Descent Systems for Bruhat Posets

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Abstract

Let (W, S) be a finite Weyl group and let $w \in W$. It is widely appreciated that the descent set
$$D(w) = \{s \in S \mid l(ws) < l(w)\}$$
determines a very large and important chapter in the study of Coxeter groups. In this paper we generalize some of those results to the situation of the Bruhat poset W^J where $J \subset S$. Our main results here include the identification of a certain subset $S^J \subset W^J$ that convincingly plays the role of $S \subset W$, at least from the point of view of descent sets and related geometry. The point here is to use this resulting *descent system* (W^J, S^J) to explicitly encode some of the geometry and combinatorics that is intrinsic to the poset W^J. In particular, we arrive at the notion of an *augmented poset*, and we identify the *combinatorially smooth* subsets $J \subset S$ that have special geometric significance in terms of a certain corresponding torus embedding $X(J)$.

Introduction

If (W, S) is a Weyl group and $w \in W$, $s \in S$, then either $ws < w$ or else $w < ws$. Hence we define
$$D(w) = \{s \in S \mid l(ws) < l(w)\},$$
the *descent set* of $w \in W$. This innocuous looking situation is at the heart of many important results in geometry, combinatorics, group theory and representation theory.

Evidently, the interest in these objects began with Solomon [14], who defines a certain subalgebra $B \subset \mathbb{Q}[W]$, and uses it to help understand the representations of W. B is often called the *descent algebra* since it can be defined in terms of descent sets. Brown [3] looks at this descent algebra and reconstitutes it as the semigroup algebra of a certain idempotent ("face") semigroup associated with the reflection arrangement of W.

The numbers $|D(w)|$ can be used to calculate the Betti number of the associated torus embedding $X(\phi)$ of W. These Betti numbers can be obtained directly from the h-vector of the associated rational, convex polytope. In [16] Stanley proves that the h-vector of any simplicial, convex polytope is a symmetric, unimodal sequence. Stembridge [13] proves that the canonical representation of W on $H^*(X(\phi);\mathbb{Q})$ is a permutation representation and, with the help of Dolgachev-Lunts [6], he computes this representation. In [2] Brenti studies these descent polynomials (i.e. the Poincaré polynomials of $X(\phi)$) as analogues of the *Eulerian polynomials*. He also looks at the q-analogues of these polynomials.

In the theory of group embeddings $|D(w)|$ is an important ingredient in the calculation of the Poincaré polynomial of a "wonderful" compactification of a semisimple group of adjoint type. See [5, 11].

In this paper we expand the entire study to include all Bruhat posets W^J, where $J \subset S$. In particular, we study the relationship between W^J and a certain torus embedding $X(J)$. This leads us to the notion of an *augmented poset* $(W^J, \leq, \{\nu_s\}_{s\in S\setminus J})$. Further scrutiny leads us to the notion of a *descent system* (W^J, S^J). These descent systems are particularly interesting if the $X(J)$ is quasi-smooth in the sense of Danilov [4]. We characterize, and then list, the possible subsets J of S for which $X(J)$ is quasi-smooth.

1 W-invariant Polytopes

Let V be a rational vector space and let $r : W \to Gl(V)$ be the usual reflection representation of the Weyl group W. Along with this goes the **Weyl chamber** $\mathcal{C} \subseteq V$ and the corresponding set of **simple reflections** $S \subseteq W$. W is generated by S, and $\mathcal{C}$ is a fundamental domain for the action of W on V. See [8] for details.

Let $\lambda \in \mathcal{C}$. In this section we describe the face lattice $\mathcal{F}_\lambda$ of the polytope $\mathcal{P}_\lambda = Conv(W \cdot \lambda)$, the convex hull of $W \cdot \lambda$ in V. It turns out that $\mathcal{F}_\lambda$ depends only on $W_\lambda = \{w \in W \mid w(\lambda) = \lambda\} = W_J =< s \mid s \in J >$, where $J = \{s \in S \mid s(\lambda) = \lambda\}$. Thus we describe $\mathcal{F}_\lambda = \mathcal{F}_J$ explicitly in terms of $J \subseteq S$.

Closely associated with these polytopes is a certain class of reductive algebraic monoids. We use what is known about this class of monoids to calculate $\mathcal{F}_J$ in terms of the underlying Dynkin diagram of (W, S).

We now recall some results first recorded in [10]. Throughout the paper we use some of the language and techniques of linear algebraic monoids. Unfortunately this theory is not widely appreciated, but luckily the main results and constructions have recently been assembled in [12]. Let M be an irreducible, normal algebraic monoid with reductive unit group G. We refer to such monoids as **reductive**. The reader can find any unproved statements about reductive monoids in [9, 12]. See Solomon's survey [15] for a less technical introduction to the subject.

If M is a reductive monoid with unit group G we let $B \subseteq G$ be a Borel subgroup of G and $T \subseteq B$ a maximal torus of G. $\overline{T}$ is the Zariski closure of T in M. $\overline{T}$ is a normal, affine torus embedding. The set of **idempotents** $E(\overline{T})$ of $\overline{T}$ is defined to be

$$E(\overline{T}) = \{e \in \overline{T} \mid e^2 = e\}.$$

There is exactly one idempotent in each T-orbit on $\overline{T}$. In the cases of interest in this paper, $E(\overline{T}) \setminus \{0\}$ can be canonically identified (as a poset) with face lattice $\mathcal{F}_\lambda$ above. We let $E_1 = E_1(\overline{T}) = \{e \in E(\overline{T}) \mid dim(Te) = 1\}$. In the above-mentioned identification, E_1 is identified with the vertices of $\mathcal{F}_\lambda$.

The $G \times G$-orbits of G are particularly interesting in this paper. Let

$$\Lambda = \{e \in \overline{T} \mid eB = eBe\}$$

be the **cross section lattice** of M relative to T and B. It is a basic fact that

$$M = \bigsqcup_{e \in \Lambda} GeG.$$

As above we let $S \subseteq W$ be the set of **simple involutions** of W relative to T and B. We regard S as the set of vertices of a graph with edges $\{(s,t) \mid st \neq ts\}$. Thus we may speak of the connected components of any subset of S.

A reductive monoid M with $0 \in M$ is called **$\mathcal{J}$-irreducible** if $M \backslash \{0\}$ has exactly one minimal $G \times G$-orbit.

Theorem 1.1. *Let M be a reductive monoid. The following are equivalent.*

1. *M is $\mathcal{J}$-irreducible.*
2. *There is an irreducible rational representation $\rho : M \to End(V)$ which is finite as a morphism of algebraic varieties.*
3. *If $\overline{T} \subseteq M$ is the Zariski closure in M of a maximal torus $T \subseteq G$ then the Weyl group W of T acts transitively on the set of minimal nonzero idempotents of $\overline{T}$.*

Notice in particular that one can construct, up to finite morphism, all $\mathcal{J}$-irreducible monoids from irreducible representations of a semisimple group. Indeed, let G_0 be semisimple and let $\rho : G_0 \to End(V)$ be an irreducible representation. Define $M_1 \subseteq End(V)$ to be the Zariski closure of $K^*\rho(G_0)$ where $K^* \subseteq End(V)$ is the set of homotheties. Finally let $M(\rho)$ be the normalization of M_1. Then, according to Theorem 1.1, $M(\rho)$ is $\mathcal{J}$-irreducible.

It turns out that, if M is $\mathcal{J}$-irreducible, there is a unique minimal nonzero idempotent $e \in E(\overline{T})$ such that $eB = eBe$, where B is the given Borel subgroup containing T. If M is $\mathcal{J}$-irreducible we say that M is **$\mathcal{J}$-irreducible of type** J if, for this idempotent e,

$$J = \{s \in S \mid se = es\},$$

where S is the set of simple involutions relative to T and B. The set J can be determined in terms of any irreducible representation satisfying condition *2* of Theorem 1.1. Indeed, let $\lambda \in X(T)_+$ be any highest weight such that $\{s \in S \mid s^*(\lambda) = \lambda\} = J$. Then $M(\rho_\lambda)$ is $\mathcal{J}$-irreducible of type J where ρ_λ is the irreducible representation of G_0 with highest weight λ. ρ_λ determines a representation of $M(\rho_\lambda)$ on V. Furthermore, any two $\mathcal{J}$-irreducible monoids with a finite, dominant morphism between them are of the same type. If e is the above-mentioned minimal idempotent then $B^-e = eB^-e$, where B^- is the Borel subgroup opposite B. eMe is a reductive monoid with idempotent set $\{0, e\}$ and thus $dim(eMe) = 1$. Hence eB^-e is also one-dimensional. Thus there exists a character $\chi : B^- \to K^*$ such that $be = ebe = \chi(b)e$ for all $b \in B^-$. It follows that B^- acts on $e(V)$ by the rule

$$\rho_\lambda(b)(v) = \rho_\lambda(b)(\rho_\lambda(e)(v)) = \chi(b)\rho_\lambda(e)(v) = \chi(b)v.$$

Therefore $L = e(V) \subseteq V$ is the unique one-dimensional $\rho_\lambda(B^-)$-stable subspace of V with weight λ. In particular, $\chi|T = \lambda$ and $P = \{g \in G_0 \mid \rho_\lambda(g)(L) = L\}$ is a parabolic subgroup of G_0 of type J.

We now describe the $G \times G$-orbit structure of a $\mathcal{J}$-irreducible monoid of type $J \subset S$. The following result was first recorded in [10].

Theorem 1.2. *Let M be a $\mathcal{J}$-irreducible monoid of type $J \subset S$.*

1. *There is a canonical one-to-one order-preserving correspondence between the set of $G \times G$-orbits acting on M and the set of W-orbits acting on the set of idempotents of $\overline{T}$. This set is canonically identified with $\Lambda = \{e \in E(\overline{T}) \mid eB = eBe\}$.*

2. *$\Lambda \setminus \{0\} \cong \{I \subseteq S \mid$ no connected component of I is contained entirely in $J\}$ in such a way that e corresponds to $I \subseteq S$ if $I = \{s \in S \mid se = es \neq e\}$. If we let $\Lambda_2 = \{e \in \Lambda \mid dim(Te) = 2\}$ then this bijection identifies Λ_2 with $S \setminus J$.*

3. *If $e \in \Lambda \setminus \{0\}$ corresponds to I, as in 2 above, then $C_W(e) = W_K$ where $K = I \cup \{s \in J \mid st = ts$ for all $t \in I\}$.*

See §7.3 of [12] for a systematic discussion of $\mathcal{J}$-irreducible monoids, in particular Lemma 7.8 of [12].

Let M be a $\mathcal{J}$-irreducible monoid of type $J \subset S$ and assume that $\rho : M \to End(V)$ is an irreducible representation which is finite as a morphism. Let G be the unit group of M with maximal torus $T \subset G$. Then let G_0 be the semisimple part of G with maximal torus $T_0 = G_0 \cap T$, and let $\rho_\lambda = \rho|G_0$, with highest weight $\lambda \in \mathcal{C}$, the rational Weyl chamber of G_0. Then, as above, $J = \{s \in S \mid s^*(\lambda) = \lambda\}$. Define

$$\mathcal{P}_\lambda = Conv(W \cdot \lambda)$$

the convex hull of $W \cdot \lambda$ in $X(T_0) \otimes \mathbb{Q}$.

We return to the situation of the beginning of this section. i.e. just reflection groups, no reductive monoids.

Corollary 1.3. *Let W be a Weyl group and let $r : W \to Gl(V)$ be the usual reflection representation of W. Let $\mathcal{C} \subseteq V$ be the rational Weyl chamber and let $\lambda \in \mathcal{C}$. Assume that $J = \{s \in S \mid s^*(\lambda) = \lambda\}$. Then the set of orbits of W on the face lattice $\mathcal{F}_\lambda$ of $\mathcal{P}_\lambda$ is in one-to-one correspondence with $\{I \subseteq S \mid$ no connected component of I is contained entirely in $J\}$.*

The subset $I \subseteq S$ corresponds to the unique face $F \in \mathcal{F}_\lambda$ with $I = \{s \in S \mid s(F) = F$ and $s|F \neq id\}$ whose relative interior F^0 has nonempty intersection with $\mathcal{C}$.

Let M be a $\mathcal{J}$-irreducible monoid of type $J \subset S$ and let $\overline{T}$ be the closure in M of a maximal torus T of G. By part *b)* of Theorem 5.4 of [12], $\overline{T}$ is a normal variety. Define

$$X(J) = (\overline{T} \setminus \{0\})/K^*.$$

The terminology is justified since $X(J)$ depends only on J and not on M or λ. The set of distinct $\mathcal{J}$-irreducible monoids associated with $X(J)$ can be identified with the set $\mathcal{C}^J = \{\lambda \in \mathcal{C} \mid C_S(\lambda) = J\}$. In the case $J = \phi$, $X(J)$ is the torus embedding studied in [2, 6, 13].

2 The Augmented Poset

In this section we define the **augmented poset** $(W^J, \leq, \{\nu_s\})$ associated with the subset J of S. Unfortunately this requires that we engage those dreaded reductive monoids. We use them to obtain some important results relating W^J to a certain finite, partially ordered set E_1 of idempotents. That done, we obtain the desired "ascent/descent" structure on the poset W^J. See Proposition 2.14. Our construction has a fundamental relationship with the extremely important *descent systems* as discussed in Theorem 2.19 and Section 4. The reader who does not want to *engage the monoids* might be a able to find his own proofs of Proposition 2.14 and Theorem 2.19 using his favourite techniques. See the table in Remark 2.20 for a handy, but brief, translation between the monoid jargon and the Bruhat poset jargon.

Lemma 2.1. *Let $e \in E_1(\overline{T})$. Then*

$$\overline{eB}\backslash\{0\} = \cup_{\tau \in X} e\tau B$$

where $X = \{\tau \in W \mid eB\tau^{-1}e \neq 0\}$.

Proof. We first show that $\overline{eB}\backslash\{0\} \subset \cup_{\tau \in W} e\tau B$. To this end, first recall $e_1 \in E_1(\overline{T})$, the unique rank-one idempotent such that $e_1 B = e_1 B e_1$. Then $e_1 G = \sqcup_{w \in W} e_1 B w B = \cup_{w \in W} e_1 w B$. Thus, if $e = \gamma e_1 \gamma^{-1} \in E_1$, one checks that

$$eG = \gamma e_1 \gamma^{-1} G = \cup_{w \in W} \gamma e_1 \gamma^{-1} \gamma w B = \cup_{\tau \in W} e\tau B.$$

Hence $\overline{eB}\backslash\{0\} \subseteq eG \subseteq \cup_{\tau \in W} e\tau B$.

Thus it suffices to show that $X = \{\tau \in W \mid e\tau \in \overline{eB}\}$. Suppose then, that $e\tau \in \overline{eB}$. Then $0 \neq e\tau\tau^{-1}e\tau \in \overline{eB}\tau^{-1}e\tau$. Thus $eB\tau^{-1}e\tau \neq 0$. Conversely, suppose that $eB\tau^{-1}e\tau \neq 0$. Then there exists $b \in B$ such that $0 \neq x = eb\tau^{-1}e\tau$. Then $0 \neq x = ex = x\tau^{-1}e\tau \in eB\tau^{-1}e\tau$. Thus $e\tau \in K^*e\tau \subseteq eM\tau^{-1}e\tau = eB\tau^{-1}e\tau \subseteq \overline{eB}$ since $B\tau^{-1}e\tau \subseteq \overline{B}$. ∎

Corollary 2.2. *Let $e \in E_1(\overline{T})$ and let $f \in E(\overline{T})$. Then*

$$\overline{eB}f = \{0\} \cup (\cup_{\tau \in X} e\tau Bf).$$

Proposition 2.3. *The following are equivalent.*

1. *$ef = e$, and for all $\tau \in X$ with $\tau^{-1}e\tau \neq e$, $e\tau Bf = 0$.*

2. *$eBf = eBe$.*

Proof. Assume *1.* Then, by Lemma 2.1, $\overline{eB}f = \{0\} \cup eBf = \{0\} \cup efBf$ and this a closed subset of M. But $efBf = eC_B(f)$, and thus $\overline{eB}f = eC_B(f) \cup \{0\}$. Hence $\overline{eB}f$ is the union of two right $C_B(f)$-orbits: $eC_B(f)$ and $\{0\}$. But $C_B(e)$ is a connected, solvable group. Thus, by Theorem 3.1 of [7], $dim(eBf) = 1$ since there exists $h \in K[\overline{eB}f]$ such that $\{0\} = h^{-1}(0)$. Since $eBe \subseteq eBf$, it follows that $eBe = eBf$.

Conversely, assume *2*. Thus $\overline{eB}f = \overline{eBe} = \{0\} \cup eBe$. But from Lemma 2.1 $\overline{eB}f = \{0\} \cup (\cup_{\tau \in X} e\tau Bf)$. Assume that $e\tau Bf \neq 0$. Then we have

$$\phi \neq e\tau Bf \backslash \{0\} \subseteq \overline{eBe} \backslash \{0\} = eBe = K^* e.$$

Thus,

$$e \in e\tau Bf \subseteq e\tau BfB \subseteq \overline{e\tau B}$$

since $BfB \subseteq \overline{B}$. But $e\tau B \subseteq \overline{eB}$ and thus $e\tau B = eB$. Hence $e\tau = e$ and finally $\tau^{-1} e\tau = e$ □

Definition 2.4. Let $e, e' \in E_1(\overline{T})$. We say that $e < e'$ if $eBe' \neq 0$ and $e \neq e'$.

We shall see in Proposition 2.9 that $e < e'$ if and only if $\overline{BeG} \subsetneq \overline{Be'G}$.

Theorem 2.5. *Let $e \in E_1$ and let $f \in E$. The following are equivalent.*

1. *$eBf = eBe$.*
2. *(a) $ef = e$.*
   (b) If $e < e'$ then $e'Bf = 0$.
3. *(a) $ef = e$.*
   (b) If $e < e'$ then $e'f = 0$.

Proof. The equivalence of *1* and *2* is a reformulation of Proposition 2.3, taking into account Definition 2.4. That *2* implies *3* is obvious. So we assume *3* and then deduce *1*. By Lemma 2.1

$$\overline{eB} \backslash \{0\} = \cup_{\tau \in X} e\tau B$$

where $X = \{\tau \in W \mid eB\tau^{-1}e \neq 0\}$. Now $ef = e$ so that $eBf = efBf$. Thus $\overline{eB}f = e\overline{fBf} = e\overline{C}f$, where $C = C_B(f)$. Thus, again by Proposition 2.3,

$$e\overline{C}f \backslash \{0\} = \cup_{\gamma \in Y} e\gamma Cf$$

where $Y = \{\gamma \in W \mid eC\gamma^{-1}e\gamma \neq 0\}$. But if $eCe' \neq 0$ then $eBe' \neq 0$ and then, by assumption, $e'f = 0$ as long as $e' \neq e$. Hence $e\gamma f = 0$ if $\gamma^{-1}e\gamma \neq e$, and thus $e\gamma Cf = e\gamma fC = 0$ for $\gamma \in Y$. Thus $eBf = \{0\} \cup eCf$, which (as in the proof of Proposition 2.3) is one-dimensional. Thus $eBf = eBe$. □

Definition 2.6. Let $e \in E_1$. Define

$$\mathfrak{X}_e = \{f \in E(\overline{T}) \mid fe = e \text{ and } fe' = 0 \text{ for all } e' > e\}.$$

Then from Theorem 2.5

$$E(\overline{T}) \setminus \{0\} = \sqcup_{e \in E_1} \mathfrak{X}_e.$$

Proposition 2.7. *The following are equivalent for $r, s \in GJ$.*

1. *$BrG \subseteq \overline{BsG}$.*

2. $Br \subseteq \overline{Bs}$.

Proof. The case "*2* implies *1*" is clear. To prove "*1* implies *2*" we shall use the fact that $B\backslash G$ is a complete variety. Since $s \in GJ$ we have that $BsB = Bs$. Thus $\overline{Bs}B = \overline{Bs}$. But then, by a result of Steinberg, $\overline{Bs}G = \overline{BsG}$ since $B\backslash G$ is a complete variety. Now the assumption of *1* is equivalent to saying that $BrG \subseteq \overline{Bs}G$. Thus we can write $r = yg^{-1}$ where $y \in \overline{Bs}$ and $g \in G$. Hence $rg \in \overline{Bs}$. Thus $BrgB \subseteq \overline{Bs}$. But $BrgB = BrBgB = BrBwB$ for some $w \in W$. But $1 \in \overline{BwB}$, and consequently $BrB \subseteq \overline{BrBwB}$. We conclude that $BrB \subseteq \overline{Bs}$. □

Recall that, for $J \subset S$,

$$W^J = \{t \in W \mid t \text{ has minimal length in } tW_J\}.$$

Define also

$$^JW = \{t \in W \mid t \text{ has minimal length in } W_Jt\}.$$

Theorem 2.8. *Let $r = ve_1, s = we_1$ where $v, w \in W^J$. The following are equivalent.*

1. *$r \leq s$ (i.e. $BrB \subset \overline{BsB}$).*

2. *$w \leq v$ (i.e. $BwB \subset \overline{BvB}$).*

Proof. We apply Corollary 8.35 of [12]. But we notice first that, in that setup, Λ is $\{e \in E(\overline{T}) \mid Be = eBe\}$ while in the present discussion, Λ is $\{e \in E(\overline{T}) \mid eB = eBe\}$. To eliminate any potential confusion we shall first state Corollary 8.35 using $\Lambda = \{e \in E(\overline{T}) \mid eB = eBe\}$.

Write

$$W_{I_1} = \{w \in W \mid we = ew = e\} \text{ and } W_{I_2} = \{w \in W \mid we = ew\},$$

and

$$W_{J_1} = \{w \in W \mid wf = fw = f\} \text{ and } W_{J_2} = \{w \in W \mid wf = fw\}.$$

Let $a = y^{-1}ex$ and $b = t^{-1}fs$ where $x \in^{I_1} W$, $y \in^{I_2} W$, $s \in^{J_1} W$ and $t \in^{J_1} W$. This is the *normal form* for the elements of $\mathcal{R}$ as in Definition 8.34 of [12]. Then (from Corollary 8.35 of [12]) the following are equivalent.

i) $a \leq b$.

ii) $ef = e$, and there exists $w \in W_{I_1}W_{J_2}$ such that $x \leq ws$ and $wt \leq y$.

In our situation $W_{I_1} = W_{I_2} = W_{J_1} = W_{J_2}$, and $x = s = 1$. So condition *ii*) becomes

ii)' $ef = e$, and there exists $w \in W_{I_1}$ such that $1 \leq w$ and $wt \leq y$.

which is equivalent to

ii)" $ef = e$ and $t \leq y$.

since $t \leq wt$ for all $w \in W_{I_1}$

Now observe that $t \leq y$ if and only if $t^{-1} \leq y^{-1}$, while $({}^I W)^{-1} = W^I$.

Thus the result follows with $v = y^{-1}$ and $w = t^{-1}$. □

Notice that this might appear counterintuitive. Think of e_1 as "large as possible on the left" and that, somehow, multiplication by w on the left makes the result smaller "on the left". Thus, if w is less than v, then ve_1 is less than we_1.

Proposition 2.9. *The following are equivalent for $e, f \in E_1$.*

1. *$e < e'$ (in the the ordering of Definition 2.4 on E_1.).*
2. *$BeG \subset \overline{Be'G}$.*

Proof. If $BeG \subset \overline{Be'G}$ we first observe that $e\overline{BeG} \neq 0$. But $e\overline{BeG} \subset \overline{Be'G}$, and thus $eBe'G \neq 0$. Hence $eBe' \neq 0$.

Conversely, if $eBe' \neq 0$ then $eBe'G \neq 0$, and thus $e\overline{Be'G} \neq 0$. But $eM = eG \cup \overline{\{0\}}$ since $e \in E_1$. Thus $e \in e\overline{Be'G} = eM$. But $e\overline{Be'G} \subset \overline{Be'G}$ since $eB \subset \overline{B}$. Thus $e \in \overline{Be'G}$ and finally $BeG \subset \overline{Be'G}$. □

Notice that $BeG = BrG$ for $r \in We_1 \cap eW = \{r\}$ (See Section 8.3 of [12]). Similarily for e' and $s \in We_1 \cap e'W = \{s\}$. Thus an equivalent statement is "$BrG \subset \overline{BsG}$" for these $r, s \in GJ$.

Theorem 2.10. *The following are equivalent for $v, w \in W^J$.*

1. *$e = ve_1v^{-1} < e' = we_1w^{-1}$ in $(E_1, <)$.*
2. *$w < v$ in $(W^J, <)$.*

Proof. By Proposition 2.9, $e < e'$ if and only if $BeG \subset \overline{Be'G}$. As in the above comment, let $BeG = BrG$ and $Be'G = BsG$ where $r, s \in GJ$.

By Proposition 2.7, $BrG \subset \overline{BsG}$ if and only if $Br \subseteq \overline{Bs}$. Then by Theorem 2.8, $Br \subseteq \overline{Bs}$ if and only if $w < v$, where $r = ev = ve_1$, $s = e'w = we_1$ and $v, w \in W^J$. □

For $e \in E_1(\overline{T})$ we let

$$\Gamma(e) = \{g \in E_2(\overline{T}) \mid ge = e, \text{ and } ge' = 0 \text{ for all } e' > e\}.$$

Corollary 2.11. *Let $g \in E_2(\overline{T})$. Suppose that $e, f \in E_1(\overline{T})$ and that $e \neq f$. Assume that $ge = e$ and $gf = f$. Then either $e > f$ or else $f > e$. In particular*

$$\Gamma(e) = \{g \in E_2(\overline{T}) \mid ge = e, \text{ and } ge' = e' \text{ for some } e' < e\}.$$

Proof. Suppose that $e \not> f$. Recall Definition 2.6. Then $g \in \mathfrak{X}_f$, since we have that $ge' = 0$ for any $e' > f$. In particular, $g \notin \mathfrak{X}_e$. Thus there exists $e' > e$ such that $ge' = e'$. But then $e' = f$ since $g \in E_2$. Thus $f > e$. □

Remark 2.12. If we think of $\leq$ as a relation on E_1 then Corollary 2.11 says that we can regard E_2 as a subrelation of $\leq$. Notice, in particular, that

$$E_2 = \sqcup_{e \in E_1} \Gamma(e).$$

Recall that $\Lambda_2 = \{e \in E_2 \mid eB = eBe\}$. It follows from part *2* of Theorem 1.2 that there is a canonical bijection

$$\Lambda_2 \cong S \setminus J.$$

This bijection is defined by

$$s \rightsquigarrow g_s,$$

where $g_s \in \Lambda_2$ is the unique idempotent such that

1. $sg_s = g_s s \neq g_s$.
2. $g_s B \subseteq B g_s$.

Since each $g \in \Gamma(e)$ is conjugate to one and only one $g_s \in \Lambda_2$ we can write

$$\Gamma(e) = \sqcup_{s \in S \setminus J} \Gamma_s(e),$$

where

$$\Gamma_s(e) = \{g \in \Gamma(e) \mid g = v g_s v^{-1} \text{ for some } v \in W\}.$$

We now translate, as completely as possible, our monoid results into results about Bruhat posets. Theorem 2.10 is the main ingredient here that makes this possible. We let

$$S^J = (W^J(S \setminus J)W^J) \cap W^J.$$

Proposition 2.13. *There is a canonical identification* $S^J \cong \{g \in E_2 \mid ge_1 = e_1\}$.

Proof. Define

$$\varphi : W_J(S \setminus J)W_J \to E_1$$

by $\varphi(w) = we_1w^{-1}$. Then $\varphi(w) = \varphi(v)$ if and only if $wW_J = wW_J$. Then φ induces an injection $\varphi : S^J \to E_1$. Now let $w = usv \in W_J(S \setminus J)W_J$ where $s \in S \setminus J$ and $v, w \in W_J$. Then

$$\varphi(w) = use_1su^{-1}.$$

Now there is a unique $g_s \in \Lambda_2$ such that $g_s e_1 = e_1$ and $g_s se_1 s = se_1 s$. Thus $g = ug_s u^{-1}$ is the unique $g \in E_2$ such that $ge_1 = e_1$ and $guse_1su^{-1} = use_1su^{-1}$. Thus we can think of S^J as a subset of $\{e \in E_1 \mid ge = e \neq e_1 \text{ and } ge_1 = e_1 \text{ for some } g \in E_2\}$, via $r = usv \rightsquigarrow e_r = use_1su^{-1}$. On the other hand if $g \in E_2(e_1) = \{g \in E_2 \mid ge_1 = e_1\}$ then, by Proposition 6.27 of [9] and Theorem 1.2, $g = ug_s u^{-1}$ for some $s \in S \setminus J$ and some $u \in W_J$. We conclude that φ induces a canonical bijection

$$S^J \cong \{e \in E_1 \mid ge = e \neq e_1 \text{ and } ge_1 = e_1 \text{ for some } g \in E_2\}.$$

via $r \rightsquigarrow e_r$. On the other hand, if $r = usv \in S^J$, write $g_r = ug_s u^{-1} \in E_2(e_1)$. Then $r = usv \rightsquigarrow g_r = ug_s u^{-1}$ determines a bijection

$$S^J \cong E_2(e_1).$$

□

Proposition 2.14. *Let $u, v \in W^J$ be such that $u^{-1}v \in S^J W_J$. Then either $u < v$ or $v < u$ in the Bruhat order $<$ on W^J.*

Proof. If $u, v \in W^J$ with $v = urc$, $r \in S^J$, $c \in W_J$, consider as in Proposition 2.13, $g_r \in E_2(e_1)$. Then let $g = ug_r u^{-1}$. Then g is the unique rank-two idempotent such that $gue_1u^{-1} = ue_1u^{-1}$ and $gve_1v^{-1} = ve_1v^{-1}$.

Recall from Theorem 2.10 that, for $u, v \in W^J$

$$ue_1u^{-1} > ve_1v^{-1} \text{ if and only if } u < v.$$

But from Corollary 2.11, for $g \in E_2$ with $ge_i = e_i$, $i = 2, 3$, either $e_2 > e_3$ or else $e_3 > e_2$. The conclusion follows. □

Definition 2.15. Let $w \in W^J$. Define

1. $D_s^J(w) = \{r \in S_s^J \mid wrc < w \text{ for some } c \in W_J\}$, and
2. $A_s^J(w) = \{r \in S_s^J \mid w < wr\}$.

$D^J(w) = \sqcup_{s \in S \setminus J} D_s^J(w)$ is the *descent set* of w and $A^J(w) = \sqcup_{s \in S \setminus J} A_s^J(w)$ is the *ascent set* of w relative to J.

By Proposition 2.14, for any $w \in W^J$, $S^J = D^J(w) \sqcup A^J(w)$.

Remark 2.16. Notice that $wrc < w$ for some $c \in W_J$ if and only if $(wr)_0 < w$, where $(wr)_0 \in wrW_J$ is the element of minimal length in wrW_J. See Example 4.5 for a revealing illustration of the fact that $S^J = D^J(w) \sqcup A^J(w)$.

Definition 2.17. For each $v \in W^J$ and each $s \in S \setminus J$ define $\nu_s(v) = |A_s^J(v)|$. We refer to $(W^J, \leq, \{\nu_s\})$ as the *augmented poset* of J. For convenience we let

$$\nu(v) = \sum_{s \in S \setminus J} \nu_s(v).$$

Example 2.18. Let (W, S) be the Weyl group of type A_3, so that $W = S_4$ and $S = \{s_1, s_2, s_3\}$. Let $J = \phi$ and write ν_i for ν_{s_i}. To keep track of all the numbers $\{\nu_i(w) \mid w \in W\}$ define

$$H(t_1, t_2, t_3) = \sum_{w \in W} t_1^{\nu_1(w)} t_2^{\nu_2(w)} t_3^{\nu_3(w)}.$$

A straightforward calculation yields

$$H(t_1, t_2, t_3) = 1 + (3t_1 + 5t_2 + 3t_3) + (3t_2t_3 + 5t_1t_3 + 3t_1t_2) + t_1t_2t_3.$$

Theorem 2.19. *Let $J \subset S$ be any proper subset. For $e = ue_1u^{-1}$, $u \in W$, we write $e = e_u$.*

1. *$E_2 \cong \{(u, v) \in W^J \times W^J \mid u < v \text{ and } u^{-1}v \in S^J W_J\}$.*
2. *Let $u \in W^J$ and $e_u = ueu^{-1} \in E_1$. Then $E_2(e_u) \cong \{v \in W^J \mid u^{-1}v \in S^J W_J\}$.*

3. *Let* $u \in W^J$ *and* $e_u = ueu^{-1} \in E_1$. *Then* $\Gamma(e_u) \cong \{v \in W^J \mid u < v \text{ and } u^{-1}v \in S^J W_J\} \cong A^J(u)$.

4. *Let* $u \in W^J$ *and* $e_u = ueu^{-1} \in E_1$. *Then* $\Gamma_s(e_u) \cong \{v \in W^J \mid u < v \text{ and } u^{-1}v \in S_s^J W_J\} \cong A_s^J(u)$.

5. *If* $w \in W^J$ *and* $s \in S \setminus J$ *then* $\nu_s(u) = |\Gamma_s(e_u)|$.

Proof. This follows from Proposition 2.13 and Proposition 2.14. □

Remark 2.20. The following table provides the reader with a summary-translation between the monoid jargon and the Bruhat poset jargon. Let $E = E(\overline{T})$ be the set of idempotents of $\overline{T}$ and let $E_i = \{f \in E \mid dim(fT) = i\} \subset E$. As above, we let $e_1 \in E_1 = E_1(\overline{T})$ be the unique element such that $e_1 B = e_1 B e_1$. For $e, e' \in E_1$ let $v, w \in W^J$ be the unique elements such that $e = ve_1 v^{-1}$ and $e' = we_1 w^{-1}$. We write $e = e_v$ and $e' = e_w$. For $e, f \in E$ we write $e \sim f$ if there exists $w \in W$ such that $wew^{-1} = f$. If $s \in S \setminus J$ let $g_s \in E_2$ be the unique idempotent such that $g_s s = sg_s$ and $g_s B = g_s B g_s$. Let $\Lambda^\times = \{I \subset S \mid \text{no component of } I \text{ is contained in } J\}$ and for $I \in \Lambda^\times$ let $I^* = I \cup \{t \in J \mid ts = st \text{ for all } s \in I\}$.

Reductive Monoid Jargon	**Bruhat Order Jargon**
$e_1 \in \Lambda_1 = \{e_1\}$	$1 \in W^J$
$e = e_v \in E_1$	The $v \in W^J$ with $e = ve_1 v^{-1}$
$e_v \leq e_w$ in E_1, i.e. $e_v B e_w \neq 0$	$w \leq v$ in W^J
$E_2 = \{g \in E \mid dim(gT) = 2\}$	$(u, v) \in W^J \times W^J$ such that $u < v$ and $u^{-1}v \in S^J W_J$
$\{g \in E_2 \mid gB = gBg\}$	$S \setminus J$
$\{g \in E_2 \mid ge_1 = e_1\}$	$S^J = (W_J(S \setminus J)W_J) \cap W^J$
$\{g \in E_2 \mid ge_1 = e_1, g \sim g_s\}$	$S_s^J = (W_J s W_J) \cap W^J$
$E_2(e_w) = \{g \in E_2 \mid ge_w = e_w\}$	$\{v \in W^J \mid w^{-1}v \in S^J W_J\}$
$\Gamma(e_w) = \{g \in E_2(e_w) \mid ge' = e' \text{ for some } e' < e_w\}$	$A^J(w) = \{r \in S^J \mid w < wr\}$
$\Gamma_s(e_w) = \Gamma(e_w) \cap \{g \in E_2 \mid g \sim g_s\}$	$A_s^J(w) = \{r \in S_s^J \mid w < wr\}$
$E(\overline{T}) \setminus \{0\}$	$\{(w, I) \mid I \in \Lambda^\times,\ w < ws \text{ if } s \in I^*\}$

The "picture" here is this. W^J is canonically identified with the set of vertices of the rational polytope $\mathcal{P}_\lambda$. On the other hand there is a canonical ordering on $E_1 = E_1(\overline{T})$ comming from the associated reductive monoid. Evidently $(E_1, \leq)$ and $(W^J, \leq)$ are anti-isomorphic as posets. Furthermore the set of edges $Edg(\mathcal{P}_\lambda)$ of $\mathcal{P}_\lambda$ is canonically identified with $E_2 = E_2(\overline{T})$. If $g(v, w) = g(w, v) \in Edg(\mathcal{P}_\lambda)$ is the edge of $\mathcal{P}_\lambda$ joining the distinct vertices $v, w \in W^J$ then either $v < w$ or else $w < v$. Given $v \in W^J$, with edges $Edg(v) = \{g \in E_2 \mid g = g(v, w) \text{ for some } w \in W^J\}$, the question of whether $v < w$ or $w < v$ is coded in the "descent system" (W^J, S^J).

3 Bruhat Posets and Simple Polytopes

Recall that if $\lambda \in \mathcal{C}$, then the rational polytope $\mathcal{P}_\lambda$ records the combinatorial properties of the orbit structure of T on $\overline{T}$. In this section we characterize, in terms of $J \subseteq S$, the conditions under which $\mathcal{P}_\lambda$ is a **simple polytope**. A polytope $\mathcal{P}$ is called *simple* if each vertex figure of $\mathcal{P}$ is a simplex, or equivalently, each vertex is the endpoint of exactly m edges $\mathcal{P}$, where m is the dimension of $\mathcal{P}$.

Definition 3.1. We refer to $X(J)$, J and $\mathcal{P}_\lambda$ as *combinatorially smooth* if $\mathcal{P}_\lambda$ is a simple polytope.

Notice that this condition is equivalent to saying that $X(J)$ is *quasi-smooth* in the sense of §14 of [4].

Let $e \in E_1$ be the unique rank-one idempotent such that $eB = eBe$. If $J \subset S$ we let $\pi_0(J)$ denote the set of connected components of J. The following theorem indicates exactly how to detect the very interesting condition of Definiton 3.1.

Theorem 3.2. *Let $\lambda \in \mathcal{C}$. The following are equivalent.*

1. *$\mathcal{P}_\lambda$ is a simple polytope.*
2. *There are exactly $|S|$ edges of $\mathcal{P}_\lambda$ meeting at λ.*
3. *$J = \{s \in S \mid s(\lambda) = \lambda\}$ has the properties*
   - *(a) If $s \in S\backslash J$, and $J \not\subseteq C_W(s)$, then there is a unique $t \in J$ such that $st \neq ts$. If $C \in \pi_0(J)$ is the unique connected component of J with $t \in C$ then $C\backslash\{t\} \subseteq C$ is a setup of type $A_{l-1} \subseteq A_l$.*
   - *(b) For each $C \in \pi_0(J)$ there is a unique $s \in S\backslash J$ such that $st \neq ts$ for some $t \in C$.*

Proof. *1* and *2* are equivalent by standard results about polytopes.

Assume that *3* holds. We now deduce from this that *2* holds. This is equivalent to the statement $|\{f \in E_2(\overline{T}) \mid fe = e\}| = |S|$. Now

$$\Lambda_2 \cong S\backslash J$$

via the correspondence $f = f_s$ if $sf = fs \neq f$. See Theorem 4.16 of [10]. So we write

$$\Lambda_2 = \{f_s \mid s \in S\backslash J\}.$$

Then from part (iii) of Proposition 6.27 of [9]

$$\{f \in E_2(\overline{T}) \mid fe = e\} = \bigcup_{w \in W_J} w\Lambda_2 w^{-1} = \bigcup_{s \in S\backslash J} Cl_{W_J}(f_s).$$

Let $s \in S\backslash J$.
Case 1: $st = ts$ for all $t \in J$.

Then $f_s w = w f_s$ for all $w \in W_J$. In this case $Cl_{W_J}(f_s) = \{f_s\}$.
Case 2: $ts \neq st$ *for some unique* $t \in J$. Let C be that unique component with $t \in C$. Thus $C_{W_J}(f_s) = W_{J\backslash\{t\}}$ and, consequently, $Cl_{W_J}(f_s) \cong W_J/W_{J\backslash\{t\}}$. But, by assumption,

$$W_J/W_{J\backslash\{t\}} \cong W_C/W_{C\backslash\{t\}} \cong S_{m+1}/S_m,$$

where $|C| = m$ and S_m is the symmetric group on m letters. Thus

$$|Cl_{W_J}(f_s)| = |S_{m+1}/S_m| = \frac{(m+1)!}{m!} = m+1.$$

Since, by assumption, each C occurs for exactly one $t \in S\backslash J$, we conclude that

$$|\{f \in E_2(\overline{T}) \mid fe = e\}| = (\sum_{C\in\pi_0(J)} (|C|+1)) + |\{s \in S\backslash J \mid st = ts \text{ for all } t \in J\}|.$$

But $(\sum_{C\in\pi_0(J)}(|C|+1)) = |J| + |\pi_0(J)|$ while $|\{s \in S\backslash J \mid st = ts \text{ for all } t \in J\}| = |S\backslash J| - |\pi_0(J)|$. Thus, $|\{f \in E_2(\overline{T}) \mid fe = e\}| = |S|$.

Assume *2*, and let $s \in S\backslash J$.
Case 1: $st = ts$ *for all* $t \in J$.
In this case $Cl_{W_J}(f_s) = \{f_s\}$.
Case 2: $st_i \neq t_i s$ *for* $i = 1, 2$ *or* 3 *where* $t_i \in C_i \in \pi_0(J)$.
$C_i \neq C_j$ if $t_i \neq t_j$ since S is a tree. Thus

$$|Cl_{W_J}(f_s)| = \sum_i |W_{C_i}/W_{C_i\backslash\{t_i\}}|.$$

It is a basic fact that, in all cases $|W_{C_i}/W_{C_i\backslash\{t_i\}}| \geq |C_i| + 1$, with equality if and only if $C_i\backslash\{t_i\} \subseteq C_i$ is a setup of type $A_{l-1} \subseteq A_l$.

In any case,

$$|\{f \in E_2(\overline{T}) \mid fe = e\}| = |S\backslash J| + \sum_{s\in S\backslash J} (\sum_{C_i \not\subseteq C_J(f_s))} |W_{C_i}/W_{C_i\backslash\{t_i\}}| - 1).$$

One checks that if the right-hand-side of this equation is equal to $|S|$ then each $C \in \pi_0(J)$ occurs exactly once in the double summation, and each $C_i\backslash\{t_i\} \subseteq C_i$ is a setup of type $A_{l-1} \subseteq A_l$. Thus *3* holds. □

Notice that, in particular, if (W,S) is an irreducible Weyl group and $J \subset S$ is a combinatorially smooth subset then each irreducible component of J contains exactly one end-node of S.

Corollary 3.3. *For each irreducible Dynkin diagram we obtain the following calculation for* $\{J \subseteq S \mid J$ *is combinatorially smooth*$\}$. *For each type the list is grouped into the different cases depending on which of the end-nodes are elements of* J.

1. A_1.

(a) $J=\phi$.

A_n, $n\geq 2$. *Let* $S=\{s_1,...,s_n\}$.

(a) $J=\phi$.
(b) $J=\{s_1,...,s_i\}$, $1\leq i<n$.
(c) $J=\{s_j,...,s_n\}$, $1<j\leq n$.
(d) $J=\{s_1,...,s_i,s_j,...s_n\}$, $1\leq i$, $i\leq j-3$ *and* $j\leq n$.

2. B_2.

(a) $J=\phi$.
(b) $J=\{s_1\}$.
(c) $J=\{s_2\}$.

B_n, $n\geq 3$. *Let* $S=\{s_1,...,s_n\}$, α_n *short.*

(a) $J=\phi$.
(b) $J=\{s_1,...,s_i\}$, $1\leq i<n$.
(c) $J=\{s_n\}$.
(d) $J=\{s_1,...,s_i,s_n\}$, $1\leq i$ *and* $i\leq n-3$.

3. C_n, $n\geq 3$. *Let* $S=\{s_1,...,s_n\}$, α_n *long.*

(a) $J=\phi$.
(b) $J=\{s_1,...,s_i\}$, $1\leq i<n$.
(c) $J=\{s_n\}$.
(d) $J=\{s_1,...,s_i,s_n\}$, $1\leq i$ *and* $i\leq n-3$.

4. D_n, $n\geq 4$. *Let* $S=\{s_1,...s_{n-2},s_{n-1},s_n\}$.

(a) $J=\phi$.
(b) $J=\{s_1,...,s_i\}$, $i\leq n-3$.
(c) $J=\{s_{n-1}\}$.
(d) $J=\{s_n\}$.
(e) $J=\{s_1,...,s_i,s_{n-1}\}$, $i\leq n-4$.
(f) $J=\{s_1,...,s_i,s_n\}$, $i\leq n-4$.

5. E_6. *Let* $S=\{s_1,s_2,s_3,s_4,s_5,s_6\}$.

(a) $J=\phi$.
(b) $J=\{s_1\}$ *or* $\{s_1,s_2\}$.

(c) $J = \{s_5\}$ or $\{s_4, s_5\}$.

(d) $J = \{s_6\}$.

(e) $J = \{s_1, s_5\}, \{s_1, s_2, s_5\}$ or $\{s_1, s_4, s_5\}$.

(f) $J = \{s_1, s_6\}$.

(g) $J = \{s_5, s_6\}$

(h) $J = \{s_1, s_5, s_6\}$.

6. E_7. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$.

(a) $J = \phi$.

(b) $J = \{s_1\}, \{s_1, s_2\}$ or $\{s_1, s_2, s_3\}$.

(c) $J = \{s_6\}$ or $\{s_5, s_6\}$.

(d) $J = \{s_7\}$.

(e) $J = \{s_1, s_6\}, \{s_1, s_2, s_6\}, \{s_1, s_2, s_3, s_6\}, \{s_1, s_5, s_6\},$ or $\{s_1, s_2, s_5, s_6\}$.

(f) $J = \{s_6, s_7\}$.

(g) $J = \{s_1, s_7\}$ or $\{s_1, s_2, s_7\}$.

(h) $J = \{s_1, s_6, s_7\}, \{s_1, s_2, s_6, s_7\}$.

7. E_8. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$.

(a) $J = \phi$.

(b) $J = \{s_1\}, \{s_1, s_2\}, \{s_1, s_2, s_3\}$ or $\{s_1, s_2, s_3, s_4\}$.

(c) $J = \{s_7\}$ or $\{s_6, s_7\}$.

(d) $J = \{s_8\}$.

(e) $J = \{s_1, s_7\}, \{s_1, s_2, s_7\}, \{s_1, s_2, s_3, s_7\}, \{s_1, s_2, s_3, s_4, s_7\},$
$\{s_1, s_6, s_7\}, \{s_1, s_2, s_6, s_7\}, \{s_1, s_2, s_3, s_6, s_7\}$ or $\{s_1, s_2, s_5, s_6\}$.

(f) $J = \{s_7, s_8\}$.

(g) $J = \{s_1, s_8\}, \{s_1, s_2, s_8\}$ or $\{s_1, s_2, s_3, s_8\}$.

(h) $J = \{s_1, s_7, s_8\}, \{s_1, s_2, s_7, s_8\}$.

8. F_4. Let $S = \{s_1, s_2, s_3, s_4\}$.

(a) $J = \phi$.

(b) $J = \{s_1\}$ or $\{s_1, s_2\}$.

(c) $J = \{s_4\}$ or $\{s_3, s_4\}$.

(d) $J = \{s_1, s_4\}$.

9. G_2. Let $S = \{s_1, s_2\}$.

(a) $J = \phi$.

(b) $J = \{s_1\}$.

(c) $J = \{s_2\}$.

Proof. This is an elementary calculation with Dynkin diagrams using Theorem 3.2. The numbering of the elements of S is as follows. For types A_n, B_n, C_n, F_4, and G_2 it is the usual numbering. In these cases the end nodes are s_1 and s_n. For type E_6 the end nodes are s_1, s_5 and s_6 with $s_3s_6 \neq s_6s_3$. For type E_7 the end nodes are s_1, s_6 and s_7 with $s_4s_7 \neq s_7s_4$. For type E_8 the end nodes are s_1, s_7 and s_8 with $s_5s_8 \neq s_8s_5$. In each case of type E_n, the nodes corresponding to $s_1, s_2, ..., s_{n-1}$ determine the unique subdiagram of type A_{n-1}. For type D_n the end nodes are s_1, s_{n-1} and s_n. The two subdiagrams of D_n, of type A_{n-1}, correspond to the subsets $\{s_1, s_2, ..., s_{n-2}, s_{n-1}\}$ and $\{s_1, s_2, ..., s_{n-2}, s_n\}$ of S. □

Remark 3.4. It is easy to check that $J \subset S$ is combinatorially smooth if an only if $X(J)$ is simplicial, or quasi smooth, in the sense of Danilov [4].

4 The Descent System (W^J, S^J)

Let (W, S) be a finite Weyl group and let $w \in W$. It is widely appreciated [1, 2, 14] that the **descent set**

$$D(w) = \{s \in S \mid l(ws) < l(w)\}$$

determines a very large and important chapter in the study of Coxeter groups. In this section we interpret the results of Sections 2 and 3 solely in the language of Coxeter groups applied to W, W^J, $J \subset S$ and the Bruhat ordering on W^J. Our main result here is the explicit identification of the subset $S^J \subset W^J$.

Definition 4.1. Let (W, S) be a weyl group and let $J \subset S$ be a proper subset. Define

$$S^J = (W_J(S \setminus J)W_J) \cap W^J.$$

We refer to (W^J, S^J) as the *descent system* associated with $J \subset S$.

Proposition 4.2. *Let (W^J, S^J) be the descent system associated with $J \subset S$. The following are equivalent.*

1. *J is combinatorially smooth.*

2. *$|S^J| = |S|$.*

Proof. The equivalence of *1* and *2* follows from Proposition 2.13 using part *2* of Theorem 3.2. □

Definition 4.3. Assume that $J \subset S$ is combinatorially smooth. Define, for $s \in S \setminus J$,

$$S^J_s = (W_J s W_J) \cap W^J.$$

Recall now, that for $s \in S \setminus J$, there is a unique $g_s \in \Lambda_2$ such that $\{s\} = \{t \in S \mid tg_s = g_s t \neq g_s\}$. Furthermore, $s \rightsquigarrow g_s$ determines a bijection between $S \setminus J$ and Λ_2. Each $g \in E_2(\overline{T})$ is conjugate to a unique g_s, $s \in S \setminus J$. See part *2* of Theorem 1.2.

Theorem 4.4. *Assume that $J \subset S$ is combinatorially smooth. Then*

1. $S^J = \sqcup_{s \in S \setminus J} S^J_s$.

2. *Let $s \in S \setminus J$. In case $st = ts$ for all $t \in J$, $S^J_s = \{s\}$. Otherwise, $S^J_s = \{s, t_1 s, t_2 t_1 s, ..., t_m \cdots t_2 t_1 s\}$ where $C = C_s = \{t_1, t_2, ..., t_m\}$, $st_1 \neq t_1 s$ and $t_i t_{i+1} \neq t_{i+1} t_i$ for for $i = 1, ..., m-1$.*

3. $S^J_s \cong \{g \in E_2 \mid ge_1 = e_1$ *and* $cgc^{-1} = g_s$ *for some* $c \in W_J\}$.

Proof. This is well-known information about the standard inclusion of Weyl groups $W_{n-1} \subset W_n$ of type A. The details are left to the reader. See also Theorem 3.2 above. □

Example 4.5. Let

$$W = < s_1, ... s_n >$$

be the Weyl group of type A_n (so that $W \cong S_{n+1}$), and let

$$J = \{s_2, ..., s_n\} \subset S = \{s_1, ..., s_n\}.$$

Then $J \subset S$ is combinatorially very smooth. One checks, using Theorem 4.4, that

$$W^J = \{1, s_1, s_2 s_1, s_3 s_2 s_1, ..., s_n s_{n-1} \cdots s_2 s_1\}.$$

Notice that

$$1 < s_1 < s_2 s_1 < ... < s_n s_{n-1} \cdots s_1.$$

In this very special example we also obtain that $S^J = W^J \setminus \{1\}$. We now calculate

$$A^J(w) = A^J_{s_1}(w)$$

for each $w \in W^J$. This a simple calculation using the generators and relations of W. One obtains,

$(s_j \cdots s_1)(s_i \cdots s_1) = [s_j \cdots s_2]$ if $1 = i \leq j$,

$(s_j \cdots s_1)(s_i \cdots s_1) = (s_{i-1} \cdots s_1)[s_j \cdots s_2]$ if $1 < i \leq j$, and

$(s_j \cdots s_1)(s_i \cdots s_1) = (s_i \cdots s_1)[s_{j+1} \cdots s_2]$ if $i > j$.

We conclude from this that

$$A^J(s_j \cdots s_1) = \{s_m \cdots s_1 \mid m > j\}.$$

Let us write $a_j = s_j \cdots s_1$ if we think of $s_j \cdots s_1 \in W^J$, and $r_j = s_j \cdots s_1$ if we think of $s_j \cdots s_1 \in S^J$. Also, if $w \in W$, we write w_0 for the element of minimal length in wW_J. By the calculation above we obtain that

$(a_j r_i)_0 = 1 < a_j$ if $1 = i \leq j$,

$(a_j r_i)_0 = a_{i-1} < a_j$ if $1 < i \leq j$, and

$(a_j r_i)_0 = a_i > a_j$ if $i > j$.

Can it get any better than this?

Example 4.6. Let

$$W =< s_1, ...s_n >$$

be the Weyl group of type A_n (so that $W \cong S_{n+1}$), and let

$$J = \{s_3, ...s_n\} \subset S.$$

Notice that $J \subset S$ is combinatorially smooth.

If $w \in W^J$ then $w = a_p$, $w = b_q$, or else $w = a_p b_q$. Here $a_p = s_p \cdots s_1$ $(1 \leq p \leq n)$ and $b_q = s_q \cdots s_2$ $(2 \leq q \leq n)$. If we adopt the useful convention $a_0 = 1$ and $b_1 = 1$, then we can write

$$W^J = \{a_p b_q \mid 0 \leq p \leq n \text{ and } 1 \leq q \leq n\}$$

with uniqueness of decomposition. Let $w = a_p b_q \in W^J$. After some tedious calculation with braid relations and reflections, we obtain that

a) $A^J_{s_1}(a_p b_q) = \{s_1\}$ if $p < q$.
$A^J_{s_1}(a_p b_q) = \phi$ if $q \leq p$.
Thus $\nu_{s_1}(a_p b_q) = 1$ if $p < q$ and $\nu_{s_1}(a_p b_q) = 0$ if $q \leq p$.

b) $A^J_{s_2}(a_p b_q) = \{s_m \cdots s_n \mid m > q\}$ if $q < n$.
$A^J_{s_2}(a_p b_q) = \phi$ if $q = n$.
Thus $\nu_{s_2}(a_p b_q) = n - q$.

It is interesting to compute the "Euler polynomial"

$$H(t_1, t_2) = \sum_{w \in W^J} t_1^{\nu_1(w)} t_2^{\nu_2(w)}$$

of the augmented poset $(W^J, \leq, \{\nu_1, \nu_2\})$ (we write ν_i for ν_{s_i}). A simple calculation yields

$$H(t_1, t_2) = \sum_{k=1}^{n} [k t_1 + (n + 1 - k)] t_2^{n-k}.$$

References

[1] A. Bjorner, F. Brenti, *"Combinatorics of Coxeter Groups"*, Graduate Texts in Mathematics 231, Springer Verlag, New York, 2005.

[2] F. Brenti, *q-Eulerian poynomials arising from Coxeter groups*, European Journal of Combinatorics (1994)15, 417-441.

[3] K. Brown, *Semigroup and ring theoretical methods in probability*, in "Representations of finite dimensional algebras and related topics in Lie theory and geometry", Fields Inst. Commun. 40, Amer.Math.Soc., Providence, RI, 2004, 3-26.

[4] V.I. Danilov, *"The Geometry of Toric Varieties"*, Russian Mathematical Surveys, 33(1978), 97-154.

[5] C. DeConcini and C. Procesi, *Complete symmetric varieties*, Springer Lecture Notes in Mathematics 131(1983), 1-44.

[6] I. Dolgachev and V. Lunts, *A character formula for the representation of a weyl group on the cohomology of the associated toric variety*, Journal of Algebra, 168(1994), 741-772.

[7] F. D. Grosshans, *The variety of points which are not semi-stable*, Illinois Journal of Math. 26(1982), 138-148.

[8] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, GTM vol. 9, Third Ed., Springer, 1980.

[9] M. S. Putcha, *"Linear algebraic monoids"*, Cambridge University Press, 1988.

[10] M. S. Putcha and L. E. Renner, *The system of idempotents and lattice of $\mathcal{J}$-classes of reductive algebraic monoids*, Journal of Algebra 116(1988), 385-399.

[11] L. E. Renner, *An explicit cell decomposition of the canonical compactification of an algebraic group*, Can. Math. Bull., 46(2003), 140-148.

[12] L. E. Renner, *"Linear Algebraic Monoids"*, Encyclopedia of Mathematical Sciences, vol. 134, Springer, 2005.

[13] J. R. Stembridge, *Some permutation representations of Weyl groups associated with the cohomology of toric varieties*, Advances in Mathematics 106(1994), 244-301.

[14] L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, Journal of Algebra 41(1976), 255-264.

[15] L. Solomon, *An introduction to reductive monoids*, "Semigroups, formal languages and groups", J. Fountain, Ed., Kluwer Academic Publishers, 1995, 295-352.

[16] R. P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics and geometry*, Ann. NY Acad. Sci. 576(1989),500-534.

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