be as in 3.1. Then* $l(r_+) + l(r_0) + l(r_-) = l(r) + 2l(e_0)$ *where* $e_0 \in WrW$ *is the unique idempotent such that* $Be_0 = e_0 Be_0$.

Proof.

$$\begin{aligned} l(r) &= \dim(BrB) - \dim(B\nu B) \\ &= \dim(Br) + \dim(rB) - \dim(Br \cap rB) - \dim(B\nu B), \text{ by 1.5 }, \\ &= \dim(Br_+) + \dim(r - B) - \dim(Br_0 \cap r_0 B) - \dim(B\nu B), \text{ by 3.3 }. \end{aligned}$$

On the other hand,

$$\begin{aligned} l(r_+) + l(r_0) + l(r_-) = {} & \dim(Br_+) + \dim(r_+ B) - 2\dim(B\nu B) + \dim(Br_0) \\ & + \dim(r_0 B) - \dim(Br_0 \cap r_0 B) - \dim(B\nu B) \\ & + \dim(Br) + \dim(r_- B) - 2\dim(B\nu B). \end{aligned}$$

Inspecting the summands, we find that

$$\begin{gathered} \dim(r_+ B) = \dim(Br_-) = \dim(e_0 B), \text{ and} \\ \dim(Br_0) = \dim(r_0 B) = \dim(B\nu B). \end{gathered}$$

Thus,

$$\begin{aligned} l(r_+) + l(r_0) + l(r_-) &= \dim(Br_+) + \dim(rB) - \dim(Br_0 \cap r_0 B) - \dim(B\nu B) \\ &\quad + 2(\dim(e_0 B) - \dim(B\nu B)) \\ &= l(r) + 2l(e_0). \end{aligned}$$

☐

3.5 Remark. There is a length function l' with $l'(r) = l'(r_+) + l'(r_0) + l'(r_-)$. Indeed,

$$l'(r) = l(r) - l(e_0).$$

However, l' will take negative values for some $r \in \mathcal{R}$.

We now compute the length of an idempotent.

3.6 Proposition. *Let $e \in E(\mathcal{R})$. Then*

(a) $\dim(BeB) = |\Phi^+| - |\{\alpha \in \Phi^+ | U_\alpha e = eU_\alpha = e\}| + \dim(eT)$.
(b) *If $e\mathcal{J}f$ then $l(e) = l(f)$.*

Proof. We first note that $Be \cong Ue \times eT$, where $U \subseteq B$ in the subgroup of unipotent elements. For if $uet = ves$ then $u^{-1}ves = et$. So it suffices to show that $ues = et$ implies $ue = e$. But in that case $ues = eues$, so we may assume $eu = ue$. On the other hand, $C_B(e)e \cong C_U(e)e \times eT$ by the structure theory of solvable groups applied to $C_B(e)e$. Similarly, $eB \cong Te \times eU$. Thus, $\dim(Be) = \dim(Ue) + \dim(eT) = \dim(U) - |\{\alpha \in \Phi^+ | U_\alpha e = \{e\}\}| + \dim(eT)$ and $\dim(eB) = \dim(U) - |\{\alpha \in \Phi^+ | eU_\alpha = \{e\}\}| + \dim(eT)$.

But $eB \cap Be = eBe$ and so $\dim(eBe) = |\{\alpha \in \Phi^+ | U_\alpha e = eU_\alpha \neq \{e\}| + \dim(eT)$.

Hence, by 1.5,$\dim(BeB) = \dim(be) + \dim(eB) - \dim(eBe) + \dim(eT) = \dim(U) - |\{\alpha \in \Phi^+ | U_\alpha e = eU_\alpha = \{e\}\}| + \dim(eT)$. But $|\{\alpha \in \Phi^+ | U_\alpha e = eU_\alpha = \{e\}\} = \frac{1}{2}|\{\alpha \in \Phi | U_\alpha e = eU_\alpha = \{e\}\}|$, and this depends only on the W-conjugacy class of e. □

3.7 Remark. By 3.6 the "length" function l' of 3.5 has the property that $l'(e) = 0$ for any $e \in E(\mathcal{R})$.

3.8 Proposition. *Let $r \in \mathcal{R}$ be such that $r = r_-$. Write $r = f_0\sigma = e\sigma$ where $f_0B \subseteq Bf_0$. Then $l(r) = l(f_0) + k$ where $k = \min\{l(\tau) | r = f_0\tau,\ \tau \in W\}$. A similar formula holds if $r = r_+$.*

Proof. Assume that, in $r = f_0\sigma = \sigma e$, σ has been chosen with minimal length such that $\sigma^{-1}f_0\sigma = e$. Without loss of generality we may assume $\sigma \neq 1$. Write $\sigma = \tau\rho$ where $l(\sigma) = l(\tau) + 1$, and $l(\rho) = 1$. So $f_0\tau \neq f_0\sigma = r$. But $\rho^{-1}(\tau^{-1}f_0\tau)\rho = \sigma^{-1}f_0\sigma = e$. Thus, $l(\tau)$ is minimal with $\tau^{-1}f_0\tau = \rho e\rho$, and so by 2.7, $f_0\tau \in O$. By induction we have $l(f_0\tau) = l(f_0) + l(\tau)$. But

$$\begin{aligned} BrB &= Bf_0\sigma B \\ &= Bf_0B\sigma B, \text{ since } Bf_0B = Bf_0, \\ &= Bf_0B\tau B\rho B, \text{ since } l(\sigma) = l(\tau) + 1 \\ &= Bf_0\tau B\rho B, \text{ since } Bf_0B = Bf_0. \end{aligned}$$

Thus, by 1.4, $l(r) = l(f_0\tau) + 1 = l(f_0) + l(\tau) + 1 = l(f_0) + l(\sigma)$. □

We now analyze the the effect on r_+, r_0 and r_- of multiplying by a simple involution.

3.9 Proposition. *Let $r \in \mathcal{R}$ and write $r = r_+r_0r_-$ as in 3.1. Let ρ be a simple involution.*

(a) $(\rho r)_- = r_-$
(b) *If $\rho r\mathcal{H}r$ then $(\rho r)_+ = r_+$*
(c) *If $\rho r \not\mathcal{H} r$ then $(\rho r)_0 = r_0$.*

Proof. (a) is obvious. For (b), write $r_+ = e\sigma \in O$. If $\rho e = e\rho$ then $\rho r\mathcal{H}r$, and so $(\rho r)_+\mathcal{H}r_+$ which implies, by 2.8, that $(\rho r)_+ = r_+$. If $\rho e \neq e\rho$ then $l(\rho)$ is minimal among $\{l(\tau) | \tau^{-1}e\tau = \rho^{-1}e\rho\}$. Thus, by 2.7, $e\rho \in O$. But then by 2.5, $\rho e = (e\rho)^* \in O$. Hence, by 2.2(a), $\rho r_+ = \rho e \cdot er_+ \in O$. But then, the decomposition $\rho r = (\rho r_+)(r_0)(r_-)$ satisfies the axioms of 3.1. □

3.10 Proposition. *Let $r, s \in \mathcal{R}$. The following are equivalent:*

(a) $BsB \subseteq \overline{BrB}$,

(b) $Bs_0B \subseteq s_+^*\overline{BrB}s_-^*$.

Proof. $BsB = Bs_+Bs_0Bs_-B$. So (a) is equivalent to $s_+Bs_0Bs_- \subseteq \overline{BrB}$. But then $Bs_0B = s_+^*s_+Bs_0Bs_-s_-^* \subseteq s_+^*\overline{BrB}s_-^*$. Conversely, if $Bs_0B \subseteq s_+^*\overline{BrB}s_-^*$, then
$s_+Bs_0Bs_- \subseteq e\overline{BrB}f$ for some $e, f \in E(\mathcal{R})$. But $e, f \in \overline{T} \subseteq \overline{B}$. So $e\overline{BrB}f \subseteq \overline{TBrBT} = \overline{BrB}$. ☐

Ultimately, one would like a complete description of the adherence order: $x \leq y$ if $BxB \subseteq \overline{ByB}$. However, there are several nontrivial pieces to this problem. First, one must determine when $GxG \subseteq \overline{GyG}$. This has been solved completely for $\mathcal{J}$*-irreducible* monoids in [6]. At the other extreme one must determine when $BxB \in \overline{ByB}$ if $x\mathcal{J}y$. One anticipates a complete solution in terms of these two extremes. The remainder of this section contains what we know about the situation $x\mathcal{J}y$.

3.11 Proposition. *Let $r, s \in \mathcal{R}$. Suppose $r \geq s$ and $r\mathcal{J}s$. Then $r_+ \geq s_+$ and $r_- \geq s_-$.*

Proof. Our assumption is that $\overline{BrB} \supseteq BsB$. Write $r = r_+r_0r_-$ and $s = s_+s_0s_-$ as in 3.1. Then $\overline{GrB} \subseteq GsB$, and so $\overline{Gr_-B} \supseteq Gs_-B$. We can write $r_- = f_0\sigma$ and $s_- = f_0\tau$, where $f_0B \subseteq Bf_0$. In any case, $f_0(\overline{Gr_-B}) \supseteq f_0Gs_-B$. So $\overline{H_{f_0}r_-B} \supseteq H_{f_0}s_-B$, where H_{f_0} is the $\mathcal{H}$-class of f_0 in M.

Claim. $\overline{H_{f_0}r_-B} = H_{f_0}\overline{r_-B}$ *(closure in Gr_-G).*

Proof. $r_- = f_0\sigma \in O$. So $f_0Br_- \subseteq r_-B$. Hence, $f_0Br_B = r_-B$. Now $f_0Bf_0 \subseteq H_{f_0}$ is a Borel subgroup, while $f_0Bf_0 \cdot r_-B = r_-B$. Thus, $f_0Bf_0 \cdot \overline{r_-B} = \overline{r_-B}$. So by [11; page 68], $H_{f_0}\overline{r_-B} \subseteq Gr_-G$ is closed.

Thus we obtain that

$$H_{f_0}\overline{r_-B} \supseteq H_{f_0}s_-B. \tag{*}$$

Write $B_{f_0} = f_0Bf_0 \subseteq H_{f_0}$, and notice that

$$\begin{aligned} H_{f_0}s_-B &= \bigcup_{\zeta \in \mathcal{R},\, \zeta\mathcal{H}_{f_0}} B_{f_0}\zeta B_{f_0}s_-B \\ &= \bigcup_{\zeta} B_{f_0}\zeta f_0 B_{f_0}s_-B \\ &= \bigcup_{\zeta} B_{f_0}\zeta s_-B. \end{aligned} \tag{**}$$

The last equality follows from $f_0Bs_-B = s_-B$.

Now by (*), $zs_- \in \overline{r_-B}$ for some $z \in H_{f_0}$. So if $b_1 \in B_{f_0}$ and $b_2 \in B$ then $b_1zs_-b_2 \in \overline{Br_-B}$. But by (**), there exist $b_1 \in B_{f_0}$ and $b_2 \in B$ such that $b_1zs_-b_2 = \zeta s_-$ for some $\zeta \in \mathcal{R}$, $\zeta\mathcal{H}f_0$. We conclude that, for some $\zeta \in \mathcal{R}$, $\zeta\mathcal{H}f_0$, $\zeta s_- \in \overline{Br_-B}$. By 3.1, we may write $s_- = \nu^*vs_-$ and $\zeta s_- = \nu^*as_-$ for some $a\mathcal{H}\nu$. So by 3.2,

$$\begin{aligned} Bs_-B &= Br^*B\nu Bs_-B, \\ B\zeta s_-B &= B\nu^*BaBs_-B \end{aligned}$$

But we may write $a = w\nu$ for some $w \in W$ and so $BaB = Bw\nu B = BwB\nu B$. Hence, $\overline{BaB} \supseteq B\nu B$ since $1 \in \overline{BwB}$. Further, $Bs_-B = B\nu^* B\nu Bs_-B \subseteq B\nu^*\overline{BaB}s_-B \subseteq \overline{B\zeta s_-B}$. Recalling that $\zeta s_- \in \overline{Br_-B}$ we finally obtain that $Bs_-B \subseteq \overline{Br_-B}$.

Similarily, $Bs_+B \subseteq \overline{Br_+B}$. ☐

3.12 Corollary. *Let $\nu \in \mathcal{R}$ be such that $B\nu = \nu B$, and let $H \subseteq M$ be the $\mathcal{H}$-class of ν in M. Suppose $r \in \mathcal{R}$ and $r\mathcal{J}\nu$. Then*

$$\overline{Br_+Hr_-B} = \overline{Br_+}H\overline{r_-B}$$

where the closure is in $G\nu G$.

Proof. Let a be the element in $\mathcal{R}$ such that $BaB \subseteq H$ is open and dense. There is such an element since $BHB = H$. Now let $x \in \overline{Br_+Hr_-B}$, say $x \in BsB$. Then $s \leq r_+ar_-$, and so by 3.11 $s_+ \leq r_+$ and $s_- \leq r_-$. But $Bs_+Hs_-B = (Bs_+)(\bigcup_{\zeta\mathcal{H}r} B\zeta B)(s_-B) = \bigcup_{\zeta\mathcal{H}\nu} Bs_+\zeta s_-B$, by 3.2. So,

$$\begin{aligned} BsB &= Bs_+Bs_0Bs_-B \\ &\subseteq Bs_+Hs_-B \\ &\subseteq \overline{Br_+}H\overline{r_-B} \end{aligned}$$

☐

3.13 Example. The reader might wonder "Does $r \geq s$, $r\mathcal{J}s$ imply $r_0 \geq s_0$?" The answer here is "no" in general. Let $M = M_3(k), T = D_3(k)^*$ and $B = T_3(k)^*$. Suppose $r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\overline{BrB} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b, c, d, e, \in k \right\}$ and so $s \in \overline{BrB}$. But $r_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \nu$, while $s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Clearly, $r_0 \not\geq s_0$. In fact, $s_0 > r_0$!

This example shows that Figure 3 of [9] incorrectly depicts the Bruhat order on the monoid of upper triangular 3×3 matrices. What was actually depicted is the $\mathcal{J}$-order on the upper triangular elements of $\mathcal{R}$. The above example makes it clear that the Bruhat order strictly refines the $\mathcal{J}$-order. I would like to thank M. Putcha for pointing this out to me.

3.14 Proposition. *Let $r \in \mathcal{R}$ with $r\mathcal{J}\nu$ and $\nu = e_0\sigma = \sigma f_0$. Then $e_0\overline{BrB}f_o \cap G\nu G = \overline{BrB} \cap H_\nu$.*

Proof. Let $x \in \overline{BrB} \cap H_\nu$. Then $x = e_0x = xf_0 = e_0xf_0$. So $x \in e_0\overline{BrB}f_0 \subseteq \overline{BrB}$. But, $e_0\overline{BrB}f_0 \cap G\nu G \subseteq e_0Mf_0 \cap G\nu G = H_\nu$. ☐

4. Fat $\mathcal{H}$-Classes.

One knows, from the general principles of solvable transformation groups, that any set of the form $BrB, r \in \mathcal{R}$, can be written as $BrB \cong G_a^k \times G_m^l$. However, it is useful to know how the factors fit together and also how this decomposition is related to r_+, r_0 and r_-.

4.1 Lemma. *Let $x \in M$. Then*

(a) *$GxB = GeB$ for some unique $e \in E(\mathcal{R})$,*

(b) *$|Gx \cap E(eB)| = 1$*

Similarly, for BxG *and* $xG \cap E(Bf)$.

Proof. By [9; Theorem 9.10] there exists a unique $r \in \mathcal{R}$ such that

$$Br \subseteq rB, \quad x \in GrB \quad \text{and} \quad Gx \cap V = T\overline{x} \tag{*}$$

where $V = \{y \in rB \mid yr^* = rr^* = e_0\}$, $r = e_0\gamma = \sigma e$ and $\overline{x} \in Gx \cap V$. So write $\overline{x} = rb = \sigma eb$. Then $rbr^* = rr^*$, or what is the same, $\sigma ebe\sigma^{-1} = \sigma e\sigma^{-1}$. Thus, $ebe = e$. Now, one checks that $\sigma^{-1}V = \{z \in eB \mid ze = e\} = E(eB)$. So $eb \in Gx \cap E(eB) \neq \emptyset$. If also, $eb_1 \in Gx \cap E(eB)$, then $\sigma eb_1 \in Gx \cap V$ and so by $(*)$, $\sigma eb_1 = t\sigma eb$ for some $t \in T$. Thus, $eb_1 = t'eb$ where $t' = \sigma^{-1}t\sigma \in T$. However, if $eb_1 \in E(eB)$ then $(t'eb)^2 = t'eb$. Hence, $ebt'e = e$. But then $ebet' = e$, while $ebe = e$. Thus, $et' = t'e = e$. ☐

4.2 Lemma. *Let* $r = e\sigma = \sigma f \in \mathcal{R}$. *Then the following are equivalent*

- (a) $r \in O$,
- (b) $Br \subseteq E(Be)rB$,
- (c) $rB \subseteq BrE(fB)$.

Proof. Assume (a). By [9; page 324], we can write $B_u = U_1 U_2 U_3$ where $U_3 e = \{e\}$, $eu = ue$ for all $U \in U_2$, and $eU_1 = \{e\}$, with $U_1 \cong U_1 e$. So if $b \in B$ then $br = be\sigma = ueb_1\sigma$, where $u \in U_3$ and $b_1 \in C_B(e)$. But if $r \in O$, then $eC_B(e) = eC_{B^\sigma}(e)$. But then $b_2\sigma \in C_{B^\sigma}(e)\sigma \subseteq \sigma B\sigma^{-1}\sigma = \sigma B$. So we may write $b_2\sigma = \sigma b_3$ for some $b_3 \in B$. Putting it together, we get $br = ueb_1\sigma = ueb_2\sigma = ue\sigma b_3 \in E(Be)rB$.

Now assume $Br \subseteq E(Be)rB$. Then $eBr \subseteq eE(Be)rB = erB = rB$. Thus, $r^*Br \subseteq r^*rB$.

4.3 Theorem. *Let* $r \in \mathcal{R}$ *and write* $r = r_+r_0r_-$ *as in 3.1, with* $r = e\sigma = \sigma f$. *Then*

$$\varphi : E(Be) \times r_+Br_0Br_- \times E(fB) \to BrB$$

$\varphi(e', x, f') = e'xf'$, *is bijective.*

Proof. Let $x = b_1rb_2 \in BrB$. Then

$$\begin{aligned} x &= b_1rb_2 = b_1r_+r_0r_-b_2, \\ &= e'r_+b_1'r_0b_2'r_-f', \text{ by 4.2 (b) and (c).} \end{aligned}$$

Hence, ζ is surjective.

Conversely, we can write $r = r_+\tau$ for some $\tau \in W$ and so $x = brb' = br_+\tau b' = uer_+b_3\tau b'$, as in the proof of 4.2. But then $ue \in E(Be) \cap xG$. So by 4.1, if we can write $x = e'r_+xr_-f'$, with $e' \in E(Be)$ and $f' \in E(fB)$, the e' and f' are unique. So assume $e'xf' = e'yf'$, with $x, y \in Br_0B$. Then $ee' = e$ and $f'f = f$. So $x = e(e'xf')f = e(e'yf')f = y$. ☐

4.4 Corollary. *Let* $H \subseteq \mathcal{R}$ *be an* $\mathcal{H}$*-class. So* $H = r_+H_\nu r_-$, *where* H_ν *is the* $\mathcal{H}$*-class of* ν. *Let* $r_+ = e\sigma$ *and* $r_- = \tau f$. *Then*

$$\varphi : E(Be) \times H_r \times E(fB) \to \bigcup_{s \in H} BsB$$

is bijective, where H *is the* $\mathcal{H}$*-class of* r *in* M.

Proof. Clearly $H_r = r_+ H_\nu r_- = r_+ \left(\bigcup_{a\mathcal{H}\nu} BaB \right) r_-$. So the result follows from 4.3, since φ is bijective on $B \times B$ orbits. $\square$

In our last application we compute the right B-orbits on M. Since $M = \bigsqcup_{r\in\mathcal{R}} BrB$, this amounts to finding, for each $r \in \mathcal{R}$, a subgroup H of B such that $H \times rB \to BrB$, $(h, rb) \mapsto hrb$, is bijective. If $r = e\sigma \in \mathcal{O}$ then the result follows from 4.2 since one can easily find a subgroup $U \subseteq B$ such that $U \underset{\cong}{\longrightarrow} Ue = E(Be)$. For the general case we find the subgroup $V \subseteq C_G(e) \cap N_G(U)$ (U as above) such that $BrB = UVrB = E(Be)VrB \cong E(Be) \times V \times rB$.

4.5 Lemma. *Let $r \in \mathcal{R}$ and write $r = ce\sigma$, where $e\sigma \in \mathcal{O}$ and $c \in C_W(e)$. Let $U \subseteq B_u$ be the closed subgroup normalized by T such that $U \underset{\cong}{\longrightarrow} Ue$. Then $Br \subseteq UC_{(B_u^-)^c \cap B_u}(e)rB$. Furthermore, $eC_{(B_u^-)^c \cap B_u}(e) \cong eC_{B_u^- \cap B_u^{c^{-1}}}(e)$.*

Proof. We can write $B = UC_B(e)U_1$, where $U_1 e = \{e\}$. Thus,

$$\begin{aligned} Br &= UC_B(e)U_1 ec\sigma = UC_B(e)ec\sigma = UeBec\sigma = UceB^{c^{-1}}e\sigma \\ &= UceC_{B_u^- \cap B^{c^{-1}}}(e)C_{B\cap B^{c^{-1}}}(e)e\sigma = UceC_{B_u^- \cap B^{c^{-1}}}(e)He\sigma, \end{aligned}$$

where $H \subseteq C_{B^\sigma}(e)$. Such an H exists since $e\sigma \in \mathcal{O}$. Indeed, if $e\sigma \in \mathcal{O}$ then $C_B(e)e = C_{B^\sigma}(e)e$, while $C_{B\cap B^{c^{-1}}}(e) \subseteq C_B(e)$. But then $\sigma\sigma^{-1}H\sigma \subseteq \sigma B$. Hence,

$$Br \subseteq UceC_{B_u^- \cap B^{c^{-1}}}(e)e\sigma\sigma^{-1}H\sigma \subseteq UeC_{(B_u^-)^c \cap B}(e)ce\sigma\sigma^{-1}H\sigma \subseteq UC_{(B_u^-)^c \cap B}(e)rB.$$

The last statement is obvious. $\square$

For $r = ce\sigma \in \mathcal{R}$, with $e\sigma \in \mathcal{O}$ and $ce = ec$, we let

$$V_r \subseteq C_{(B_u^-)^c \cap B_u}(e)$$

be the unique subgroup normalized by T such that $V_r \underset{\cong}{\longrightarrow} eV_r = eC_{(B_u^-)^c \cap B_u}(e)$.

4.6 Lemma. *Let $r = ce\sigma \in \mathcal{R}$ be as above and write $r = r_+ r_0 r_-$ as in 3.1. Then $l(r_0) = \dim(V_r)$.*

Proof. Recall from 3.4 that $l(r_+) + l(r_0) + l(r_-) = l(r) + 2l(e_0)$. So $l(r_+\nu r_-) + 2l(e_0) = l(r_+) + l(\nu) + l(r_-) = l(r_+) + l(r_-)$. Hence, $l(r_+\nu r_-) + l(r_0) + 2l(e_0) = l(r_+) + l(r_0) + l(r_-) = l(r) + 2l(e_0)$. So we obtain $l(r) = l(r_+ r_-) + l(r_0)$, and consequently, $\dim(BrB) = \dim(Br_+\nu r_- B) + l(r_0)$.

On the other hand, we can write $r_0 = e_0 w\nu = we_0\nu = w\nu$. So $l(r_0) = \dim(Br_0) - \dim(Br_0 \cap r_0 B)$, while $Br_0 \cap r_0 B = e_0 Be_0 w\nu \cap w\nu B = e_0 Be_0 w\nu \cap wB\nu = w(e_0 w^{-1} Bwe_0 \cap e_0 Be_0)\nu = w(e_0 C_{B\cap B^{w^{-1}}}(e_0))\nu$. So $l(r_0) = \dim(e_0 C_{B_u^- \cap B^{w^{-1}}}(e_0))$. Hence, $l(r_0) = l_{e_0}(e_0 w)$, where l_{e_0} represents the length function of $e_0 C_G(e_0)$ relative to $e_0 Be_0$ and $e_0 T$. But by definition, $\dim V_r = l_e(ec)$. So we must show that $l_{e_0}(e_0 w) = l_e(ec)$.

Let $I = \{\alpha \in \triangle | \sigma_\alpha e_0 = e_0 \sigma_\alpha \neq e_0\}$. Then by [4; Proposition 10.9 and Theorem 10.10], I is canonically identified with the set of simple reflections of $e_0 Be_0$. By Proposition 3.8 we can write $r = \zeta r_0 \tau$, with ζe_0, $f_0\tau \in \mathcal{O}$ and $\zeta e_0 = e\zeta$. But then $\zeta I \zeta^{-1}$ is the set of simple reflections of $\zeta(e_0 Be_0)\zeta^{-1} = eB^\zeta e$. But $e\zeta \in \mathcal{O}$ and so $eB^\zeta e = eBe$. Hence, $\zeta I\zeta^{-1}$ is the set of simple reflections of eBe. Therefore, $l_e(ec) = l_e(\zeta e_0 w \zeta^{-1}) = l_{e_0}(e_0 w)$. $\square$

4.7 Theorem. *Let $r = e\sigma \in \mathcal{R}$. Then $\varphi : E(Be) \times V_r \times rB \to BrB$, $\varphi(x, y, z) = xyz$, is bijective.*

Proof. By 4.5, φ is surjective. Further, both sides have the same dimension. Indeed, by 4.2 $\dim(Br_+\nu r_-B) = \dim(E(Be)) + \dim(rB)$, while $\dim(BrB) = \dim(Br_+\nu r_-B) + \dim(V_r)$ from the proof 4.6.

Consider $\varphi^{-1}(r) = \{(e', v, rb) | e'vrb = r\}$ and let $(e', v, rb) \in \varphi^{-1}(r)$. Then by 4.1, $e' = e$. Also $vrb = r$, since $ev = ve$. But then $vrB \subseteq rB$. So let $\tau \in T$. Then $tvt^{-1}rB = tvrBt^{-1} = rB$. Thus, $K = V \cap \{v \in B_u | vrB = rB\}$ is a closed subgroup of B_u normalized by T. Hence, either $K = \{1\}$ or $\dim(K) > 0$. But all fibres of φ have the same dimension. Hence, $K = \{1\}$. But then $v = 1$. So $(e', v, rb) = (e, 1, r)$. Similarly, all other fibres are singletons. ☐

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO CANADA N6A 5B7