THE BLOCKS OF SOLVABLE ALGEBRAIC MONOIDS

Lex E. Renner

0. Introduction. In this paper we determine the blocks of certain solvable algebraic monoids. A *block* (generally) can be thought of as an equivalence class of irreducible representations. The equivalence relation in this setup is generated by declaring irreducible representations (ρ, V) and (φ, W) in the same block if there exists an indecomposable representation (ψ, U) such that both (ρ, V) and (φ, W) occur as factors in a composition series of (ψ, U).

The blocks of $M_n(k)$ were first determined by Schur [6]. Assuming $k = \mathbb{C}$, Schur proves that any polynomial representation $\rho : M_n(k) \to M_n(k)$ is completely reducible. Thus, each block consists of exactly one irreducible representation. So assume ρ above is irreducible. It is then easy to see that

$$p(\operatorname{diag}(t, \underset{(n)}{\ldots}, t)) = \operatorname{diag}(t^r, \underset{(m)}{\ldots}, t^r)$$

for some $r \geq 0$. In this case we say that the *degree* of ρ is r. Finally, Schur proves that there exists a canonical one to one correspondence

$$\{\rho \mid \text{irreducible, degree}(\rho) = r\} \longleftrightarrow \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0, \sum_{i=1}^{n} \lambda_i = r\}.$$

$\lambda = (\lambda_1, \ldots, \lambda_n)$ can be regarded as the highest weight of $\rho = \rho_\lambda$ relative to an appropriate choice of maximal torus T and Borel subgroup B. Indeed, the set of dominant weights of $M_n(k)$ relative to $T =$ diagonals and $B =$ upper triangulars is $Y^+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\} \subseteq X(T) = \mathbb{Z}^n$. Furthermore, by the main results of [5], the irreducible representations of the normal monoid $M_n(k)$ are canonically identified with Y^+. (The irreducible representation (ρ, V) corresponds to the $\lambda \in X(T)$ with $\rho(t)(v) = \lambda(t)v$ for all $t \in T$, where $v \in V$ spans the B-stable one-dimensional subspace of V.) Notice however, that while the classification of irreducible representations is "characteristic free", the classification of blocks is not so straightforward for reductive monoids if $\operatorname{char}(k) = p > 0$. Donkin [1] calculates the blocks of $M_n(k)$ if $\operatorname{char}(k) = p > 0$. The end results depend heavily on the action of the affine p-Weyl group.

In this paper we treat the case of *solvable polarized* monoids (M, θ). This means that M is irreducible with a zero element $0 \in M$, and has a solvable unit group G. $\theta : k^* \hookrightarrow G$ is a central one-parameter subgroup with $\lim\limits_{t \to 0} \theta(t) = 0 \in M$. Our results complement those of Schur. We find (see Theorem 3.5 below) that there exists a canonical one-to-one correspondence

$$\{\text{blocks of } M \text{ of } \theta\text{-degree } = r\} \longleftrightarrow \{\text{characters of } Z(M) \text{ of } \theta\text{-degree } = r\}.$$

Here, $Z(M)$ is the scheme theoretic centre of M. We illustrate these results with two examples. Example 3.8 shows that $Z(M)$ need not be a reduced subscheme of M.

1. Basic structure of $\mathcal{O}(M)$.

Let M be an irreducible algebraic monoid over the algebraically closed field k. Our general reference here is [3]. We assume further that M has a zero element $0 \in M$ with $0 \in \overline{Z(M)}$. We refer to an irreducible monoid with these properties as *polarizable.* It is easy to see that M is polarizable if and only if there exists a one-parameter subgroup $\theta : k^* \to Z(G)$ such that $0 \in \overline{\theta(k^*)}$. (We can, and shall, assume that any such θ is a closed embedding.) We refer to (M, θ) as a *polarized monoid*, and θ as a *polarization of* M.

Polarized monoids arise naturally from group representations. Let $\rho : G \to Gl(V)$ be a rational representation of the connected algebraic group G. Then $ZGl(V) \cong k^*$, and so $M(\rho) =: \overline{\rho(G)ZGl(V)} \subseteq \mathrm{End}(V)$ is canonically polarized via $\theta = i : ZGl(V) \hookrightarrow G(M(\rho))$.

The other "obvious" class of polarized monoids is the class of finite dimensional k-algebras. Let N be such an algebra. Then $\phi : k^* \to N$, defined by $\phi(\alpha) = \alpha I_N$, determines a canonical polarization on A.

Let $(M, \theta), (N, \phi)$ be polarized monoids. A morphism $\varphi : M \to N$ of algebraic monoids is called a *θ-morphism* if $\theta(0) = 0$ and $\varphi(\mathrm{Image}(\theta)) \subseteq \mathrm{Image}(\phi)$. The *$\theta$-degree* of a θ-morphism φ is the degree of $\varphi|\mathrm{Im}(\theta)$. So $\varphi(\theta(\alpha)) = \phi(\alpha^n)$ if φ is of θ-degree n. We are mainly interested here in θ-morphisms $\varphi : (M, \theta) \to (N, \phi)$ where N is the multiplicative monoid of a finite dimensional k-algebra, and as mentioned above, $\phi : k^* \to N$ is defined by $\phi(\alpha) = \alpha 1_N$. Evidently (by 1.4 below), there is an initial object of this sort, in each positive degree, for any polarized monoid (M, θ).

If M is an algebraic monoid, we denote by $\mathcal{O}(M)$ the affine coordinate algebra of M. This is not our customary notation (although it is common), but we also encounter group algebras and monoid algebras which we denote by $k[X]$ and $k[S]$. I want to avoid unappealing notational accidents such as $k[X(T)] = k[T]$ where the LHS is a group algebra (covariant in $X(T)$) and the RHS is an affine coordinate algebra (contravariant in T).

Let (M, θ) be a polarized monoid. Define $\mu : k^* \times M \to M$ by

$$\mu(s, x) = \theta(s)x.$$

By standard results about one-parameter subgroups, μ induces a direct sum decomposition,

$$\mathcal{O}(M) = \bigoplus_{n \geq 0} \mathcal{O}_n(M)$$

where each $\mathcal{O}_n(M)$ is finite dimensional over k, and for each $m, n \geq 0$, $\mathcal{O}_n(M)\mathcal{O}_n(M) \subseteq \mathcal{O}_{m+n}(M)$. Indeed, $\mathcal{O}_n(M) = \{f \in \mathcal{O}(M) \mid \lambda_s f(s) = s^n f(x)$ for all $s \in k^*$ and all $x \in M\}$, where $\lambda_s f(x) = f(sx)$.

1.1 Proposition. *Let $\Delta : \mathcal{O}(M) \to \mathcal{O}(M) \otimes \mathcal{O}(M)$ be the morphism induced by multiplication $m : M \times M \to M$. Then $\Delta(\mathcal{O}_n(M)) \subseteq \mathcal{O}_n(M) \otimes \mathcal{O}_n(M)$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
k^* \times k^* \times M \times M \arrow[r, "\mathrm{id} \times m"] \arrow[d, "\gamma"'] & k^* \times k^* \times M \arrow[d, "\nu"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}$$

where $m(x,y) = xy, \nu(s,t,x) = stx$ and $\gamma(s,t,x,y) = (sx,ty)$. Then, for all $\chi \in X(k^* \times k^*) = \mathbb{Z} \oplus \mathbb{Z}, \Delta(\mathcal{O}(M)_\chi) \subseteq \mathcal{O}(M \times M)_\chi$, where $\mathcal{O}(M)_\chi = \{f \in \mathcal{O}(M) \mid f(stx) = \chi(s,t)f(x)$ for all $s,t \in k^*$ and $x \in M\}$ and $\mathcal{O}(M \times M)_\chi = \{f \in \mathcal{O}(M \times M) \mid f(sx,ty) = \chi(s,t)f(x,y)$ for all $s,t \in k^*$ and $x,y \in M\}$. But if $f(stx) = \chi(s,t)f(x)$ for all $s,t \in k^*$ and $x \in M$, then $\chi(s,t) = (st)^n$ for some $n \geq 0$ since ν factors through $\zeta : k^* \times k^* \times M \to K^* \times M, \zeta(s,t,x) = (st,x)$ via

$$\begin{tikzcd}
k^* \times k^* \times M \arrow[rr, "\nu"] \arrow[dr, "\zeta"'] & & M \\
& k^* \times M \arrow[ur, "\delta"'] &
\end{tikzcd}$$

where $\delta(s,x) = sx$. Thus, if $\mathcal{O}(M)_\chi \neq 0$, then $\chi = (n,n) \in X(k^* \times k^*) = \mathbb{Z} \oplus \mathbb{Z}$ for some $n \geq 0$. But then $\mathcal{O}(M \times M)_\chi = (\mathcal{O}(M) \otimes \mathcal{O}(M))_{(n,n)} = \mathcal{O}_n(M) \otimes \mathcal{O}_n(M)$. Conclude that $\Delta(\mathcal{O}_n(M)) \subseteq \mathcal{O}_n(M) \otimes \mathcal{O}_n(M)$. ■

1.2 Proposition. *Let $V \subseteq \mathcal{O}(M)$ be a finite dimensional, k-linear subspace. Then the following are equivalent:*

(a) $\Delta(V) \subseteq V \otimes V$.

(b) For any $g \in G, l_g(V) \subseteq V$ and $r_g(V) \subseteq V$, where $l_g(f)(x) = f(gx)$ and $r_g(f)(x) = f(xg)$ for $x \in M$.

(c) There exists a finite dimensional associative k-algebra S and a morphism $\rho : M \to S$ of algebraic monoids such that $\rho(M) \subseteq S$ spans and $\rho^* : \mathcal{O}(S) = Symm_k(S^*) \to \mathcal{O}(M)$ induces an isomorphism $\rho^* : S^* \to V$.

Proof. Assume (a). Then surely $\Delta(V) \subseteq \mathcal{O}(G) \otimes V$ and $\Delta(V) \subseteq V \otimes \mathcal{O}(G)$. Thus, by two applications of [2; Proposition 8.6(b)], V is stable under r_g and l_g. Conversely, if (b) holds, then again by two applications of [loc. cit.] we obtain $\Delta(V) \subseteq V \otimes \mathcal{O}(G) \cap \mathcal{O}(G) \otimes V$. But $V \otimes \mathcal{O}(G) \cap \mathcal{O}(G) \otimes V = V \otimes V$, so (a) holds.

To finish the proof, first notice that if $\rho : M \to S$ is a morphism of algebraic monoids, with S as above, then $\rho(M) \subseteq S$ spans if and only if $\rho^*|S^* : S^* \to \mathcal{O}(M)$ is injective. This condition is necessary, for if $\rho(M)$ does not span S let $T = \mathrm{Span}(\rho(M)) \subseteq S$, which is a k-algebra. Hence, ρ factors as

$$\begin{tikzcd}
M \arrow[rr, "\rho"] \arrow[dr] & & S \\
& T \arrow[ur, "\subsetneq"'] &
\end{tikzcd}$$

Taking the dual shows that $\rho^*|S^*$ cannot be injective. On the other hand, let $U = \rho^*(S^*) \subseteq \mathcal{O}(M)$. Since ρ is a morphism, we obtain the following commutative diagram upon restriction of $\rho^* : \mathcal{O}(S) \to \mathcal{O}(M)$ to S^*.

$$\begin{tikzcd}
S^* \arrow[r, "\rho^*"] \arrow[d, "\mu^*"'] & \mathcal{O}(M) \arrow[d, "\Delta"] \\
S^* \otimes S^* \arrow[r, "\rho^* \otimes \rho^*"'] & \mathcal{O}(M) \otimes \mathcal{O}(M)
\end{tikzcd} \tag{*}$$

Here $\mu : S \times S \to S$ is the linear map induced by multiplication on S. But then $\Delta(U) \subseteq \text{Image}(\rho^* \otimes \rho^*) = U \otimes U$. So if $\rho^*|S^*$ is not injective we obtain a factorization of $\rho : M \to S$ as

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr] & & S \\
& T \arrow[ur, "\subsetneq"'] &
\end{tikzcd}$$

where $T = U^*$. Thus, the condition is sufficient.

Now suppose $\Delta(V) \subseteq V \otimes V$. Dualizing the commutative diagram

$$\begin{tikzcd}
\text{Symm}_k(V) \arrow[r, "i"] \arrow[d, "\text{Symm}_k(\Delta|V)"'] & \mathcal{O}(M) \arrow[d, "\Delta"] \\
\text{Symm}_k(V) \otimes \text{Symm}_k(V) \arrow[r] & \mathcal{O}(M) \otimes \mathcal{O}(M)
\end{tikzcd}$$

yields a morphism $\rho = i^* : M \to S = V^*$ of algebraic monoids, where S has the algebra structure induced from $\text{Symm}(\Delta|V)$. By the above remarks, $\rho(M) \subseteq S$ spans. So (a) implies (c). Finally, if (c) holds for $\rho : M \to S$ with $V = \text{Image}(\rho^*|S^*)$ then the diagram in (*) shows that $\Delta(V) \subseteq V \otimes V$. ■

1.3 Proposition. *Assume that (M, θ) is a polarized monoid, where S is a finite-dimensional k-algebra and $\rho(M) \subseteq S$ spans. Then $S = \prod_{n \geq 0} S_n$ (finite direct product of k-algebras) in such a way that each $\rho_n = pr_n \circ \rho : M \to S_n$ is a θ-morphism of degree n. Furthermore, $\rho_n(M) \subseteq S_n$ spans.*

Proof. Let $V = \text{Image}(\rho^*(S^*)) \subseteq \mathcal{O}(M))$ be as in 1.2. By 1.2(b), V is stable under left and right translation by G, in particular by elements of $\theta(k^*) \subseteq G$. So $r_\alpha(V) \subseteq V$ for all $\alpha \in \theta(k^*)$. Now recall the grading

$$\mathcal{O}(M) = \bigoplus_{n \geq 0} \mathcal{O}_n(M) \quad \text{where}$$

$$\mathcal{O}_n(M) = \{ f \in \mathcal{O}(M) \mid r_\alpha f = \alpha^n f \text{ for all } \alpha \in k^* \}.$$

Hence, $V = \bigoplus_{n \geq 0} V_n$ is homogeneous, since our grading is induced from right (or left) translation. But then

$$\begin{aligned}
\Delta(V_n) &\subseteq (V \otimes V) \cap (\mathcal{O}_n(M) \otimes \mathcal{O}_n(M)) \\
&= (V \cap \mathcal{O}_n(M)) \otimes (V \cap \mathcal{O}_n(M)) \\
&= V_n \otimes V_n.
\end{aligned}$$

It follows from 1.2 that $\rho_n : M \to S_n = V_n^*$ is a morphism of algebraic monoids. Furthermore, $S = \prod_{n \geq 0} S_n$ since $\Delta|V = \bigoplus_{n \geq 0} \Delta|V_n$. It follows that $\rho_n = pr_n \circ \rho : M \to S_n$ is a θ-morphism of degree n. ■

1.4 Proposition. *Let (M, θ) be a polarized monoid. Then there is an initial object $\rho_n : M \to S_n(M)$ in the category of θ-morphisms of degree n. Furthermore, $\rho_n(M) \subseteq S_n(M)$ spans.*

Proof. Let $\rho_n : M \to S_n(M)$ be the morphism guaranteed, via Proposition 1.2, by $V = \mathcal{O}_n(M)$. So $S_n(M) = \mathcal{O}_n(M)^*$. Furthermore , by the proof of 1.3, ρ_n is a θ-morphism of degree n. It follows directly from 1.2 that ρ_n is initial among θ-morphisms of degree n. ■

1.5 Proposition. *Let (M, θ) and (N, ϕ) be polarized monoids. Suppose $\varphi : M \to N$ is a θ-morphism of degree $k \geq 0$. Then for each $n \geq 0$*

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\rho_{nk}"'] & N \arrow[d, "\rho_n"] \\
S_{nk}(M) \arrow[r, "\varphi_n"'] & S_n(N)
\end{tikzcd}$$

commutes, where φ_n is the k-linear dual of the induced map $\varphi^* : \mathcal{O}_n(N) \to \mathcal{O}_{nk}(M)$. Furthermore, if φ is dominant then each φ_n is surjective.

Proof. It is clear, from our previous comments, that $\rho_n : M \to S_n(M) = \mathrm{Hom}(\mathcal{O}_n(M), k)$ is defined by $\rho_n(x)(f) = f(x)$. Furthermore, the following diagram commutes:

$$\begin{tikzcd}
M \times M \arrow[r, "\rho_n \otimes \rho_n"] \arrow[d, "m"'] & \mathrm{Hom}(\mathcal{O}_n(M) \otimes \mathcal{O}_n(M), k) \arrow[d, "\Delta^*"] \\
M \arrow[r, "\rho_n"'] & \mathrm{Hom}(\mathcal{O}_n(M), k)
\end{tikzcd}$$

where $\Delta^*(\gamma)(f) = \gamma(\Delta(f))$ for all $\gamma \in \mathrm{Hom}(\mathcal{O}_n(M) \otimes \mathcal{O}_n(M), k)$ and $f \in \mathrm{Hom}(\mathcal{O}_n(M), k)$.

In any case, $\varphi : M \to N$ induces $\varphi^* : \mathcal{O}_n(N) \to \mathcal{O}_{nk}(M)$ which satisfies $\varphi^*(f)(x) = f(\varphi(x))$ for all $f \in \mathcal{O}_n(N)$ and $x \in M$. So $\varphi_n : S_{nk}(M) \to S_n(N)$ satisfies $\varphi_n(\zeta)(f) = \zeta(\varphi^*(f))$ for $\zeta \in S_{nk}(M) = \mathrm{Hom}(\mathcal{O}_{nk}(M), k), f \in \mathcal{O}_n(N)$. Finally we obtain

$$\begin{aligned}
\varphi_n(\rho_{nk}(x))(f) &= \rho_{nk}(\varphi^*(f)) \\
&= \varphi^*(f)(x) \\
&= f(\varphi(x)) \\
&= \rho_n(\varphi(x))(f).
\end{aligned}$$

So our diagram commutes.

If φ is dominant then $\varphi^* : \mathcal{O}(N) \to \mathcal{O}(M)$ is injective. But then each $\varphi_n : S_{nk}(M) \to S_n(N)$ is surjective. ■

1.6 Proposition. *Let $X \subseteq M$ be a closed, θ-stable subset and consider $I_n(X) = \{f \in \mathcal{O}_n(M) \mid f|X = 0\} \subseteq \mathcal{O}_n(M) = S_n(M)^*$. This inclusion yields a canonical identification $I_n(X) = (S_n(M)/Span(\rho_n(X)))^* \subseteq S_n(M)^*$. In particular, $\mathcal{O}(X) \cong \bigoplus Span(\rho_n(X))^*$ is the reduced coordinate algebra of X.*

Proof. Recall that $\rho_n : M \to S_n(M) = \mathcal{O}_n(M)^*$ is defined by $\rho_n(x)(f) = f(x)$. Also, $\mathcal{O}_n(M)$ is identified with $S_n(M)^*$ via $\mathcal{O}_n(M) \to S_n(M)^*, f \longmapsto \overline{f}$ $(\overline{f}(\gamma) = \gamma(f)$ for $\overline{f} \in \mathcal{O}_n(M)$ and $\gamma \in S_n(M) = \mathcal{O}_n(M)^*)$. In any case,

$$\begin{aligned} I_n(X) &= \{f \in \mathcal{O}_n(M) \mid f(x) = 0 \text{ for all } x \in X\} \\ &= \{f \in \mathcal{O}_n(M) \mid f(\rho_n(x)) = 0 \text{ for all } \ x \in X\} \\ &= \{f \in S_n(M)^* \mid f|\mathrm{Span}(\rho_n(X)) = 0\} \\ &= (S_n(M)/\mathrm{Span}(\rho_n(X))^* \subseteq S_n(M)^*. \end{aligned}$$

The second equality results since $f(x) = f(\rho_n(x))$ for all $x \in M$ and $f \in \mathcal{O}_n(M)$. But then $\mathcal{O}_n(X) = S_n(M)^*/(S_n(M)/\mathrm{Span}(\rho_n(X)))^* = \mathrm{Span}(\rho_n(X))^*$. ■

1.7 Proposition. *For all $m, n \geq 0$, the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\rho_{m+n}"] \arrow[d, "d"'] & S_{m+n}(M) \arrow[d, "\mu_{m,n}"] \\ M \times M \arrow[r, "\rho_m \otimes \rho_n"'] & S_m(M) \otimes S_n(N) \end{tikzcd}$$

where $d(x) = (x, x)$ and $\mu_{m,n}$ is the projection onto the (m, n)-summand of $S_{m+n}(M) \underset{\mu}{\longrightarrow} S_{m+n}(M \times M)$.

Proof. By 1.5,

$$\begin{tikzcd} M \arrow[r, "\rho_{m+n}"] \arrow[d, "d"'] & S_{m+n}(M) \arrow[d, "\mu"] \\ M \times M \arrow[r, "\rho'_{m+n}"'] & S_{m+n}(M \times M) \end{tikzcd}$$

commutes. But $S_{m+n}(M \times M) = \bigoplus\limits_{k+l=m+n} S_k(M) \otimes S_l(M)$ and $\mu_{m,n}$ is obtained from μ by composing with the projection onto $S_m(M) \otimes S_n(M)$, while $\rho_m \otimes \rho_n$ is obtained from ρ'_{m+n} by composing with the same projections. ■

We now turn our attention to blocks. Let A be a finite dimensional associative k-algebra. The *blocks* of A are the obvious summands in the decomposition $A = \bigoplus\limits_{e \in Z_1} eAe$, where Z_1 is the set of primative central idempotents of A. We denote the blocks of A by $Bl(A)$ and identify them with Z_1.

1.8 Proposition.

(a) *Let A and B be finite dimensional associative k-algebras and suppose $\varphi : A \to B$ is a surjective morphism. Then for all $e \in Z_1(B) = Bl(B)$ there exists a unique $f \in Z_1(A) = Bl(A)$ such that $\varphi(f)e = e$.*

(b) *With e, f and φ as in (a), define $Bl(\varphi)(e) = f$. Then $Bl(\varphi) : Bl(B) \to Bl(A)$ defines a contravariant functor from the category of finite dimensional algebras and surjections to the category of finite sets.*

Proof. We can write $1 = \sum\limits_{f \in I} f$ where $I = Z_1(A)$ and $1 \in A$ is the identity element. Thus, $e = \varphi(1)e = \sum \varphi(f)e$. Thus, $\varphi(f) \neq 0$ for some $f \in I$. But then $\varphi(f)e = e$

since e is a primitive idempotent of $Z(A)$. If also $\varphi(f')e = e$ for $f' \in I$, then $\varphi(ff') = \varphi(f)\varphi(f') \neq 0$ which forces $f = f'$. So f is unique, and this proves (a).

For (b), consider $A \xrightarrow[\psi]{} B \xrightarrow[\phi]{} C$ and let $e \in Bl(C), f \in Bl(B), g \in Bl(A)$. Assume $\varphi(f)e = e$ and $\psi(g)f = f$. Then $\varphi(\psi(g))\varphi(f) = \varphi(\psi(g)f) = \varphi(f)$, and so $\varphi(\psi(g))\varphi(f)e = \varphi(f)e = e$. In particular, $\varphi(\psi(g))e \neq e$ which implies $\varphi(\psi(g))e = e$, since e is primitive. Hence, $Bl(\psi)(Bl(\varphi)(e)) = Bl(\psi)(f) = g = Bl(\varphi \circ \psi)(e)$. This proves (b). ■

If (M, θ) is a polarized monoid we define

$$Bl(M) = \bigsqcup_{n \geq 0} Bl(S_n(M)).$$

1.9 Corollary. *Let (M, θ) and (N, ϕ) be polarized monoids, and let $\varphi : M \to N$ be a dominant θ-morphism. Then φ induces a map of sets $Bl(\varphi) : Bl(N) \to Bl(M)$. This is a contravariant functor from polarized monoids and dominant θ-morphisms to sets.*

Proof. By 1.5(a) φ induces commutative diagrams

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\rho_{nk}"'] & N \arrow[d, "\rho_n"] \\
S_{nk}(M) \arrow[r, "\varphi_n"'] & S_n(N)
\end{tikzcd}$$

for each $n \geq 0$. The φ_n's are unique since Image(ρ_m) spans S_m for each $m \geq 0$. So this construction is functorial with the appropriate degree shift. Furthermore, by 1.5(b), φ_n is surjective. So by 1.8(a), φ_n induces $Bl(\varphi_n) : Bl(S_n(N)) \to Bl(S_{nk}(M))$, and we define $Bl(\varphi) : Bl(N) \to Bl(M)$ to be the disjoint union of $\{Bl(\varphi_n)\}_{n \geq 0}$. By 1.8(b) this construction is functorial. ■

Let A be a finite dimensional k-algebra. A k-subalgebra $D \subseteq A$ is called a *toral* subalgebra if $D \cong k \times \cdots \times k$ as k-algebras. It is well known (following Wedderburn) that any two maximal, toral subalgebras D, $D' \subseteq A$ are conjugate under A^*, the unit group of A.

2. Blocks and weights for solvable monoids.

In this section we assume that (M, θ) is a polarized monoid with solvable unit group G. We refer to M as a *solvable polarized monoid.* We find that the block structure of such monoids can be nicely unravelled using the weight spaces of a suitable torus action on $\mathcal{O}(M)$.

2.1 Proposition. *For each $n \geq 0, S_n(M)$ is a solvable k-algebra.*

Proof. We can embed $S_n(M)$, as a k-subalgebra of $M_n(k)$ for some $n \geq 0$. So we obtain $\rho_n : M \to S_n(M) \subseteq M_n(k)$. Now M has a solvable (dense) unit group G, so by [2; Theorem 17.6], $\rho_n(G) \subseteq T_n(k)$ for an appropriate choice of basis, where $T_n(k)$ is the associated upper triangular subalgebra of $M_n(k)$. Since $T_n(k)$ is linear, it is Zariski closed in $M_n(k)$. Hence, $\overline{\rho_n(G)} \subseteq T_n(k)$, and so by continuity, $\rho_n(M) \subseteq T_n(k)$. But $\rho_n(M)$ spans $S_n(M)$ by Proposition 1.2. Thus, $S_n(M) \subseteq T_n(k)$. We conclude that $S_n(M)$ is solvable. ■

2.2 Proposition. *Let $T \subseteq G$ be a maximal torus and let $D_n = Span(\rho_n(T)) \subseteq S_n(M)$. Then $D_n \subseteq S_n(M)$ is a maximal toral subalgebra.*

Proof. It is well known that D_n is a toral subalgebra. Since G is solvable, we can write $G = TG_n$ where $G_n \subseteq G$ is the subgroup of unipotent elements. Then $\rho_n(G) = \rho_n(T)\rho_n(G_n) \subseteq D_n(1+\mathcal{N}) \subseteq D_n \oplus \mathcal{N}$, where $\mathcal{N} \lhd S_n(M)$ is the ideal of nilpotent elements. But then $B = D_n \oplus \mathcal{N}$ is a k-subalgebra of $S_n(M)$ with $\rho_n(G) \subseteq B$. Since $\rho_n(G)$ spans $S_n(M)$ we must have $B = S_n(M)$. But then $D_n \subseteq S_n(M)$ is a maximal toral subalgebra. ■

2.3 Proposition. *Consider the linear torus action $\mu_n : T \times T \times S_n(M) \to S_n(M)$ defined by $\mu_n(s,t,x) = \rho_n(s)x\rho_n(t)$. Then each weight space, say $0 \neq {}^{\alpha}S_n^{\beta} = \{x \in S_n(M) \mid \rho_n(s)x\rho_n(t) = \alpha(s)\beta(t)x \text{ for all } s,t \in T\}$, corresponds to a pair of primitive idempotents $(e,f) \in E(D_n) \times E(D_n)$ with $eS_nf \neq 0$. This correspondence is a bijection via $(e,f) \longleftrightarrow eS_nf = {}^{\alpha}S_n^{\beta} \longleftrightarrow (\alpha,\beta)$.*

Proof. Let $X_1(D_n) = \{\chi_e : D_n \to k \mid \chi_e(t) = et \quad \text{for all} \quad t \in D_n\}$ where $e \in E_1(D_n)$, the set of primitive idempotents of D_n. Consider $\gamma : D_n \times S_n \to S_n$ defined by $\gamma(t,x) = tx$. Then $\Phi(\gamma) = X_1(D_n)$ where, by definition, $\Phi(\gamma) = \{\alpha \in X(D_n) \mid \gamma(t,x) = \alpha(t)x \text{ for all } t \in D_n \text{ and some } x \in S_n, x \neq 0\}$. By 2.2 $\rho_n(T) \subseteq D_n$ spans this algebra. So consider the diagram

$$\begin{tikzcd}
X(D_n) \arrow[rr, "\rho_n^*"] \arrow[dr] & & X(T) \\
& X(D_n') \arrow[ur] &
\end{tikzcd}$$

where $X(D_n')$ is the free abelian monoid with basis $\rho_n^*(X_1(D_n))$. This diagram defines $D_n' \subseteq D_n$ as a subalgebra with $\rho_n(T) \subseteq D_n'$. But $\rho_n(T)$ spans D_n, so $D_n' = D_n$. This implies that ρ_n^* is injective when restricted to $X_1(D_n)$. Notice also that the set of weights of $l : T \times S_n \to S_n, l(t,x) = \rho_n(t)x$, is $\Phi(l) = \rho_n^*(X_1(T))$. Similarly, the set of weights of $r : T \times S_n \to S_n, r(t,x) = x\rho_n(t)$, is also given by $\Phi(r) = \rho_n^*(X_1(T))$. A little more calculation applied to the action $\nu : T \times T \times S_n \to S_n, \nu(s,t,x) = \rho_n(s)x\rho_n(t)$, yields

(i) $\Phi(\nu) \subseteq (\rho_n \times \rho_n)^*(X_1(D_n) \times X_1(D_n))$, and

(ii) $(\rho_n \times \rho_n)^* : X_1(D_n) \times X_1(D_n) \to X(T \times T)$ is injective. Furthermore, $\Phi(\nu) = (\rho_n \times \rho_n)^*(\Phi(\mu))$ where $\mu : D_n \times D_n \times S_n \to S_n, \mu(s,t,x) = sxt$. Direct calculation yields that $\Phi(\mu) = \{(\chi_e, \chi_f) \mid eS_nf \neq 0\}$. The sought after bijection is

$$(\rho_n \times \rho_n)^* | \Phi(\mu) : \Phi(\mu) \to \Phi(\nu).$$ ■

The morphism $\rho_n : M \to S_n(M)$ induces $\rho_n^* : \mathcal{O}(S_n(M)) = S_n(M)^* \to \mathcal{O}_n(M)$. Define $\gamma : T \times T \times \mathcal{O}(M) \to \mathcal{O}(M)$ by $\gamma(s,t,x) = f(sxt)$. One checks that

$$ {}^{\alpha}\mathcal{O}_n(M)^{\beta} = ({}^{\alpha}S_n(M)^{\beta})^* \quad \text{and} $$
$$ {}^{\alpha}\mathcal{O}_n(M)^{\beta} \otimes {}^{\delta}\mathcal{O}_m(M)^{\gamma} \to {}^{\alpha+\delta}\mathcal{O}_{n+m}(M)^{\beta+\gamma}. $$

We define

$$\mathcal{S} = \{(\alpha,\beta) \in X(T) \times X(T) \mid {}^{\alpha}\mathcal{O}_n(M)^{\beta} \neq 0 \text{ some } n \geq 0\} \text{ and}$$
$$\Delta(\overline{T}) = \{(\alpha,\beta) \in X(\overline{T}) \times X(\overline{T}) \mid \alpha = \beta\}.$$

2.4 Proposition.

(a) $\Delta(\overline{T}) \subseteq \mathcal{S} \subseteq X(\overline{T}) \times X(\overline{T})$.

(b) *$\mathcal{S}$ is a finitely generated submonoid of $X(\overline{T}) \times X(\overline{T})$.*

(c) *$\mathcal{S} = \varphi_g^*(\overline{TgT}) \cap X(\overline{T} \times \overline{T})$, for some $g \in M$, where $\varphi_g : \overline{T} \times \overline{T} \to M$ is defined by $\varphi_g(s,t) = sgt$.*

Proof. For (a), consider the unique morphism $\pi : M \to \overline{T}$ such that $\pi|\overline{T} = id$ (see [4; Theorem 2.3]). This induces $\mathcal{O}(\overline{T}) \hookrightarrow \mathcal{O}(M)$ which is $T \times T$-equivariant. Furthermore, $\mathcal{O}(\overline{T}) = \bigoplus_{\alpha \in X(\overline{T})} {}^{\alpha}\mathcal{O}(\overline{T})^{\beta}$ with $\dim({}^{\alpha}\mathcal{O}(\overline{T})^{\beta}) = 1$. Hence ${}^{\alpha}\mathcal{O}(M)^{\alpha} \neq 0$, since ${}^{\alpha}\mathcal{O}(\overline{T})^{\alpha} \subseteq {}^{\alpha}\mathcal{O}(M)^{\alpha}$.

For (b), let $\{x_1, \ldots, x_m\} \subseteq \mathcal{O}(M)$ be a set of algebra generators which are $T \times T$-homogeneous. So $x_i \in {}^{\alpha_i}\mathcal{O}(M)^{\beta_i}$ for some $(\alpha_i, \beta_i) \in \mathcal{S}$. Then any nonzero homogeneous $f \in {}^{\alpha}\mathcal{O}(M)^{\beta}$ can be expressed as a linear combination of the monomials $\{x_1^{k_1} \cdot \ \ldots \ \cdot x_m^{k_m}\}$. Thus, $(\alpha, \beta) = \sum_{i=1}^{m} k_i(\alpha_i, \beta_i)$ for appropriate $\{k_i\}$ appearing in the expression of f.

For (c), let $\overline{TgT} \subseteq M$ be any $T \times T$-orbit closure. So $\mathcal{O}(M) \twoheadrightarrow \mathcal{O}(\overline{TgT})$, and for each $(\alpha, \beta) \in \mathcal{S}$, ${}^{\alpha}\mathcal{O}(M)^{\beta} \twoheadrightarrow {}^{\alpha}\mathcal{O}(\overline{TgT})^{\beta}$. Thus $\varphi_g^*(\overline{TgT}) \cap X(\overline{T} \times \overline{T}) \subseteq \mathcal{S}$. Let $\{(\alpha_i, \beta_i)\}_{i=1}^{m} \subseteq \mathcal{S}$ be a set of generators and let $J_i \subseteq \mathcal{O}(M)$ be the ideal generated by ${}^{\alpha_i}\mathcal{O}(M)^{\beta_i} \neq 0$. If $V_i = \{x \in M \mid f(x) = 0 \text{ for all } f \in J_i\}$ then $V_i \subseteq M$ is a proper closed subset. Hence, $V = \bigcup_{i=1}^{m} V_i \subseteq M$ is closed and proper. So let $g \in M \backslash V$. Then for each $i = 1, \ldots, m, J_i \nsubseteq \operatorname{Ker}(\mathcal{O}(M) \to \mathcal{O}(\overline{TgT}))$. Hence, ${}^{\alpha_i}\mathcal{O}(\overline{TgT})^{\beta_i} \neq 0$ for $i = 1, \ldots, m$. But then if $(\alpha, \beta) \in \mathcal{S}$, say $(\alpha, \beta) = \sum k_i(\alpha_i, \beta_i)$, take $f_i \neq 0$ with $f_i \in {}^{\alpha_i}\mathcal{O}(\overline{TgT})^{\beta_i}$. Then $0 \neq f = \prod f_i^{k_i} \in {}^{\alpha}\mathcal{O}(\overline{TgT})^{\beta}$. Hence $\mathcal{S} \subseteq \varphi_g^*(\overline{TgT}) \cap X(\overline{T} \times \overline{T})$. ■

We now consider the relationship between the blocks of each $S_n(M)$ and the monoid $\mathcal{S}$. Let $D_n \subseteq S_n$ be a maximal toral subalgebra and let $E_1(D_n) \subseteq D_n$ be the primitive idempotents. Define a relation on $E_1(D_n)$ as follows:

$$e \sim f \quad \text{if} \quad eS_nf \neq 0 \quad \text{or} \quad fS_ne \neq 0.$$

This generates an equivalence relation on $E_1(D_n)$, also denoted by $\sim$. So $e \sim f$ if there exist $e_1, e_2, \ldots, e_s \in E_1(D_n)$ such that $e_1 = e, e_s = f$ and for each $i = 1, \ldots s-1$ either $e_iS_ne_{i+1} \neq 0$ or else $e_{i+1}S_ne_i \neq 0$. It is well known that the blocks of S_n are in one-to-one correspondence with these equivalence classes as follows.

Let $E_1(D_n) = E_1 \cup \cdots \cup E_r$, where each $E_i \subseteq E_1(D_n)$ is an equivalence class. Then the primitive, central idempotents of S_n are obtained as follows:

$$Z_1 = \{f_1, \ldots, f_r\} \quad \text{where} \quad f_i = \sum_{e \in E_i} e.$$

These are primitive idempotents of $Z(S_n)$ not of S_n.

For $\alpha, \beta \in X(\overline{T})$ we write $\alpha \to \beta$ if $(\alpha, \beta) \in \mathcal{S}$ and $\alpha \leftarrow \beta$ if $(\beta, \alpha) \in \mathcal{S}$

2.5 Proposition. *The equivalence on* $X(\overline{T})$ *generated by* $\rightarrow$ *is the same as the equivalence generated by* $\leftarrow$. *This relation can be described as follows:*

$$\alpha \sim \beta \text{ if there exists } \gamma_1, \ldots, \gamma_{2m-1} \in X(\overline{T})$$

for $m \geq 1$ *such that* $\alpha = \gamma_1 \rightarrow \gamma_2 \leftarrow \cdots \leftarrow \gamma_{2m-1} = \beta$.

Proof. The indicated $\sim$ is an equivalence relation as follows:

(i) $\alpha \sim \alpha$ since $\Delta(\overline{T}) \subseteq \mathcal{S}$ by 2.4(a).

(ii) If $\alpha = \gamma_1 \rightarrow \gamma_2 \leftarrow \cdots \leftarrow \gamma_{2m-1} = \beta$ then $\gamma_{2m-1} \rightarrow \gamma_{2m-2} \leftarrow \cdots \rightarrow \gamma_2 \leftarrow \gamma_1$.

(iii) If $\gamma_1 \rightarrow \gamma_2 \leftarrow \cdots \rightarrow \gamma_{2m-1}$ and $\gamma_{2m-1} \rightarrow \gamma_{2m} \leftarrow \cdots \leftarrow \gamma_{2n-1}$ then $\gamma_1 \rightarrow \gamma_2 \leftarrow \cdots \leftarrow \gamma_{2m-1} \rightarrow \gamma_{2m} \leftarrow \cdots \leftarrow \gamma_{2n-1}$.

Now any equivalence relation containing $\rightarrow$ also contains $\leftarrow$ by the symmetry condition. So $\rightarrow$ and $\leftarrow$ generate the same equivalence relation. Certainly, it is contained in $\sim$ since if $\alpha \rightarrow \beta$ then $\alpha \rightarrow \beta \leftarrow \beta$ and so $\alpha \sim \beta$. Conversely, if $\alpha \sim \beta$, then by transitivity, it is in the relation generated by $\rightarrow$. ■

2.6 Lemma. *Suppose* $\alpha \sim \beta$ *and* $\lambda \sim \delta$. *Then for some* $m \geq 1$,

$$\alpha = \gamma_1 \rightarrow \gamma_2 \leftarrow \cdots \leftarrow \gamma_{2m-1} = \beta \quad \text{and}$$
$$\lambda = \zeta_1 \rightarrow \zeta_2 \leftarrow \cdots \leftarrow \zeta_{2m-1} = \delta.$$

Proof. Initially, we may have $\alpha = \gamma_1 \rightarrow \gamma_2 \leftarrow \cdots \leftarrow \gamma_{2m-1} = \beta$ and $\lambda = \zeta_1 \rightarrow \zeta_2 \leftarrow \cdots \leftarrow \zeta_{2n-1} = \delta$, where without loss of generality, $m > n$. But we can fix that as follows. Consider $\gamma = \zeta_1 \rightarrow \zeta_1 \leftarrow \cdots \leftarrow \zeta_{2n-1} = \delta \rightarrow \delta \leftarrow \cdots \leftarrow \delta$, where δ occurs $2(m-n)+1$ times. ■

2.7 Corollary. *Suppose* $\alpha \sim \beta$ *and* $\lambda \sim \delta$. *Then* $\alpha\lambda \sim \beta\delta$.

Proof. We can assume, by 2.6, that

$$\alpha = \gamma_1 \rightarrow \gamma_2 \leftarrow \cdots \leftarrow \gamma_{2m-1} = \beta, \quad \text{and}$$
$$\lambda = \zeta_1 \rightarrow \zeta_2 \leftarrow \cdots \leftarrow \zeta_{2m-1} = \delta.$$

But if $\gamma_i \rightarrow \gamma_{i+1}$ and $\zeta_i \rightarrow \zeta_{i+1}$ then, by definition, $(\gamma_i, \gamma_{i+1}), (\zeta_i, \zeta_{i+1}) \in \mathcal{S}$, a semigroup. So $(\gamma_i\zeta_i, \gamma_{i+1}\zeta_{i+1}) \in \mathcal{S}$; or what is the same, $\gamma_i\zeta_i \rightarrow \gamma_{i+1}\zeta_{i+1}$. Similarly for $\leftarrow$. Thus,

$$\alpha\lambda = \gamma_1\zeta_1 \rightarrow \gamma_2\zeta_2 \leftarrow \cdots \leftarrow \gamma_{2m-1}\zeta_{2m-1} = \beta\delta. \qquad ■$$

2.8 Theorem. $X(\overline{T})/\!\sim\, \cong Bl(M)$. *Furthermore,* $Bl(M)$ *is a finitely generated monoid and* $X(\overline{T}) \twoheadrightarrow Bl(M), \alpha \longmapsto [\alpha]$, *is a morphism of monoids.*

Proof. $X(\overline{T})/\!\sim$ is a monoid by 2.7. Again by 2.7, $X(\overline{T}) \rightarrow X(\overline{T})/\!\sim$ is a homomorphism.

We recall from 1.9, that $Bl(M) = \bigsqcup_{n \geq 0} Bl(S_n(M))$. From the discussion preceeding 2.5, $Bl(S_n(M))$ is identified with $E_1(D_n)/\!\sim$ where $\sim$ is the equivalence relation generated by

$$e \sim f \text{ if } eS_n(M)f \neq 0$$

But $E_1(D_n)$ is identified, via Proposition 2.3, with $\mathcal{S}_n = \{(\alpha,\beta) \in X(\overline{(T)} \mid {}^{\alpha}\mathcal{O}_n(M)^{\beta} \neq 0\} \subseteq \mathcal{S}$. Furthermore, $eS_n(M)f = ({}^{\alpha}\mathcal{O}_n(M)^{\beta})^*$ defines this bijection via $(e,f) \longleftrightarrow (\alpha,\beta)$. It follows immediately that under this bijection, between $\mathcal{S}$ and $\bigsqcup\limits_{n\geq 0} E_1(D_n)$, the equivalence relation on $X(\overline{T})$ generated by $\to$ (as in 2.5) corresponds to the above equivalence relation on $\bigsqcup\limits_{n\geq 0} E_1(D_n)$. Thus $X(\overline{T})/\!\sim\; \cong \bigsqcup\limits_{n\geq 0} E_1(D_n)/\!\sim\; \equiv Bl(M)$. ■

3. Main Results.

In this section we assume that (M,θ) is a polarized solvable monoid. The action of G on M, defined by $\mathrm{int}(g)(x) = gxg^{-1}$, induces an action $\mathrm{int}^*(g)(f)(x) = f(g^{-1}xg)$. Furthermore, $\mathrm{int}^*(g)(\mathcal{O}_n(M)) \subseteq \mathcal{O}_n(M)$ for all $n \geq 0$, since $g^{-1}\theta(\alpha)xg = \theta(x)g^{-1}xg$ for all $g \in G, x \in M$ and $\alpha \in k^*$. We define

$$V_n \subseteq \mathcal{O}_n(M)$$

as the unique, minimal subspace such that $\mathrm{int}^*(g)(V_n) \subseteq V_n$ for all $g \in G$, and G acts trivially on $\mathcal{O}_n(M)/V_n$. Define

$$J \lhd \mathcal{O}(M)$$

as the homogeneous ideal generated by $V = \bigoplus\limits_{n\geq 0} V_n$.

3.1 Proposition.

(a) $\Delta(J) \subseteq \mathcal{O}(M) \otimes J + J \otimes \mathcal{O}(M)$.

(b) Let $S \subseteq M$ be a closed subscheme. Then the following are equivalent

(i) $S \subseteq Z$, *where* $Z = Spec(\mathcal{O}(M)/J) \subseteq M$

(ii)
$$\begin{tikzcd} S \times M \arrow[r, "m"] \arrow[d, "\tau"'] & M \\ M \times S \arrow[ur, "m"'] & \end{tikzcd}$$
commutes, where $\tau(s,x) = (x,s)$ *and* $m(x,y) =$ y.

Proof. By definition, $\mathcal{O}_n(M)/V_n$ is dual to $Z_n \subseteq S_n(M)$, the center of $S_n(M)$. So

$$\begin{tikzcd} Z_n \otimes Z_n \arrow[r, hook] \arrow[d] & S_n(M) \otimes S_n(M) \arrow[d, "\Delta^*"] \\ Z_n \arrow[r, hook, "\varphi_n"'] & S_n(M) \end{tikzcd}$$

commutes for all $n \geq 0$. Dualizing this diagram, we obtain for each $n \geq 0$, a commutative diagram

$$\begin{tikzcd} \mathcal{O}_n(M) \otimes \mathcal{O}_n(M) \arrow[r, two heads, "p_n \otimes p_n"] & (\mathcal{O}_n(M)/V_n) \otimes (\mathcal{O}_n(M)/V_n) \\ \mathcal{O}_n(M) \arrow[u, "\Delta"] \arrow[r, two heads] & \mathcal{O}_n(M)/V \arrow[u, "\overline{\Delta}"'] \end{tikzcd}$$

But then $\Delta(\mathrm{Ker}(p_n)) \subseteq \mathrm{Ker}(p_n \otimes p_n)$. On the other hand $\ker(p_n) = V_n$ and $\mathrm{Ker}(p_n \otimes p_n) = \mathcal{O}_n \otimes V_n + V_n \otimes \mathcal{O}_n$. Indeed, $\mathrm{Ker}(p_n \otimes p_n) \supseteq \mathcal{O}_n \otimes V_n + V_n \otimes \mathcal{O}_n$, but these two subspaces have the same dimension since $(\mathcal{O}_n(M) \otimes V_n) \cap (V_n \otimes \mathcal{O}_n(M)) = V_n \otimes V_n$. So $\Delta(V_n) \subseteq \mathcal{O}_n(M) \otimes V_n + V_n \otimes \mathcal{O}_n(M)$ for all $n \gg 0$, and thus $\Delta(V) \subseteq \mathcal{O}(M) \otimes V + V(\otimes)\mathcal{O}(M)$.

To prove (a), first notice that $J = \sum_{f \in \mathcal{O}} fV$. So $\Delta(J) = \Delta(\Sigma fV) \subseteq \Sigma\Delta(fV)$. But

$$\begin{aligned}
\Delta(fV) &= \Delta(f)\Delta(V) \\
&\subseteq \Delta(f)(\mathcal{O}(M) \otimes V + V \otimes (\mathcal{O}(M)) \\
&= (\Sigma g_i \otimes h_i)(\mathcal{O}(M) \otimes V + V \otimes \mathcal{O}(M)), \text{ where } \Delta(f) = \Sigma g_i \otimes h_i \\
&\subseteq \Sigma(\mathcal{O}(M) \otimes h_i V + g_i V \otimes \mathcal{O}(M)) \\
&\subseteq \mathcal{O}(M) \otimes J + J \otimes \mathcal{O}(M).
\end{aligned}$$

This proves (a).

For (b), first assume (ii). Then the diagram

$$\begin{tikzcd}
S \times G \arrow[r, "m"] \arrow[d, "\tau"'] & M \\
G \times S \arrow[ur, "m"'] &
\end{tikzcd}$$

commutes. So $sg = gs$ for all $g \in G(R)$ and $s \in S(R)$, where R is any commutative k-algebra. This implies that $S \subseteq M$ is a closed subscheme for which the action of G is trivial. But by definition, $Z = \mathrm{Span}(\mathcal{O}(M)/J)$ is the largest closed subscheme with this property. Hence, $\mathcal{I}(S) \supseteq J$, where $\mathcal{O}(S) = \mathcal{O}(M)/\mathcal{I}(S)$.

Conversely, assume (i). Then $\mathcal{I}(S) \supseteq J$ and so G acts trivially on $\mathcal{O}(S) = \mathcal{O}(M)/\mathcal{I}(S)$. Thus

$$\begin{tikzcd}
S \times G \arrow[r, "m_1"] \arrow[d, "\tau"'] & M \\
G \times S \arrow[ur, "m_2"'] &
\end{tikzcd}$$

commutes. But then

$$\begin{tikzcd}
\mathcal{O}(S \times G) & \mathcal{O}(M) \arrow[l, "m_1^*"'] \arrow[dl, "m_2^*"] \\
\mathcal{O}(G \times S) \arrow[u, "\tau^*"] &
\end{tikzcd}$$

commutes. However, m_1^* (resp. m_2^*) factors through $\mathcal{O}(S \times M) \subseteq \mathcal{O}(S \times G)$ (resp. $\mathcal{O}(M \times S) \subseteq \mathcal{O}(G \times S)$), while $\tau^*(\mathcal{O}(S \times M)) \subseteq \mathcal{O}(M \times S)$. So we obtain

$$\begin{tikzcd}
\mathcal{O}(S \times M) & \mathcal{O}(M) \arrow[l, "m_1^*"'] \arrow[dl, "m_2^*"] \\
\mathcal{O}(M \times S) \arrow[u, "\tau^*"] &
\end{tikzcd}$$

This concludes the proof. ■

We refer to $Z = \mathrm{Spec}(\mathcal{O}(M)/J) \subseteq M$ as the *center* of M. By 3.1(a) Z is a submonoid scheme of M.

Now let $\overline{T} \subseteq M$ be as usual and consider $h : X(\overline{T}) \to X(\overline{T})/{\sim} \cong Bl(M)$ as determined by 2.8. So $h(\alpha) = h(\beta)$ if ${}^{\alpha}\mathcal{O}(M)^{\beta} \neq 0$. Define

$$Y =: \mathrm{Spec}(k[X(\overline{T})/{\sim}]) \subseteq \overline{T}$$

where "$k[\cdots]$" denotes "monoid algebra over k".

3.2 Theorem. *Let Y be as above. Then $Y \subseteq Z$. In particular, $Y \subseteq M$ is inde-*

Proof. Define $l : \overline{T} \times M \to M$ by $l(x, y) = xy$ and $r : M \times \overline{T} \to M$ by $r(x, y) = xy$. So we obtain $l^* : \mathcal{O}(M) \to \mathcal{O}(\overline{T}) \otimes \mathcal{O}(M)$ and $r^* : \mathcal{O}(M) \to \mathcal{O}(M) \otimes \mathcal{O}(\overline{T})$. Furthermore,

$$\begin{aligned} {}^{\alpha}\mathcal{O}(M) &= \{f \in \mathcal{O}(M) \mid l^*(f) = \alpha \otimes f\} \quad \text{and} \\ \mathcal{O}(M)^{\beta} &= \{f \in \mathcal{O}(M) \mid r^*(f) = f \otimes \beta\}. \end{aligned}$$

Consider now, $h : \mathcal{O}(\overline{T}) \to \mathcal{O}(Y) = k[X(\overline{T})/{\sim}]$, where $h(\alpha) = h(\beta)$ if ${}^{\alpha}\mathcal{O}(M)^{\beta} \neq 0$. Define

$$\begin{aligned} L : \mathcal{O}(M) &\xrightarrow[l^*]{} \mathcal{O}(\overline{T}) \otimes \mathcal{O}(M) \xrightarrow[h\otimes \mathrm{id}]{} \mathcal{O}(Y) \otimes \mathcal{O}(M), \quad \text{and} \\ R : \mathcal{O}(M) &\xrightarrow[r^*]{} \mathcal{O}(M) \otimes \mathcal{O}(\overline{T}) \xrightarrow[\mathrm{id}\otimes h]{} \mathcal{O}(M) \otimes \mathcal{O}(Y). \end{aligned}$$

One checks easily that for $f \in {}^{\alpha}\mathcal{O}(M)$, $L(f) = h(\alpha) \otimes f$ and for $f \in \mathcal{O}(M)^{\beta}$, $R(f) = f \otimes R(\beta)$. On the other hand, ${}^{\alpha}\mathcal{O}(M)^{\beta} = {}^{\alpha}\mathcal{O}(M) \cap \mathcal{O}(M)^{\beta}$. So if $f \in {}^{\alpha}\mathcal{O}(M)^{\beta}$ then

$$\begin{aligned} L(f) &= h(\alpha) \otimes f, \\ R(f) &= f \otimes h(\beta), \quad \text{and} \\ h(\alpha) &= h(\beta). \end{aligned}$$

We conclude that the diagram

$$\begin{tikzcd} \mathcal{O}(M) \arrow[r, "L"] \arrow[dr, "R"'] & \mathcal{O}(Y) \otimes \mathcal{O}(M) \arrow[d, "\tau"] \\ & \mathcal{O}(M) \otimes \mathcal{O}(X) \end{tikzcd}$$

commutes, where $\tau(g \otimes h) = h \otimes g$. Indeed, by the above remarks $\tau \circ L \mid {}^{\alpha}\mathcal{O}(M)^{\beta} = R \mid {}^{\alpha}\mathcal{O}(M)^{\beta}$. But $\mathcal{O}(M) = \bigoplus_{(\alpha,\beta)\in\mathcal{S}} {}^{\alpha}\mathcal{O}(M)^{\beta}$ and so $\tau \circ L = R$ as claimed. But then 3.1(b) applies, and we conclude that $Y \subseteq Z$. ■

3.3 Theorem. *The commutative diagram*

$$\begin{tikzcd} \mathcal{O}(M) \arrow[r] \arrow[dr] & \mathcal{O}(Z) \arrow[d] \\ & \mathcal{O}(Y) \end{tikzcd}$$

identifies $\mathcal{O}_n(Y)^*$ *with* $Z_n(M)_s$*, the maximal toral subalgebra of the center of* $S_n(M)$*.*

Proof. By 3.2 $Y \subseteq Z$, so we obtain the above diagram. Thus $J \subseteq \mathcal{I}(Y)$ with J as in 3.1. But then $\mathcal{O}_n(M)/J_n \longrightarrow \mathcal{O}_n(M)/\mathcal{I}_n(Y) = \mathcal{O}_n(Y)$. Dualizing, we obtain $S_n(Y) \subseteq (\mathcal{O}_n(M)/J_n)^* \subseteq (\mathcal{O}_n(M)/V_n)^* \subseteq Z_n(M)$. So $S_n(Y) \subseteq Z_n(M)_s$ since $S_n(Y)$ is a toral algebra. But $\dim(S_n(Y)) = |X_n(\overline{T})/{\sim}\ |$ where $X_n(\overline{T}) = \{\alpha \in X(\overline{T}) \mid \theta\text{-degree}(\alpha) = n\}$, and $\sim$ is the equivalence relation of Proposition 2.5. On the other hand, $\dim(Z_n(M)_s = |Bl(S_n(M))| = |E_1(D_n)/{\sim}\ |$, where $T \subseteq D_n \subseteq S_n(M)$ and D_n is a maximal toral subalgebra. But as we noted in the proof of 2.8, there is a bijection $X_n(\overline{T}) \longleftrightarrow E_1(D)$ which identifies the above two equivalence relations. Hence, $\dim(S_n(Y)) = \dim(Z_n(M)_s)$, and so $S_n(Y) = Z_n(M)_s$. ■

3.4 Proposition. $Y = \bigcap_{g\in G} g\overline{T}g^{-1}$.

Proof. By 3.3, $Y \subseteq Z$ the center of M. So for any $g \in G, Y = gYg^{-1}$. Thus, if $Y \subseteq \overline{T}$ as guaranteed by 3.2, then $Y = gYg^{-1} \subseteq g\overline{T}g^{-1}$. Hence, $Y \subseteq \bigcap_{g\in G} g\overline{T}g^{-1} = V$. So consider the morphism $i^* : \mathcal{O}(V) \twoheadrightarrow \mathcal{O}(Y)$ induced from this inclusion. By Proposition 1.5, we obtain for each $n \geq 0$, a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, hook, "i"] \arrow[d, "\rho_n"'] & V \arrow[d, "\rho_n"] \\
S_n(Y) \arrow[r, "i_n"'] & S_n(V)
\end{tikzcd}$$

Furthermore, each i_n is injective, being dual to $\mathcal{O}_n(V) \twoheadrightarrow \mathcal{O}_n(Y)$. Now, even though V may not be reduced as a scheme, it is a subscheme of $\overline{T}$ and so each $S_n(V)$ is a toral k-algebra. Furthermore, $V \subseteq Z$ since V is rigid, and is normalized by the connected group G. Hence, each $S_n(V)$ is a central, toral subalgebra of $S_n(M)$. Hence, by Theorem 3.3, $S_n(Y) = S_n(V)$. ■

3.5 Theorem. *There is a canonical bijection*

$$Bl_n(M, \theta) \cong X_n(Y) = X_n(Z)$$

where $X_n(Y) = \{x : Y \to k \mid \theta\text{-}degree(x) = n\}$.

Proof. By definition, $Bl_n(M, \theta) = \{\text{blocks of } S_n(M)\}$. So if $\varphi : M \to B$ is a block of M, of θ-degree n, then we obtain (from 1.5) a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\rho_n"'] & B \arrow[d, "\rho_1 \cong"] \\
S_n(M) \arrow[r, "\varphi_1"'] \arrow[dr, "\gamma"'] & S_1(B) \arrow[d, "\cong"] \\
 & e_\varphi S_n(M)
\end{tikzcd}$$

where $\gamma(x) = e_\varphi x$ and $e_\varphi \in E_1(Z_n(M))$. But we then obtain

$$\begin{tikzcd}
Z_n(M) \arrow[r, hook] \arrow[d, "\varphi'"'] & S_n(M) \arrow[d, "\varphi_1"] \\
Z(B) \arrow[r, hook] \arrow[d, "\pi"'] & S_1(B) = B \\
k &
\end{tikzcd}$$

φ' exists and is unique since φ_1 is surjective. π is reduction modulo the nilradical of $Z(B)$. This determines a bijection

$$Bl_n(M, \theta) \longrightarrow Alg_k(Z_n(M), k)$$

defined by $\varphi \mapsto \pi \circ \varphi'$. But

$$\begin{aligned} Alg(Z_n(M), k) &= Alg(Z_n(M)_s, k) \\ &= Alg(\mathcal{O}_n(M)^*, k) \subseteq \mathcal{O}_n(Y) \\ &= X_n(Y) = X_n(Z). \end{aligned} \qquad \blacksquare$$

It is useful to know if there is something "extra" one can say if (M, θ) is a linear associative k-algebra with the canonical polarization $\theta : k^* \to M, \theta(\alpha) = \alpha 1$.

3.6 Proposition. *Let (M, θ) be a solvable linear associative k-algebra with its canonical polarization θ. Then Z, as defined in 3.1(b)(i), is isomorphic to $Z(M)$, the conventional center of M.*

Proof. $\mathcal{O}(M) = \mathrm{Symm}_k(U)$ where U is the k-linear dual of M. Then $\mathcal{O}(Z) = \mathcal{O}(M)/J$, where $J = \langle \bigoplus_{n \geq 0} V_n \rangle$, as in 3.1. Consider $J_1 = \langle V_1 \rangle \subseteq J$. Then, as in the proof of 3.1(a), $\Delta(J_1) \subseteq \mathcal{O}(M) \otimes J_1 + J_1 \otimes \mathcal{O}(M)$. So $\mathcal{O}(M) \twoheadrightarrow \mathcal{O}(M)/J_1$ represents the closed subscheme $Z_1 \subseteq M$. Clearly $Z \subseteq Z_1$ since $J_1 \subseteq J$. But $V_1 \subseteq \mathcal{O}_1(M)$ and so $Z_1 \subseteq M$ is a subalgebra with $\mathcal{O}(Z_1) \cong \mathrm{Symm}_k(\mathcal{O}_1(M)/V_1)$. But by definition of V_1, G acts trivially on $\mathcal{O}_1(M)/V_1$ and so $Z_1 \subseteq M$ is a central subscheme, in the sense of 3.1(b). Hence $Z_1 \subseteq Z$. $\blacksquare$

3.7 Example. We define polarizable, solvable monoids M and M' as follows:

$$\begin{gathered} M = \{(u, (r, s)) \mid u, r, s \in k\} \quad \text{with} \\ (u, (r, s))(v(k, l)) = (klu + r^2 v, (rk, sl)), \quad \text{and} \end{gathered}$$

$$\begin{gathered} M' = \{(u, (r, s)) \mid u, r, s \in k\} \quad \text{with} \\ (u, (r, s))(v, (k, l)) = (lu + rv, (rk, sl)). \end{gathered}$$

Define $\varphi : M' \to M$ by $\varphi(u, (r, s)) = (ru, (r, s))$. One checks that φ is a birational θ-morphism of degree one (we can omit reference to θ and ϕ here since each monoid is uniquely polarized). Furthermore, φ induces an isomorphism $\varphi : \overline{T}' \xrightarrow[\cong]{} \overline{T}$.

We now compute the center of each monoid. Clearly,

$$Z(M) = \{(0, (r, r)) \mid r \in k\}$$

since M' is isomorphic to the linear associative algebra of upper triangular 2×2 matrices.

On the other hand, let $(u, (r, s)), (v, (k, l)) \in M$. Then

$$\begin{gathered} (u, (r, s))(v, (k, l)) = (klu + r^2 v, (rk, sl)), \quad \text{and} \\ (v, (k, l))(u, (r, s)) = (rsv + k^2 u, (rk, sl)). \end{gathered}$$

So $(u, (r, s)) \in Z(M)$ if and only if, for all $(v, (k, l)), klu + r^2 v = rsv + k^2 u$. So if $r = 0$ then $klu = k^2 u$ for all k, l, thus forcing $u = 0$. Furthermore, we must have $r^2 v = rsv$ for all v; or what is the same $r = s$ or $r = 0$. We conclude that

$$\begin{aligned} Z(M) &= \{(u, (r, s)) \in M \mid u = 0 \text{ and } r = s \text{ or } r = 0\} \\ &= \{(0, (r, r)) \mid r \in k\} \cup \{(0, (0, s)) \mid s \in k\}. \end{aligned}$$

In particular, $Z(M') \subsetneq Z(M)$, so that M and M' will have different block structure even though $\varphi : M' \to M$ is a birational equivalence with $\overline{T}' \xrightarrow[\cong]{} \overline{T}$.

3.8 Example. We define a polarizable monoid N as follows:

$$N = \{(u,(r,s)) \mid u,r,s \in k\} \quad \text{with multiplication}$$
$$(u,(r,s))(v,(k,l)) = (k^2lu + r^3v, (rk, sl)).$$

One checks that $\overline{T} = \{(0,(\alpha,\beta)) \mid \alpha,\beta \in k\}$ is the closure in N of the maximal torus $T = \{((0,(\alpha,\beta)) \mid \alpha\beta \neq 0\}$. Assume $(0,(r,s)) \in \overline{T}$ is central. Then we must have

$$(0,(r,s))(r,(k,l)) = (v,(k,l))(0,(r,s)) \quad \text{for all} \quad v,k,l.$$

Thus, $r^3v = sr^2v$ for all v. So $r^3 = sr^2$. It follows that, with Y as in 3.2 (or 3.4) we obtain

$$\begin{aligned}\mathcal{O}(Y) &= k[U,R,S]/(U, R^3 - SR^2)\\ &\cong k[R,S]/(R^3 - SR^2).\end{aligned}$$

Our main purpose here is to show that $\mathcal{O}(Y)$ is not reduced as a k-algebra. Indeed, if $x = \overline{R}$ and $y = \overline{S}$ in $\mathcal{O}(Y)$ let $f = x(x-y) \in \mathcal{O}(Y)$. Then $f \neq 0$, yet $f^2 = x^2(x-y)^2 = (x^3 - yx^2)(x-y) = 0$.

One can obtain from this presentation of $\mathcal{O}(Y)$, a formula for the number of blocks of N, of each degree. In fact,

$$\begin{aligned}|Bl_0(N)| &= 1\\ |Bl_1(N)| &= 2\\ |Bl_n(N)| &= 3 \quad \text{if} \quad n \geq 2.\end{aligned}$$

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Department of Mathematics
The University of Western Ontario
London, Ontario N6A 5B7
Canada