

# DISTRIBUTION OF PRODUCTS IN FINITE MONOIDS II: ALGEBRA

Lex E. Renner

## 1 Introduction

Let  $S$  be a finite monoid and let  $a \in S$ . Define

$$H_n(a) = \{(x_1, \dots, x_n) \in S^n \mid x_1 \cdot \dots \cdot x_n = a\}$$

and let

$$h_n(a) = |H_n(a)| .$$

To get a grip on the behaviour of  $\{h_n(a) \mid n \geq 1\}$  we define

$$h(a) = \sum_{n \geq 1} h_n(a) t^{n-1} .$$

$h(a)$  was studied in [2] from a combinatorial perspective. We found that it could be pictured as a generalization of the generating function for the *Zeta Polynomial*  $Z(P, n)$  of a finite poset  $(P, \geq)$  [3; §3.11].

In this paper we continue our study of  $\{h(a) \mid a \in S\}$ . But this time we approach it from a more algebraic perspective. For example

- (i) What is the recurrence formula for  $\{h_1(a), h_2(a), \dots\}$  and how are these recurrence formulae (as  $a \in S$  varies) related? (See Proposition 2.4.)
- (ii) Is there a canonical collection  $\{R_0, R_1, \dots, R_{n-1}\}$  of rational functions that forms a basis of  $\text{Span}_{\mathbb{Q}}\{h(a) \mid a \in S\}$ ? (See Proposition 2.2.)
- (iii) What role does  $\mathbb{Q}[S]$  (the monoid algebra) play in the discussion? (See Section 2.)

We consider the example  $S = M_n(\mathbb{F}_q)$  in detail. In a future paper we shall use the results of this paper to obtain explicit calculations for a large class of finite monoids.

## 2 The Element $\alpha$

Let  $S$  be a finite monoid and let  $\mathbb{Q}[S]$  be the monoid algebra of  $S$  over  $\mathbb{Q}$ . Define

$$\alpha = \sum_{s \in S} s \in \mathbb{Q}[S] .$$

**2.1 Lemma.** (i)  $\alpha^n = \sum_{s \in S} h_n(s)s$  where  $h_n(s) = |\{(x_1, \dots, x_n) \in S^n \mid x_1 \cdot \dots \cdot x_n = s\}|$ .

$$(ii) \frac{\alpha}{1 - \alpha t} = \sum_{s \in S} \left( \sum_{n \geq 1} h_n(s)t^{n-1} \right) s =: \sum_{s \in S} h(s)s \text{ where } h(s) \in \mathbb{Q}(t).$$

*Proof.* (i) results from gathering the terms of a straightforward calculation. Everything in (ii) is a formality except the fact that  $h(s) \in \mathbb{Q}(t)$ . But that follows from [2; Theorem 2.2].  $\square$

A well-known theorem of undergraduate algebra says that we can write

$$\begin{aligned} \alpha^n &= A_0 + A_1\alpha + \dots + A_{n-1}\alpha^{n-1} \\ \text{where } \{1, \alpha, \dots, \alpha^{n-1}\} &\subseteq \mathbb{Q}[S] \text{ is linearly independent,} \\ \text{and } \{A_i\} &\subseteq \mathbb{Q} . \end{aligned}$$

Furthermore, the  $A_i$ 's are unique. We write

$$\min(\alpha) = X^n - A_{n-1}X^{n-1} - \dots - A_1X - A_0 .$$

**2.2 Proposition.**  $\frac{\alpha}{1 - \alpha t} = R_0 + R_1\alpha + \dots + R_{n-1}\alpha^{n-1}$  where

$$\begin{aligned} R_0 &= \frac{A_0 t^{n-1}}{D} \\ R_{n-1} &= \frac{t^{n-2}}{D} \\ R_{n-2} &= \frac{t^{n-3}}{D} - A_{n-1} \frac{t^{n-2}}{D} \\ &\vdots \\ R_2 &= \frac{t}{D} - A_{n-1} \frac{t^2}{D} - \dots - A_3 \frac{t^{n-2}}{D} \\ R_1 &= \frac{1}{D} - A_{n-1} \frac{t}{D} - \dots - A_2 \frac{t^{n-2}}{D} \end{aligned}$$

and

$$D = 1 - t(A_0 t^{n-1} + \dots + A_{n-1}) .$$

Furthermore,  $A_0 = 0$  if  $S$  has more than one invertible element. So  $R_0 = 0$  in that case.

*Proof.* Write  $\frac{\alpha}{1-\alpha t} = R_0 + R_1\alpha + \cdots + R_{n-1}\alpha^{n-1}$ . So  $\alpha = (1-\alpha t)(R_0 + R_1\alpha + \cdots + R_{n-1}\alpha^{n-1})$ . Expand this out using  $\alpha^n = A_0 + A_1\alpha + \cdots + A_{n-1}\alpha^{n-1}$ , and we obtain the advertised formulae for  $R_i$ ,  $i = 0, 1, \dots, n-1$ .

To complete the proof we only need to show that  $A_0 = 0$  if  $G(S) = \{s \in S \mid sg = gs \doteq 1 \text{ for some } g \in S\}$  contains more than one element.

If  $s \in G(S)$ , one checks easily that  $h_n(s) = g^{n-1}$  where  $g = |G(S)|$ . So  $h(s) = \frac{1}{1-gt}$ . Now let  $\alpha = \sum_{s \in S} s \in \mathbb{Q}[S]$  and let  $\bar{\alpha} \in \mathbb{Q}[S]/\langle S \setminus G(S) \rangle \cong \mathbb{Q}[G]$ . With this identification

$$\bar{\alpha} = \sum_{s \in G(S)} s,$$

so that  $\min(\bar{\alpha}) = X^2 - gX$  as long as  $g > 1$ . But if  $f \in \mathbb{Q}[x]$  and  $f(\alpha) = 0$  then  $f(\bar{\alpha}) = 0$ . So  $\min(\bar{\alpha}) \mid \min(\alpha)$ . But  $X \nmid \min(\bar{\alpha})$ , and so the constant term of  $\min(\alpha)$  is zero.  $\square$

**2.3 Remark.** (a) *A simple calculation shows that*

$$\text{Span}_{\mathbb{Q}}\{R_0, \dots, R_{n-1}\} = \begin{cases} \text{Span}_{\mathbb{Q}}\left\{\frac{1}{D}, \frac{t}{D}, \dots, \frac{t^{n-2}}{D}\right\} & \text{if } A_0 = 0 \\ \text{Span}\left\{\frac{1}{D}, \frac{t}{D}, \dots, \frac{t^{n-1}}{D}\right\} & \text{if } A_0 \neq 0 \end{cases}$$

*In particular,  $\{R_1, \dots, R_{n-1}\}$  is linearly independent over  $\mathbb{Q}$  (and if  $A_0 \neq 0$  then  $\{R_0, \dots, R_{n-1}\}$  is linearly independent).*

(b) *An easy calculation shows that if we can write*

$$\frac{\alpha}{1-\alpha t} = \sum_{i=1}^m h_i(t)x_i$$

*where  $\{x_i\} \subseteq \mathbb{Q}[S]$  is a  $\mathbb{Q}$ -basis, and  $\{h_i(t)\} \subseteq \mathbb{Q}(t)$ , then*

$$\text{Span}_{\mathbb{Q}}\{h_i(t)\}_{i=1}^m = \text{Span}_{\mathbb{Q}}\{R_i\}_{i=1}^{n-1}.$$

*It follows that  $(D) = \{g \in \mathbb{Q}[t] \mid gh(a) \in \mathbb{Q}[t]\}$ .*

(c) *We can also express the  $A_i$ 's in terms of the  $R_i$ 's. As before, expand out*

$$\alpha = (1-\alpha t)(R_0 + R_1\alpha + \cdots + R_{n-1}\alpha^{n-1})$$

*and again use  $\alpha^n = A_0 + A_1\alpha + \cdots + A_{n-1}\alpha^{n-1}$ . We then obtain*

$$\alpha^n = \frac{R_0}{tR_{n-1}} + \left(\frac{R_1 - 1 - tR_0}{tR_{n-1}}\right)\alpha + \left(\frac{R_2 - tR_1}{tR_{n-1}}\right)\alpha^2 + \cdots + \left(\frac{R_{n-1} - tR_{n-2}}{tR_{n-1}}\right)\alpha^{n-1}.$$

Thus,

$$\begin{aligned}
A_0 &= \frac{R_0}{tR_{n-1}} \\
A_1 &= \frac{R_1 - 1 - tR_0}{tR_{n-1}} \\
A_2 &= \frac{R_2 - tR_1}{tR_{n-1}} \\
&\vdots \\
A_{n-1} &= \frac{R_{n-1} - tR_{n-2}}{tR_{n-1}}
\end{aligned}$$

because  $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\} \subseteq \mathbb{Q}(t)[S]$  is linearly dependent over  $\mathbb{Q}(t)$ .

**2.4 Proposition.** Let  $f(X) = X^d - (B_0 + B_1X + \dots + B_{d-1}X^{d-1})$ . Then the following are equivalent, where  $\alpha = \sum_{s \in S} s$ .

(i)  $f(\alpha) = 0$ .

(ii)  $\min(\alpha) \mid f$ .

(iii)  $h_{d+m}(s) = B_0h_m(s) + B_1h_{m+1}(s) + \dots + B_{d-1}h_{m+d-1}(s)$  for all  $m \geq 0$ . Here we use the convention

$$h_0(s) = \begin{cases} 1 & \text{if } s = 1 \\ 0 & \text{if } s \neq 1 \end{cases} .$$

(iv) (a)  $h(s) t^d f(1/t) \in \mathbb{Q}[t]$  for any  $s \in S$ .

(b) If  $t^k \mid \min(\alpha)$  then  $t^k \mid f$ .

*Proof.* (i) and (ii) are equivalent by elementary algebra.

(iii) is equivalent to saying

$$\sum_{s \in S} h_d(s)s = B_0 + B_1 \left( \sum_{s \in S} h_1(s)s \right) + \dots + B_{d-1} \left( \sum_{s \in S} h_{d-1}(s)s \right)$$

so that (using  $\sum_{s \in S} h_\ell(s)s = \alpha^\ell$ )

$$\alpha^d = B_0 + B_1\alpha + \dots + B_{d-1}\alpha^{d-1} .$$

Hence, (i) and (iii) are equivalent.

Before we consider (iv) notice that if  $g(t) = t^k \prod_{i=1}^s (t - \alpha_i)^{n_i}$  then  $t^{\deg(g)} g(1/t) = \prod_{i=1}^s (1 - \alpha_i t)^{n_i}$ . It follows that  $g(t) \mid f(t)$  if and only if  $t^{\deg(g)} g(1/t) \mid t^d f(1/t)$ , and  $t^k \mid f$  whenever  $t^k \mid g$ . But from 2.2,  $D = t^n \min(\alpha)(1/t)$ , while from Remark 2.3 (b),  $h(s)t^d f(1/t) \in \mathbb{Q}[t]$  if and only if  $D \mid t^d f(1/t)$ . Thus, (iv) is equivalent to (ii).  $\square$

**2.5 Theorem.** With  $\{R_i(t) \mid i = 0, \dots, n-1\}$  as in 2.2 we obtain

$$h(a) = h_0(a)R_0(t) + h_1(a)R_1(t) + \dots + h_{n-1}(a)R_{n-1}(t) .$$

*Proof.* Let  $h_i = h_i(a)$ ,  $h = h(a)$ . Then

$$\begin{aligned} h &=: \sum_{i \geq 1} h_i t^{i-1} \\ &= h_1 + h_2 t + \dots + h_{n-1} t^{n-2} \\ &\quad + (\epsilon + A_1 h_1 + \dots + A_{n-1} h_{n-1}) t^{n-1} \\ &\quad + (A_0 h_1 + \dots + A_{n-1} h_n) t^n \\ &\quad + (A_0 h_2 + \dots + A_{n-1} h_{n+1}) t^{n+1} \\ &\quad \vdots \end{aligned}$$

(using 2.4 with  $f = \min(\alpha)$ , where  $\epsilon = A_0$  if  $a = 1$  and  $\epsilon = 0$  if  $a \neq 1$ )

$$\begin{aligned} &= h_1 + h_2 t + \dots + h_{n-1} t^{n-2} \\ &\quad + A_0 t^n h + \delta \\ &\quad + A_1 t^{n-1} h \\ &\quad + A_2 t^{n-2} (h - h_1) \\ &\quad \vdots \\ &\quad + A_{n-1} t (h - (h_1 + h_2 + \dots + h_{n-2})) \end{aligned}$$

(where  $\delta = A_0 t^{n-1}$  if  $a = 1$  and zero otherwise).

Thus, gathering terms

$$\begin{aligned} h(1 - t(A_0 t^{n-1} + \dots + t A_{n-2} + A_{n-1})) &= h_1 + h_2 t + \dots + h_{n-1} t^{n-2} + \delta \\ &\quad - A_2 t^{n-2} h_1 \\ &\quad - A_3 t^{n-3} (h_1 + h_2 t) \\ &\quad \vdots \\ &\quad - A_{n-1} t (h_1 + h_2 t + \dots + h_{n-2} t^{n-3}) \\ &= h_1 + (h_2 - A_{n-1} h_1) t + (h_3 - A_{n-1} h_2 - A_{n-2} h_1) t^2 \\ &\quad + \dots + (h_{n-1} - A_{n-1} h_{n-2} - \dots - A_2 h_1) t^{n-2} + \delta \\ &= h_1 (1 - A_{n-1} t - \dots - A_2 t^{n-2}) \\ &\quad + h_2 (t - A_{n-1} t^2 - \dots - A_3 t^{n-2}) \\ &\quad \vdots \\ &\quad + h_{n-1} t^{n-2} + \delta . \end{aligned}$$

The conclusion follows immediately from 2.2. □

**2.6 Remark.** *In practice, one should calculate  $h(a)$  using a slight variation of Theorem 2.5. Indeed, for  $a \in S$ , let*

$$\Omega_a = \{x \in S \mid a \notin SxS\} .$$

*Then  $\Omega_a$  is the two-sided ideal of elements that cannot be involved in any product resulting in  $a$ . Thus we can replace  $S$  by the Rees quotient  $S/\Omega_a$  [1; page 17]. Usually, the minimal polynomial of  $\alpha$  will have lower degree for  $S/\Omega_a$  than for  $S$ .*

### 3 How To Find $\text{Min}(\alpha)$

The results of 2.4 and 2.5 imply that, if we can compute  $\text{min}(\alpha)$ , everything else will fall into place; once we calculate  $\{h_1(a), \dots, h_{n-1}(a) \mid a \in S\}$ . In this section we consider the problem of actually finding  $\text{min}(\alpha)$  in terms of salient properties of  $S$ . We first recall some results from [2].

Let  $S$  be a finite monoid and let  $a \in S$ . Define

$$\begin{aligned} \bar{a} &= \{b \in S \mid aS = bS\} \\ &= \{a_1, \dots, a_s\} . \end{aligned}$$

Define

$$R_{\bar{a}} = \begin{pmatrix} r(a_1/a_1) & r(a_2/a_1) & \dots & \dots \\ r(a_1/a_2) & r(a_2/a_2) & & \\ \vdots & & & \end{pmatrix}$$

where

$$r(b/a) = |\{x \in S \mid a = bx\}| .$$

We say  $b \geq a$  if  $a = bx$  for some  $x \in S$ . By [2; Theorem 2.2] we have

$$h(a_i) = \frac{1}{\det(I_s - tR)} \left( P_{a_i} + t \sum_{b > a} P(b/a_i) h(b) \right) \quad (*)$$

where

$$\begin{pmatrix} P_{a_1} \\ \vdots \\ P_{a_s} \end{pmatrix} = \text{Adj}(I_s - tR) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} P(b/a_1) \\ \vdots \\ P(b/a_s) \end{pmatrix} = \text{Adj}(I_s - tR) \begin{pmatrix} r(b/a_1) \\ \vdots \\ r(b/a_s) \end{pmatrix}$$

The following result gives us a crude starting point in our pursuit of  $\text{min}(\alpha)$ .

**3.1 Proposition.** Let  $S/\mathcal{R} = S/\sim$ , where  $a \sim b$  if  $aS = bS$ . For  $\bar{a} \in S/\mathcal{R}$  define  $R_{\bar{a}}$  as above. Let  $\alpha = \sum_{s \in S} s$ . Then  $\min(\alpha)$  divides  $t^k \left( \prod_{\bar{a} \in S/\mathcal{R}} \det(tI_s - R_{\bar{a}}) \right)$  for some  $k \geq 0$ .

*Proof.* By the above formula (\*), if  $D$  is the generator of  $\{f \in \mathbb{Q}[t] \mid fh(a) \in \mathbb{Q}[t] \text{ for all } a \in S\} \triangleleft \mathbb{Q}[t]$ . Then  $D$  divides  $f(t) = \prod_{\bar{a} \in S/\mathcal{R}} \det(I_s - tR_{\bar{a}})$ . Thus, if  $\deg(D) = m$  and  $\deg(f) = n$  then  $t^m D(1/t)$  divides  $t^n f(1/t)$ . But  $\min(\alpha) = t^k t^m D(1/t)$  for some  $k \geq 0$  by remark 2.3.2 and the proof of 2.2. Thus,  $\min(\alpha)$  divides  $t^{k+n} f(1/t) = t^k \left( \prod_{\bar{a} \in S/\mathcal{R}} \det(tI_s - R_{\bar{a}}) \right)$ .  $\square$

A finite monoid  $S$  is  $\mathcal{R}$ -homogeneous [2] if

- (i)  $a_1 \sim a_2 \sim a_3 \sim a_4$  implies  $r(a_1/a_2) = r(a_3/a_4)$
- (ii)  $a_1 \sim a_2 < b$  implies  $r(b/a_1) = r(b/a_2)$ .

By [2; Theorem 5.2] we have the following result:

Let  $S$  be  $\mathcal{R}$ -homogeneous and let  $a \in S$ . Then

$$h(a) = \frac{1}{1 - s_a r_a t} \left( 1 + t \sum_{b > a} r(b/a) h(b) \right) \quad (**)$$

where  $r_a = r(a/a)$  and  $s_a = |\{x \in S \mid x \sim a\}|$

**3.2 Proposition.** Let  $S$  be  $\mathcal{R}$ -homogeneous.

(i) The following are equivalent.

- (a)  $\min(\alpha)(0) \neq 0$ .
- (b)  $\{s \in S \mid st = 1 \text{ for some } t \in S\} =: G(S) = \{1\}$ .

(ii) If  $G(S) \not\cong \{1\}$  then  $X \mid \min(\alpha)$  yet  $X^2 \nmid \min(\alpha)$ .

*Proof.* From the proof of 2.2 we see that  $h(s) = \frac{1}{1-gt}$  for  $s \in G(S)$ , where  $g = |G(S)|$ . We also showed that if  $S = G(S)$ , then

$$\min(\alpha) = \begin{cases} X^2 - gX & \text{if } |G(S)| = g > 1 \\ X - 1 & \text{if } |G(S)| = 1 \end{cases}.$$

So we proceed inductively using 2.4. Let  $a \in S \setminus G(S)$ . So  $a < 1$ . Assume inductively that  $f(X) = X^n - (B_0 + B_1X + \cdots + B_{n-1}X^{n-1})$ , and assume further that

- (i)  $h_n(b) = B_0 h_0(b) + B_1 h_1(b) + \cdots + B_{n-1} h_{n-1}(b)$  for any  $b > a$ .
- (ii)  $X^2 \nmid f$ .

It follows from (\*\*) above that

$$h_{m+1}(a) - r h_m(a) = \sum_{b>a} r(b/a)h_m(b) \text{ for any } m \geq 0, \text{ where } r = s_a r_a$$

since we have adopted the convention that

$$h_0(a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1 \end{cases} .$$

So we calculate

$$\begin{aligned} h_{m+1}(a) - r h_m(a) &= \sum_{b>a} r(b/a)[b_0 h_0(b) + \cdots + B_{n-1} h_{n-1}(b)] \\ &= \sum_{i=0}^{n-1} B_i \left( \sum_{b>a} r(b/a) h_i(b) \right) \\ &= \sum_{i=0}^{n-1} B_i [h_{i+1}(a) - r h_i(a)] . \end{aligned}$$

Thus,  $h(a)$  “satisfies” the polynomial

$$g(X) = (X - r)(X^n - (B_0 + B_1 X + \cdots + B_{n-1} X^{n-1})) .$$

But if  $b > a$  then  $h(b)$  “satisfies” this polynomial because  $f(X)$  is a factor.

Inductively, this procedure manufactures a polynomial

$$h(X) = (X - r_n)(X - r_{n-1}) \cdots (X - r_1) f_0(X)$$

where

$$f_0(X) = \begin{cases} X^2 - gX & \text{if } |G(S)| = g > 1 \\ X - 1 & \text{if } |G(S)| = 1 \end{cases}$$

whose coefficients provide a linear recurrence for  $\{h(a) \mid a \in S\}$  in the sense of Proposition 2.4 (iii). It follows from 2.4 that  $\min(\alpha) \mid h(x)$ . The other conclusions follow from this and our formula for  $h(X)$ .  $\square$

**3.3 Proposition.** *Let  $S$  be  $\mathcal{R}$ -homogeneous and let  $I(S) = \{r \in \mathbb{N} \mid r = s_a r_a \text{ for some } a \in S\}$ . Then*

$$\min(\alpha) = \begin{cases} X \prod_{r \in I(S)} (X - r)^{n_r} , & \text{if } |G(S)| > 1 \\ \prod_{r \in I(S)} (X - r)^{n_r} , & \text{if } |G(S)| = 1 \end{cases}$$

$$\text{where } 1 \leq n_r \leq \max \left\{ m \mid \begin{array}{l} s_{a_i} r_{a_i} = r \text{ for some chain} \\ a_0 < a_1 < \cdots < a_n \end{array} \right\}$$

*Proof.* All that remains here is to prove the inequalities involving  $\{n_r \mid r \in I(S)\}$ . The upper bound amounts to showing that for any  $a \in S$  the pole of

$$h(a) = \frac{1}{1 - r_a s_a t} \left( 1 + t \sum_{b>a} r(b/a) h(b) \right)$$

at  $1/r$  has order less than or equal to

$$n_r(a) = \max \left\{ n \mid \begin{array}{l} s_{a_i} r_{a_i} = r \text{ for some chain} \\ a_0 < a_1 < \dots < a_n \end{array} \right\}$$

But this follows immediately (by induction) once we assume it is true for  $h(b)$ ,  $b > a$ .

To see that each  $n_r > 0$ , consider

$$L_\alpha : \mathbb{Q}[S] \longrightarrow \mathbb{Q}[S]$$

defined by  $L_\alpha(a) = a\alpha$ . Clearly,  $\min(\alpha) = \min(L_\alpha)$ .

So we compute. Let  $a \in S$ . Then  $a\alpha = a \sum_{s \in S} s = \sum_{e \sim a} r(a/e)e + \sum_{x < a} m_x x$  where  $r(a/a) = |\{s \mid as = e\}|$  and  $m_x$  is some nonnegative integer. But  $S$  is  $\mathcal{R}$ -homogeneous, and so  $r(a/e) = r_a = r_e$ . Thus,

$$a\alpha = r_a \left( \sum_{e \sim a} e \right) + \sum_{x < a} m_x x .$$

So let  $A = \sum_{e \sim a} e \in \mathbb{Q}[S]$ . Then

$$A\alpha = r_a s_a A + \sum_{x < a} \ell_x x$$

where  $s_a = |\{e \in S \mid e \sim a\}|$  and  $\ell_x$  is some nonnegative integer.

Notice that  $V = \text{Span}_{\mathbb{Q}}\{x \mid x < a\}$  is  $L_\alpha$ -stable. Thus,  $L_\alpha$  has an eigenvector in  $\mathbb{Q} \cdot A \oplus V \subseteq \mathbb{Q}[S]$  with the eigenvalue  $r_a s_a$ . Hence  $X - r_a s_a$  is a factor of  $\min(\alpha)$ .  $\square$

**3.4 Corollary.** (a) *Suppose  $S$  is  $\mathcal{R}$ -homogeneous. Then the following are equivalent.*

(i)  $\alpha$  is semisimple.

$$(ii) \min(\alpha) = \begin{cases} X \prod_{i=1}^n (X - r_i), & |G(S)| > 1 \\ \prod_{i=1}^n (X - r_i), & |G(S)| = 1 \end{cases} \quad \text{where } \{r_i\} = \{s_a r_a \mid a \in S\}.$$

(iii) For any  $a \in S$ ,  $h(a)$  has simple poles.

(b) *Furthermore, if  $S$  has the property “ $a \leq b$  and  $r_a s_a = r_b s_b$  together imply that  $a \sim b$ ” then  $\alpha$  is semisimple.*

### 3.0.1 Example

In this example we consider the example  $S = M_n(\mathbb{F}_q)$  in detail, where  $\mathbb{F}_q$  is the field with  $q$  elements.

Let  $a \in S$ . Then for some  $g, h \in G(S)$ ,

$$gah = e_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

for some  $0 \leq i \leq a$  (where  $i = \text{rank}(e_i)$ ). It follows easily that

$$r(a/a) = r(e_i/e_i) = |\{x \in S \mid e_i = e_i X\}|.$$

An elementary calculation verifies

$$r_i = r(e_i/e_i) = q^{n(n-i)}.$$

Next we need to find

$$s_i = |\{b \in S \mid b \sim e_i\}| = |e_i G \ell_n(\mathbb{F}_q)|.$$

Another simple calculation yields

$$s_i = (q^n - 1)(q^n - q) \cdot \dots \cdot (q^n - q^{n-1}).$$

So

$$s_i r_i = q^{n(n-i)} (q^n - 1)(q^n - q) \cdot \dots \cdot (q^n - q^{n-1}).$$

Notice that  $s_i r_i \neq s_j r_j$  if  $i \neq j$ . Thus by 3.4 above,  $\alpha$  is semisimple, and so

$$\min(\alpha) = X \prod_{i=0}^n (X - q^{n(n-i)} (q^n - 1) \cdot \dots \cdot (q^n - q^{n-1})) \quad \text{and}$$

$$D = \prod_{i=0}^n (1 - q^{n(n-i)} (q^n - 1) \cdot \dots \cdot (q^n - q^{n-1}) X).$$

Furthermore, by (\*\*) of section 3

$$h(e_i) = \frac{1}{1 - r_i s_i t} \left( 1 + t \sum_{b > e_i} r(b/e_i) h(b) \right).$$

We use this to find an explicit formula of the form

$$h(e_i) = \frac{1}{1 - r_i s_i t} \left( 1 + t \sum_{j > i} A_j h(e_j) \right).$$

We first calculate

$$r(b/e_i) = |\{x \in S \mid bx = e_i\}|.$$

Suppose  $bx = e_i$ . Then

$$\{x \mid bx = e_i\} = x + \{y \mid by = 0\}.$$

Thus  $r(b/e_i) = q^d$  where  $d = \dim_{\mathbb{F}_q}(\{y \mid by = 0\})$ . An elementary calculation shows that  $d = n(n - j)$ , where  $j = \text{rank}(b)$ . To find the sought after formulae, it remains to find

$$\left| \left\{ b \in S \mid \begin{array}{l} bx = e_i \text{ for some } x \\ \text{rank}(b) = j \end{array} \right\} \right| \quad \text{for each } j > i.$$

Define

$$X_j = \left\{ b \in S \mid \begin{array}{l} bx = e_i \text{ for some } x \\ \text{and rank}(b) = j \end{array} \right\} \quad \text{for each } j > i.$$

Define

$$P_i = \{g \in G \mid ge_i = e_i g e_i\}$$

where

$$G = \text{Gl}_n(\mathbb{F}_q).$$

Notice also that

$$P_i X_j G = X_j \quad \text{for all } j > i.$$

After a little calculation we conclude the following:

Let

$$\mathcal{R}_n = \left\{ A \in M_n(\mathbb{F}_q) \mid \begin{array}{l} A \text{ is a } 01 \text{ matrix with at} \\ \text{most one nonzero entry in} \\ \text{each row or column} \end{array} \right\}.$$

and let

$$\mathcal{X}_j = \{r \in \mathcal{R}_n \mid e_i \in r\mathcal{R}_n, \text{rank}(r) = j\}.$$

Then

$$X_j = P_i \mathcal{X}_j G.$$

Now

$$\mathcal{X} =: \bigcup_{j>i} \mathcal{X}_j = \bigcup_{e \in E(\mathcal{X})} eW$$

and after a little more calculation we obtain

$$\mathcal{X} = C_W(e_i) \Lambda(\mathcal{X}) W$$

where  $W \subseteq \mathcal{R}_n$  is the unit group and

$$\Lambda(\mathcal{X}) = \{e_{i+1}, e_{i+2}, \dots, e_n = 1\}.$$

Thus,

$$X_j = P_i e_j G$$

and

$$X = \bigcup_{j=i+1}^n X_j .$$

But

$$|X_j| = (q^j - 1) \cdot \dots \cdot (q^j - q^{j-1}) \begin{bmatrix} u \\ j \end{bmatrix}_q \begin{bmatrix} n-i \\ j-i \end{bmatrix}_q$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(q^n - 1) \cdot \dots \cdot (q - 1)}{[(q^j - 1) \cdot \dots \cdot (q - 1)][(q^{n-j} - 1) \cdot \dots \cdot (q - 1)]} .$$

So

$$\begin{aligned} h(e_i) &= \frac{1}{1 - r_i s_i t} \left( 1 + t \sum_{b > e_i} r(b/e_i) h(b) \right) \\ &= \frac{1}{1 - r_i s_i t} \left( 1 + t \sum_{j=i+1}^n \sum_{\substack{b > e_i \\ \text{rank}(b)=j}} r(b/e_i) h(b) \right) \\ &= \frac{1}{1 - r_i s_i t} \left( 1 + t \sum_{j=i+1}^n q^{n(n-j)} \left| \left\{ b \mid \begin{array}{l} bx = e_i \text{ for some } x \\ \text{rank}(b) = j \end{array} \right\} \right| h(b) \right) \\ &= \frac{\left( 1 + t \sum_{j=i+1}^n q^{n(n-j)} (q^j - 1) \cdot \dots \cdot (q^j - q^{j-1}) \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-i \\ j-i \end{bmatrix}_q h(e_j) \right)}{1 - q^{n(n-i)} (q^n - 1) \cdot \dots \cdot (q - 1) t} . \end{aligned}$$

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L. E. Renner  
 University of Western Ontario  
 London, Canada  
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