THE CENTRALISER OF A SEMISIMPLE ELEMENT ON A REDUCTIVE ALGEBRAIC MONOID

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1 Introduction

Let G be a simply connected algebraic group and let $s \in G$ be a semisimple element. It is well known that $C_G(s) = \{g \in G \mid gs = sg\}$ is a connected subgroup of G which is uniquely determined up to conjugacy by a certain subset of the extended Dynkin diagram of G.

If M is a reductive monoid [4] with unit group G, the situation is more complicated. Is $C_M(s) = M_s$ always irreducible? (NO) If not, can we still obtain some numerical/combinatorial identification of these monoids? (YES) What sort of structure does the monoid M_s have?

The purpose of this paper is to answer the above questions in detail, and to supply some illustrative examples. Our three main results are as follows.

Let $B \subseteq G$ be a Borel subgroup with maximal torus $T \subseteq B$. Suppose $s \in T$ and M is a reductive algebraic monoid with unit group G. Let

$$\begin{aligned} G_s &= \{g \in G \mid gs = sg\} \\ B_s &= \{g \in B \mid gs = sg\} \\ R_s &= \{x \in \overline{N_G(T)} \mid xs = sx\} \end{aligned}$$

and define

$$\mathcal{R}_s = R_s/T = \{xT = Tx \mid x \in R_s\} .$$

Theorem 2.6: $M_s = \bigsqcup_{x \in \mathcal{R}_s} B_s x B_s$, *disjoint union.*

Theorems 3.1 and 3.2

(a) M_s *is a regular monoid.*

(b) $\mathcal{R}_s$ *is a finite inverse monoid.*

Theorem 3.3 *The following are equivalent.*

(a) M_s *is irreducible.*

(b) $\mathcal{R}_s$ *is unit regular.*

2 Main Result

Let M be a reductive monoid with unit group G [4]. We assume throughout that G is simply connected. By the results of [6] this ensures that for any semisimple element s of G, $C_G(s) = \{g \in G \mid sg = gs\}$ is connected. So let $B \subseteq G$ be a Borel subgroup with maximal torus $T \subseteq B$. We may assume that $s \in T$.

We now establish our notation and recall the relevant background results. Let

$$R = \{x \in M \mid xT = Tx\}.$$

Then $R = \overline{N_G(T)} \subseteq M$ (Zariski closure) and $\mathcal{R} = \{xT = Tx \mid x \in R\}$ is a finite inverse monoid with unit group $W = N_G(T)/T$, the *Weyl group.* If $x, y \in R$ and $x \equiv y$ in $\mathcal{R}$ then $BxB = ByB$. So $BxB \subseteq M$ is well-defined for $x \in \mathcal{R}$.

2.1 Theorem [5]. $M = \bigsqcup_{x \in \mathcal{R}} BxB$, *disjoint union.*

Our purpose here is to find an analogue of Theorem 2.1 for

$$M_s = \{x \in M \mid xs = sx\} .$$

So we let

$$\begin{aligned} G_s &= \{x \in M \mid xs = sx\} = C_G(s) \\ B_s &= C_B(s) \\ R_s &= C_R(s), \text{ and} \\ \mathcal{R}_s &= \{xT = Tx \in \mathcal{R} \mid xT \cap R_s \neq \phi\} . \end{aligned}$$

Notice that if $xT \cap R_s \neq \phi$ then $sxt = xts$ for $t \in T$. It follows easily that $xT \subseteq R_s$. So, indeed,

$$\mathcal{R}_s = \{xT \in \mathcal{R} \mid xT \subseteq R_s\} .$$

2.2 Lemma. *Let $r \in \mathcal{R}$. Then $BrB \cong rT \times k^a$ for some $a \geq 0$.*

Proof. Let $V = \{u \in U \mid urB \subseteq rB\}$ where $U \subseteq B$ is the unipotent part of B. Then it follows easily that $V = \{u \in U \mid urB = rB\} = \{u \in U \mid u\overline{rB} = \overline{rB}\}$, that $V \subseteq U$ is closed, and that $T \subseteq N_G(V)$. By [2; Proposition 28.1], $V = \prod_{U_\alpha \subseteq V} U_\alpha$, where $U_\alpha \subseteq U$ is a root subgroup, $\alpha \in \Phi_+$. Let $X = \prod_{U_\alpha \subseteq U} U_\alpha$. Then

$$X \times V \longrightarrow U$$

$$(u, v) \longmapsto xv$$

is an isomorphism [2, Proposition 28.1]. Thus,

$$\begin{aligned} BrB &= UTrB \\ &= UrTB \\ &= UrB \\ &= XVrB \\ &= XrB\ . \end{aligned}$$

Thus, $\varphi : X \times rB \longrightarrow BrB$ is surjective. But φ is also injective. Indeed, suppose that $xrb_1 = yrb_2$. Then $rb_1 = x^{-1}yrb_2$ and we obtain $rB = x^{-1}yrB$. Hence $x^{-1}y \in V$, and so $x^{-1}y = v \in V$. Thus $xV = yV$, and so $x = y$ since $x \times V \cong U$. So $BrB \cong X \times rB$.

Now let $Z = \{u \in U \mid rTu = rT\}$. As for V, $Z = \prod\limits_{U_\alpha \subseteq Z} U_\alpha$. So let $Y = \prod\limits_{U_\alpha \not\subseteq Z} U_\alpha$. As above, it follows that

$$rB = rTY \cong rT \times Y\ .$$

We conclude that

$$BrB \cong X \times rB \cong X \times rT \times Y\ .$$

But $X \times Y \cong k^a$ for some $a \geq 0$. ☐

2.3 Lemma. *There is a unique morphism of algebraic varieties $\psi : BrB \to rT$ such that*

(a)
$$\begin{tikzcd} rT \arrow[r, "i"] \arrow[d, "\alpha"'] & BrB \arrow[r, "\psi"] \arrow[d, "\alpha"'] & rT \arrow[d, "\beta"'] \\ rT \arrow[r, "i"] & BrB \arrow[r, "\psi"] & rT \end{tikzcd}$$
commutes where α is defined by $\alpha(x) = sxs^{-1}$ and β is induced from α.

(b) i is the inclusion.

(c) $\psi \circ i$ is an isomorphism.

Proof. $BrB = X \times rT \times Y$. So define $\psi(x, rt, y) = rt$. ψ is unique since it is the quotient morphism for the action $U \times U \times BrB \to BrB$, $(u, v, x) \mapsto uxv^{-1}$. The diagram commutes as long as β exists. But, for any $t \in T$, $s(UrtU)s^{-1} = UrtU$. ☐

2.4 Corollary. $(BrB)_s = \{b_1rb_2 \in BrB \mid sb_1rb_2s^{-1} = b_1rb_2\}$. *Then*

$$(BrB)_s \neq \phi \Leftrightarrow (rT)_s \neq \phi\ .$$

Proof. Assume $(BrB)_s \neq \phi$. Then $\psi((BrB)_s) \neq \phi$. But then $(rT)_s \neq \phi$ since $\psi \circ i$ is an $int(s)$-equivariant isomorphism. Conversely, if $(rT)_s \neq \phi$ then $(rT)_s \subseteq (BrB)_s$, and so $(BrB)_s \neq \phi$. ☐

2.5 Proposition.

$$(BrB)_s = \begin{cases} \phi, & r \notin \mathcal{R}_s \\ B_srB_s, & r \in \mathcal{R}_s \end{cases}$$

Proof. If $r \notin \mathcal{R}_s$ then $(rT)_s = \phi$ by definition, so $(BrB)_s = \phi$ by 2.4. So let $r \in \mathcal{R}_s$. We must show that $(BrB)_s = B_s r B_s$. Clearly, $B_s r B_s \subseteq (BrB)_s$. In the proof of Lemma 2.2 we showed that $BrB = X \times rT \times Y$. So

$$\begin{aligned} (BrB)_s &= (X \times rT \times Y)_s \\ &= X_s \times rT \times Y_s, \text{ since } sXs^{-1} = X \text{ and } sYs^{-1} = Y \\ &\subseteq B_s r B_s \end{aligned}$$

□

2.6 Theorem. *$M_s = \bigsqcup_{r \in \mathcal{R}_s} B_s r B_s$, disjoint union.*

Proof.

$$\begin{aligned} M_s &= \Big(\bigsqcup_{r \in \mathcal{R}} BrB\Big)_s, \text{ by 2.1} \\ &= \bigsqcup_{r \in \mathcal{R}_s} (BrB)_s \\ &= \bigsqcup_{r \in \mathcal{R}_s} B_s r B_s, \text{ by 2.5.} \end{aligned}$$

□

3 The Structure of $\mathcal{R}_s$ and M_s

In this section we examine in more detail $\mathcal{R}_s$ and M_s. But first we recall three definitions. A semigroup S is *regular* if for any $x \in S$ there exists $a \in S$ such that $xax = x$. S is *unit regular* if for any $x \in S$ there exists a unit $a \in S$ such that $xax = x$. A semigroup S is inverse if for any $x \in S$ there is a unique $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$.

3.1 Proposition. *$\mathcal{R}_s$ is a finite inverse monoid.*

Proof. $\mathcal{R}_s \subseteq \mathcal{R}$ which is finite by [5; Theorem 3.2.1]. So $\mathcal{R}_s$ is finite. Also, it is easily verified that $\mathcal{R}_s$ is a semigroup of $\mathcal{R}$. So let $x \in \mathcal{R}_0$ and let $r^* \in \mathcal{R}$ be the unique inverse (in $\mathcal{R}$) of r. Now $srts^{-1} = rt$ for some $t \in T$. Thus, $st^{-1}r^*s^{-1} = (srts^{-1})^* = (rt)^* = t^{-1}r^*$, and so $(r^*T)_s = (Tr^*)_s \neq \phi$, proving that $r^* \in \mathcal{R}_s$. □

3.2 Proposition. *M_s is a regular, algebraic monoid.*

Proof. By Theorem 2.6 we have $M_s = \bigsqcup_{r \in \mathcal{R}_s} B_s r B_s$. Clearly, M_s is a closed submonoid of M. Now let $x = b_1 r b_2 \in M_s$, where $b_1, b_2 \in B_s$ and $\mathcal{R}_s$. Define $a = b_2^{-1} r^* b_1^{-1} \in M_s$ where $r^* \in \mathcal{R}_s$ is the unique inverse of r. A simple calculation proves that $xax = x$ and $axa = a$. Thus, M_s is regular. □

3.3 Theorem. *The following are equivalent:*

(a) M_s *is irreducible.*

(b) $\mathcal{R}_s$ *is unit regular.*

Proof. Recall that $R_s = \{x \in \overline{N_G(T)} \mid xs = sx\}$. A simple calculation verifies that $\mathcal{R}_s$ is unit regular iff R_s is unit regular.

Assume M_s is irreducible. Then $M_s = \overline{C_G(s)}$ (Zariski closure). Now let $r \in R_s$, so that $rs = sr$ and $rT = Tr$. But also we have $r \in M_s$. Hence, $r \in \overline{N_{C_G(s)}(T)}$ and so $R_s = \overline{N_{C_G(s)}(T)}$. The latter is unit regular by [3; Theorem 13] and [4; Theorem 7.3].

Conversely, assume R_s is unit regular. So if $r \in \mathcal{R}_s$ there exist $\sigma \in N_{C_G(s)}(T)$ and $e \in I(\overline{T}) = \{f \in \overline{T} \mid f^2 = f\}$ such that $r = e\sigma$. Now let $x \in M_s$. Then $x = b_1 r b_2$, for some $r \in R_s$ and $b_1, b_2 \in B_s$. But $r = e\sigma$ as above, so that $x = b_1 e\sigma b_2 \in B_s\overline{T}N_{C_G(s)}(T)B_s \subseteq \overline{C_G(s)}$. Thus, $M_s = \overline{C_G(s)}$. ☐

4 Examples

4.1 Example. *Let* $M = M_n(k)$ *and let* $s \in M$ *be semisimple. Then* s *is conjugate to a matrix of the form*

$$\begin{pmatrix} \lambda_1 I_{n_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_s I_{n_s} \end{pmatrix}$$

where $\lambda_i \neq \lambda_j$ *if* $i \neq j$, $n_1 \geq \cdots \geq n_s$, *and* $\sum_{j=1}^{m} n_j = n$. *Then*

$$M_s \cong \prod_{i=1}^{m} M_{n_i}(k)$$

and

$$\mathcal{R}_s \cong \left\{ \begin{pmatrix} A_{n_1} & & 0 \\ & \ddots & \\ 0 & & A_{n_m} \end{pmatrix} \;\middle|\; \begin{array}{l} A_{n_i} \text{ is an } n_i \times n_i \text{ 0-1 matrix} \\ \text{with at most one nonzero en-} \\ \text{try in each row or column.} \end{array} \right\}$$

For this M, *any choice of* s *yields an irreducible* M_s.

4.2 Example. *Let* $\rho : S\ell_2 \times S\ell_2 \to G\ell_6$ *be defined by*

$$\rho(A,B) = \begin{pmatrix} A \otimes {}^t\!B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

Let $G_1 = \rho(S\ell_2 \times S\ell_2)$ *and let* $G = \{tg \mid t \in ZG\ell_6,\ g \in G_1\}$. *Then*

$$M = \overline{G} \subseteq M_6(k)$$

is a reductive algebraic monoid with unit group G *and maximal torus closure*

$$\overline{T} = \left\{ \ diag(w,x,y,z,r,s) \ \middle| \ \begin{array}{l} wz = xy = rs \\ r^2 = xz \\ s^2 = wy \end{array} \right\} .$$

We can calculate $E(\overline{T})$ *to obtain*

$$E(\overline{T}) = \{0,1\} \cup \{e_i \mid i = 1, \dots, 8\}$$

where

$$\begin{array}{ll} e_1 = (1,0,0,0,0,0) & e_5 = (1,1,0,0,0,0) \\ e_2 = (0,1,0,0,0,0 & e_6 = (0,0,1,1,0,0) \\ e_3 = (0,0,1,0,0,0) & e_7 = (1,0,1,0,0,1) \\ e_4 = (0,0,0,1,0,0) & e_8 = (0,1,0,1,1,0) \end{array}$$

It follows from [4; Theorem 10.7] *that the partially ordered set* $\{GxG \mid x \in G\}$ *is* $\{0, J_1, J_2, J_3, G\}$, *where*

$$J_1 = Ge_1G, \quad J_2 = Ge_5G \quad \textit{and} \quad J_3 = Ge_7G\ .$$

Furthermore, $J_3 > J_1$ *and* $J_2 > J_1$.

The Weyl group of G *is* $W = \{w_1, w_2, w_3, w_4\}$ *where*

$$w_1 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & 0 \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix} = \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$w_2 = \left(\begin{array}{cc|cc|cc} 0 & 1 & & 0 & & \\ -1 & 0 & & & & \\ \cline{1-4} & & 0 & 1 & & \\ & & -1 & 0 & & \\ \cline{3-6} & & & & 0 & 1 \\ & 0 & & & -1 & 0 \end{array}\right) = \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

$$w_3 = \left(\begin{array}{cccc|cc} & 0 & & 1 & & \\ & & -1 & & & \\ & -1 & & & 0 & \\ & & 0 & & & \\ 1 & & & & & \\ \hline & 0 & & & 0 & 1 \\ & & & & -1 & 0 \end{array}\right) = \rho\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$$

$$w_4 = \left(\begin{array}{c|c|c} 0 & \begin{matrix}-1 & 0\\ 0 & -1\end{matrix} & \\ \cline{1-2} \begin{matrix}-1 & 0\\ 0 & -1\end{matrix} & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} \end{array}\right) = \rho\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

Let $s \in T \subseteq G$ be semisimple. Then

$$s = a\rho(u, v)$$

where $u = diag(\alpha, \alpha^{-1})$ and $v = diag(\beta, \beta^{-1})$. So $s = a\ diag(\alpha\beta^{-1}, \alpha\beta, \alpha^{-1}\beta^{-1}, \alpha^{-1}\beta, \beta, \beta^{-1})$. After straightforward but tedious calculations, for example,

$$sw_3s^{-1} = \left(\begin{array}{cccc|cc} & 0 & & \alpha^2\beta^{-2} & & \\ & & -\alpha^2\beta^2 & & & \\ & -\alpha^{-2}\beta^{-2} & & & & 0 \\ \alpha^{-2}\beta^2 & & 0 & & & \\ \hline & & & & & \beta^2 \\ & 0 & & & -\beta^{-2} & \end{array}\right),$$

we arrive at the following possibilities for $\mathcal{R}_s$.

Case 1: $\beta = \pm\alpha$, $\alpha = \pm 1$.

$$\text{Then } \mathcal{R}_s = \mathcal{R}\ .$$

Case 2: $\beta = \pm\alpha$, $\alpha = \pm i$.

$$\text{Then } \mathcal{R}_3 = E(\overline{T}) \cup \{w_3e_j \mid j = 1, 2, \dots, 6\}\ .$$

Case 3: $\beta = \pm\alpha$ and $\alpha \neq \pm 1, \pm i$.

$$\text{Then } \mathcal{R}_3 = E(\overline{T}) \cup \{w_3e_1, w_3e_4\}\ .$$

Case 3′: $\beta = \pm\alpha^{-1}$, $\alpha \neq \pm 1, \pm i$ gives a set $\mathcal{R}_s$ conjugate to the $\mathcal{R}_s$ of Case 3.

Case 4: $\beta \neq \pm\alpha$, $\alpha = \pm 1$.

$$\text{Then } \mathcal{R}_s = E(\overline{T}) \cup w_4(E(\overline{T})\ .$$

Case 4′: $\beta \neq \pm\alpha^{-1}$, $\beta = \pm 1$ gives a set $\mathcal{R}_s$ conjugate to the set $\mathcal{R}_s$ of Case 4.

Case 5: $\beta \neq \pm\alpha$, $\beta \neq \pm\alpha^{-1}$, $\alpha \neq \pm 1$, $\beta \neq \pm 1$.

$$\text{Then } \mathcal{R}_s = E(\overline{T})\ .$$

In Case 1, $M_s = M$.
In Case 4 or Case $4'$, M_s is irreducible with unit group $k^*S\ell_2$.
In Case 5, $M_s = \overline{T}$.
In Case 2, Case 3 and Case $3'$, M_s is reducible.

Remarks: The monoid M_s is not of the type discussed in [7], unless of course it is irreducible. This leads to a number of basic questions about M_s:

(i) Which spherical varieties (for $(C_G(s) \times C_G(s))$ can occur as an irreducible component of M_s?

(ii) Does the inverse monoid $\mathcal{R}_s$ satisfy some analogue Tit's axiom "$sBx \subseteq BxB \cup BsxB$ if $s \in S$, $x \in \mathcal{R}$"?

(iii) Is there an analogue of the "type map" $\lambda : U \to 2^S$ of [7]? Recall that (irreducible) reductive monoids are "Monoids of Lie type" in the sense of [7]. This means that one can describe their semigroup-theoretic structure in terms of the BN pair of G and the type map.

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